Серия «Математика»
2018. Т. 26. С. 35—46

Онлайн-доступ к журналу:
http://mathizv.isu.ru

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УДК 517.9
MSC 35K57,35B10,34C25
DOI https://doi.org/10.26516/1997-7670.2018.26.35

On Periodic Solutions of a Nonlinear Reaction-Diffusion System

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Abstract. We consider a system of three parabolic partial differential equations of a special reaction-diffusion type. In this system, the terms that describe diffusion are identical and linear with constants coefficients, whereas reactions are described by homogenous polynomials of degree 3 that depend on three parameters. The desired functions are considered to be dependent on time and an arbitrary number of spatial variables (a multi-dimensional case). It has been shown that the reaction-diffusion system under study has a whole family of exact solutions that can be expressed via a product of the solution to the Helmholz equations and the solution to a system of ordinary differential equations with homogenous polynomials, taken from the original system, in the right-hand side. We give the two first integrals and construct a general solution to the system of three ordinary differential equations, which is represented by the Jacobi elliptic functions. It has been revealed that all particular solutions derived from the general solution to the system of ordinary differential equations are periodic functions of time with periods depending on the choice of initial conditions. Additionally, it has been shown that this system of ordinary differential equations has blow-up on time solutions that exist only on a finite time interval. The corresponding values of the first integrals and initial data are found through the equality conditions. A special attention is paid to a class of radially symmetric with respect to spatial variables solutions. In this case, the Helmholz equation degenerates into an non-autonomous linear second-order ordinary differential equation, which general solution is found in terms of the power functions and the Bessel functions. In a particular case of three spatial variables the general solution is expressed using trigonometric or hyperbolic functions.

Keywords: reaction-diffusion system, exact solutions, reduction to a system of ODEs, periodic solutions, Jacobi elliptic functions.

1. Introduction

Mathematical models of the reaction-diffusion type are usually described by systems of parabolic nonlinear differential equations [2; 10–13; 15; 16]. Construction of solutions to such a class of equations is quite challenging, which is why one of the successfully used approaches is based on the reduction to a system of ordinary differential equations (ODEs) that can be performed under certain conditions [6–8]. The survey [9] demonstrated that, from the viewpoint of applications in chemical technology and the qualitative theory of differential equations, the existence and construction of exact periodic solutions are relevant and important issues for models of reaction-diffusion systems with distributed parameters. The study of these problems on the basis of various methods and approaches is currently ongoing and quite successful (see, for example, [14] and the references therein).

This paper considers a reaction-diffusion system of three parabolic equations of a special type, where the terms describing diffusion are linear and identical, whereas the reactions are described by homogenous polynomials of degree 3. We show that this system has a family of exact solutions that can be expressed through the solution to the Helmholz equation and the solution to a system of ODEs with homogenous polynomials in the right-hand side. A general solution to such a system of ODEs is constructed, and it can be represented by the special Jacobi functions. It is revealed that all solutions derived from the general solution to the system of ODEs are time-periodic functions with the periods depending on the choice of initial conditions. Additionally, we show the presence of blow-up on time solutions and single out a class of radially symmetric with respect to spatial variables solutions.

The results obtained in the present paper are closely connected to the research that has been carried out by the authors in the area of periodic solutions to reaction-diffusion systems [3; 4]. However, these results are new, because the current research focuses on a new object, and for the first time we managed to obtain the representation of dependence of solutions to the reaction-diffusion system on spatial variables using the solution to the Helmholz equation.

2. Problem Statement and Auxiliary Results

Consider the following nonlinear reaction-diffusion system of equations

$$\frac{\partial U_1}{\partial t} = K\Delta U_1 + \alpha U_1 + a(\mathbf{x})\, U_1^2\left(\lambda U_2 - \mu U_3\right),$$

$$\frac{\partial U_2}{\partial t} = K\Delta U_2 + \alpha U_2 + a(\mathbf{x})\, U_2^2\left(-\lambda U_1 + \sigma U_3\right), \tag{2.1}$$

$$\frac{\partial U_3}{\partial t} = K\Delta U_3 + \alpha U_3 + a(\mathbf{x})\, U_3^2 \left(\mu U_1 - \sigma U_2\right).$$

Here $t \in \mathbb{R}$ is time, $\mathbf{x} \in \mathbb{R}^n$ are independent spatial variables, $U_i(t, \mathbf{x})$ are the desired functions that we interpret as concentrations of interacting substances, Δ is the Laplace operator. $K > 0$ is a constant diffusion coefficient, α, λ, μ, σ are real nonzero parameters. The given function $a(\mathbf{x})$ describes the dependence of reaction rates on spatial coordinates, caused, for example, by temperature differences or by the presence of physical fields. This function can be considered as a control, the choice of which allows us to influence the reactions rates at various points of space.

The main problem is to show that for a certain choice of $a(x)$ the reaction-diffusion system (2.1) has a whole family of exact solutions that can be represented through the solutions to the Helmgolz equation and to a system of ODEs. Another goal is to single out a class of periodic with respect to time solutions and a class of solutions existing on a finite time interval (the so-called blow-up solutions).

Before we formulate the main result, we will prove a lemma for an autonomous system of ODEs of the following form

$$\dot{X}_1 = X_1^2 \left(\lambda X_2 - \mu X_3\right), \quad \dot{X}_2 = X_2^2 \left(-\lambda X_1 + \sigma X_3\right), \quad \dot{X}_3 = X_3^2 \left(\mu X_1 - \sigma X_2\right), \tag{2.2}$$

which has the following independent first integrals

$$I_1 = X_1 X_2 X_3 = C_1, \quad I_2 = \frac{\sigma}{X_1} + \frac{\mu}{X_2} + \frac{\lambda}{X_3} = C_2. \tag{2.3}$$

Here $C_1 \neq 0$, $C_2 \neq 0$ are arbitrary real constants.

Lemma 1. *Let parameters of* (2.2) *satisfy the condition*

$$\lambda\mu\sigma > 0. \tag{2.4}$$

Then, in the area $X_1 X_2 X_3 \left(\frac{\sigma}{X_1} + \frac{\mu}{X_2} + \frac{\lambda}{X_3}\right) > 27\lambda\mu\sigma$ *the system* (2.2) *has a general solution*

$$X_1(t) = \frac{z_1^* + \sigma}{C_2} - \frac{z_1^* - z_2^*}{C_2} \operatorname{sn}^2(T, k), \tag{2.5}$$

$$X_2(t) = \frac{C_1 C_2^2}{2\lambda} \cdot \frac{z_1^* - (z_1^* - z_2^*) \operatorname{sn}^2(T,k) + \delta\, P \operatorname{sn}(T,k) \operatorname{cn}(T,k) \operatorname{dn}(T,k)}{\left(z_1^* + \sigma - (z_1^* - z_2^*) \operatorname{sn}^2(T,k)\right)^2}, \tag{2.6}$$

$$X_3(t) = \frac{C_1 C_2^2}{2\mu} \cdot \frac{z_1^* - (z_1^* - z_2^*) \operatorname{sn}^2(T,k) - \delta\, P \operatorname{sn}(T,k) \operatorname{cn}(T,k) \operatorname{dn}(T,k)}{\left(z_1^* + \sigma - (z_1^* - z_2^*) \operatorname{sn}^2(T,k)\right)^2}. \tag{2.7}$$

Here $C_1 \neq 0$, $C_2 > 0$ *are arbitrary real constants satisfying* $C_1 C_2^3 > 27\lambda\mu\sigma$; $z_1^* > z_2^* > z_3^*$ *are the real roots of the cubic equation*

$$z^3 - \frac{C_1 C_2^3 - 12\lambda\mu\sigma}{4\lambda\mu}\, z^2 + 3\sigma^2 z + \sigma^3 = 0. \tag{2.8}$$

The functions $\operatorname{sn}(T,k)$, $\operatorname{cn}(T,k)$, $\operatorname{dn}(T,k)$ *are the elliptic sine, cosine, and Jacobi delta amplitude, correspondingly;* $k = \sqrt{\dfrac{z_1^* - z_2^*}{z_1^* - z_3^*}}$ *is an absolute value of the elliptic function,*

$$P = 2\sqrt{\frac{\lambda\mu}{C_1 C_2^3}}\,(z_1^* - z_2^*)\,\sqrt{z_1^* - z_3^*}, \quad T = -\delta\sqrt{\frac{\lambda\mu\, C_1}{C_2}}\sqrt{z_1^* - z_3^*}\,(t - C_3)\,,$$

$\delta = \pm 1$, C_3 *is an arbitrary constant.*

Proof. Using (2.3), express $X_2(t)$, $X_3(t)$ through the function $X_1(t)$:

$$X_2(t) = \frac{C_1 C_2\, X_1(t) - \sigma C_1 + \delta\sqrt{\Omega}}{2\lambda\, X_1^2(t)}, \quad X_3(t) = \frac{C_1 C_2\, X_1(t) - \sigma C_1 - \delta\sqrt{\Omega}}{2\mu\, X_1^2(t)}. \tag{2.9}$$

For convenience, we employ the following notations

$$\Omega = C_1\, \left(\sigma^2 C_1 - 2\sigma\, C_1 C_2\, X_1(t) + C_1 C_2^2\, X_1^2(t) - 4\lambda\mu\, X_1^3(t)\right), \quad \delta = \pm 1.$$

Taking into account (2.9), the system of ODEs (2.2) is reduced to a single equation with respect to $X_1(t)$: $\dot{X}_1 = F\,(X_1)$, where $F\,(X_1)$ has the form

$$F\,(X_1) = \frac{C_1\, \left[(C_2\, X_1 - \sigma)\left(C_1 C_2\, X_1 - \sigma C_1 + \delta\sqrt{\Omega}\right) - 4\lambda\mu\, X_1^3\right]}{C_1 C_2\, X_1 - \sigma C_1 + \delta\sqrt{\Omega}}.$$

Introduce the change of variables: $z = C_2\, X_1(t) - \sigma$ or $X_1(t) = \dfrac{1}{C_2}(z + \sigma)$. Then we have

$$\Omega = \frac{4\lambda\mu C_1}{C_2^3}\left(-z^3 + \frac{C_1 C_2^3 - 12\lambda\mu\sigma}{4\lambda\mu}\, z^2 - 3\sigma^2 z - \sigma^3\right), \tag{2.10}$$

while the function $F\,(X_1)$ can be rewritten as

$$F\,(X_1) = C_1\, z - \frac{4\lambda\mu C_1}{C_2^3} \cdot \frac{(z+\sigma)^3}{C_1\, z + \delta\sqrt{\Omega}}.$$

Multiply the nominator and denominator of the last term by $C_1\, z - \delta\sqrt{\Omega}$. Then, in view of $\delta^2 = 1$, we obtain

$$F\,(X_1) = C_1\, z - \frac{4\lambda\mu C_1}{C_2^3} \cdot \frac{(z+\sigma)^3 (C_1\, z - \delta\sqrt{\Omega})}{C_1^2\, z^2 - \Omega}.$$

Since $C_1^2\, z^2 - \Omega = \dfrac{4\lambda\mu C_1}{C_2^3}\,(z+\sigma)^3$, we finally get $F\,(X_1) = \delta\sqrt{\Omega}$, where Ω is defined by (2.10). In this case, the ODE $\dot{X}_1 = F\,(X_1)$ for the new variable z takes the form

$$\dot{z} = 2\delta\sqrt{\frac{\lambda\mu C_1}{C_2}\left(-z^3 + \frac{C_1 C_2^3 - 12\lambda\mu\sigma}{4\lambda\mu}\, z^2 - 3\sigma^2 z - \sigma^3\right)}.$$

Therefore, the problem of finding the function $X_1(t)$ is reduced to the calculation of the following integral

$$\Phi(z) = \frac{\delta}{2}\int \frac{dz}{\sqrt{\frac{\lambda\mu C_1}{C_2}\left(-z^3 + \frac{C_1 C_2^3 - 12\lambda\mu\sigma}{4\lambda\mu}\, z^2 - 3\sigma^2 z - \sigma^3\right)}}. \tag{2.11}$$

If the conditions of the theorem are satisfied, then the discriminant of (2.8) has the form

$$\mathrm{D} = \frac{\sigma^3 C_1^2 C_2^6\left(C_1 C_2^3 - 27\lambda\mu\sigma\right)}{16\lambda^3\mu^3}$$

and is positive, which is why the cubic equation (2.8) has three real roots $z_1^* > z_2^* > z_3^*$ and the integral (2.11) is reduced to the quadrature

$$\sqrt{\frac{C_2}{\lambda\mu\, C_1}}\cdot\frac{\delta\, F\,(\varphi, k)}{\sqrt{z_1^* - z_3^*}} = -\,(t - C_3)\,,$$

$$\varphi = \arcsin\sqrt{\frac{z_1^* - z}{z_1^* - z_2^*}}, \quad k = \sqrt{\frac{z_1^* - z_2^*}{z_1^* - z_3^*}}.$$

Here $F\,(\varphi, k) = \int\limits_0^{\varphi} \dfrac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}$ is an elliptic integral of the first kind. By inversing this integral, we find the function $X_1(t)$, which has the form (2.5). In turn, using (2.5) and (2.9), we obtain the functions $X_2(t)$, $X_3(t)$, which are expressed by the formulas (2.6), (2.7). The Lemma is proved. □

It can be readily seen that each particular solution obtained from the general solution (2.5)–(2.7) to the system of ODEs (2.2) is periodic with the period τ depending on the parameters λ, μ, σ and on the choice of the constants C_1, C_2. This follows from the periodic behavior of the Jacobi functions $\mathrm{sn}(T,k)$, $\mathrm{cn}(T,k)$, $\mathrm{dn}(T,k)$ and from the fact that T is linearly dependent on t. As well-known [1], for a real argument T the functions $\mathrm{sn}(T,k)$ and $\mathrm{cn}(T,k)$ have a real period $4\mathcal{K}$, and the function $\mathrm{dn}(T,k)$ has a real period $2\mathcal{K}$, where $\mathcal{K}$ is a complete first kind elliptic integral of the form

$$\mathcal{K} = \int\limits_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}, \quad k^2 = \frac{z_1^* - z_2^*}{z_1^* - z_3^*},$$

and the periodic relationships are as follows

$$\mathrm{sn}(T+4\mathcal{K},k)=\mathrm{sn}(T,k),\quad \mathrm{cn}(T+4\mathcal{K},k)=\mathrm{cn}(T,k),$$

$$\mathrm{dn}(T+2\mathcal{K},k)=\mathrm{dn}(T,k).$$

Here the real numbers z_1^*, z_2^*, z_3^*, defining the absolute value of the elliptic function k and satisfying the chain of inequities $z_1^*>z_2^*>z_3^*$, are the roots of the cubic equation (2.8), whose coefficients depend on the parameters λ, μ, σ and on the constants C_1, C_2. Therefore, the value of the complete elliptic integral $\mathcal{K}$, the real periods $4\mathcal{K}$, $2\mathcal{K}$ of the obtained elliptic functions will depend on the parameters and constants given above. The value of the period τ with respect to the original time scale t has the form$\tau=\dfrac{4\mathcal{K}}{\theta}$, where $\theta=\sqrt{\dfrac{\lambda\mu\, C_1}{C_2}}\sqrt{z_1^*-z_3^*}$.

3. The Main Result: Reduction to a System of ODEs

The main result is represented by the following theorem on reduction of the original system of reaction-diffusion parabolic equations (2.1) to the system of ODEs (2.2) with a periodic solution.

Theorem 1. *Let in the reaction-diffusion system* (2.1) *the function* $a(\mathbf{x})$ *be written in the form* $a(\mathbf{x})=\dfrac{1}{f^2(\mathbf{x})}$, *where* $f(\mathbf{x})$ *is a non-vanishing in the area* $\mathbf{x}\in\mathcal{D}\subset\mathbb{R}^n$ *solution to the Helmholz equation*

$$K\Delta f(\mathbf{x})=-\alpha f(\mathbf{x}). \tag{3.1}$$

Then, in the area $\mathbf{x}\in\mathcal{D}\subset\mathbb{R}^n$ *the reaction-diffusion system* (2.1) $\mathbf{x}\in\mathcal{D}\subset\mathbb{R}^n$ *has an exact periodic with respect to time solution*

$$U_i(\mathbf{x},t)=f(\mathbf{x})X_i(t),\quad i=\overline{1,3}, \tag{3.2}$$

where the functions $X_i(t)$, $i=\overline{1,3}$ *are defined by the formulas* (2.5)–(2.7).

Proof. Substitute the functions (3.2) into (2.1). After some trivial transformations we will arrive at

$$f(\mathbf{x})\dot{X}_1=\left[K\Delta f(\mathbf{x})+\alpha f(\mathbf{x})\right]X_1(t)+a(\mathbf{x})f^3(\mathbf{x})X_1^2\left(\lambda X_2-\mu X_3\right),$$

$$f(\mathbf{x})\dot{X}_2=\left[K\Delta f(\mathbf{x})+\alpha f(\mathbf{x})\right]X_2(t)+a(\mathbf{x})f^3(\mathbf{x})X_2^2\left(-\lambda X_1+\sigma X_3\right),$$

$$f(\mathbf{x})\dot{X}_3=\left[K\Delta f(\mathbf{x})+\alpha f(\mathbf{x})\right]X_3(t)+a(\mathbf{x})f^3(\mathbf{x})X_3^2\left(\mu X_1-\sigma X_2\right).$$

Here we can see that if the function $f(\mathbf{x})$ satisfies the Helmholz equation (3.1), then by choosing the function $a(\mathbf{x})$ according to the condition of the Theorem, the last equations are reduced to an autonomous system of ODEs (2.2). Due to Lemma 1, the system (2.2) has a general solution represented by the periodic Jabobi functions and expressed through the formulas (2.5)–(2.7). The Theorem is proved. □

In addition to the solution that can be represented via the elliptic Jacobi functions by the formulas (2.5)–(2.7), the system (2.2) has a particular solution expressed through the trigonometric functions. Therefore, the following statement takes place.

State 1. *If $\lambda\mu\sigma > 0$, then the system of ODEs* (2.2) *has a particular solution*

$$X_1(t) = \frac{\sigma}{C_2} \cdot \frac{3\left(4\cos^2(T) - 3\right)}{4\cos^2(T)},$$

$$X_2(t) = \frac{\mu}{C_2} \cdot \frac{6\cos(T)\left[-9\cos(T) + 8\cos^3(T) - 3\sqrt{3}\,\delta\,\sin(T)\right]}{9 - 24\cos^2(T) + 16\cos^4(T)}, \tag{3.3}$$

$$X_3(t) = \frac{\lambda}{C_2} \cdot \frac{6\cos(T)\left[-9\cos(T) + 8\cos^3(T) + 3\sqrt{3}\,\delta\,\sin(T)\right]}{9 - 24\cos^2(T) + 16\cos^4(T)}.$$

Here $T = \delta\dfrac{9\sqrt{3}\lambda\mu\sigma}{2\,C_2^2} \cdot (t - C_3)$, *$C_3$ is an arbitrary constant.*

This particular solution can be derived, if the initial data $X_1(t_0)$, $X_2(t_0)$, $X_3(t_0)$ at some initial instant $t_0 \geq 0$ are chosen so that for the constants C_1, C_2 of the first integrals (2.3) the equality $C_1C_2^3 - 27\lambda\mu\sigma = 0$ holds. In this case, the cubic equation (2.8) takes the form

$$z^3 - \frac{15}{4}\sigma z^2 + 3\sigma^2 z + \sigma^3 = 0,$$

and has the roots $z^*_{1,2} = 2\sigma$, $z^*_3 = -\dfrac{\sigma}{4}$. It should be noted that in this case the corresponding Cauchy problem for the system (2.2) does not have a global solution on the interval $[t_0, +\infty)$, i.e. the solutions explode.

4. Radially Symmetric with Respect to Spatial Variables Solutions

It is convenient to look for the most trivial multidimensional solutions to the Helmholz equation (3.1) in the class of radially symmetric functions, i.e. $f(\mathbf{x}) = f(r)$, $r = ||\mathbf{x}|| \triangleq \left(\sum\limits_{k=1}^{n} x_k^2\right)^{1/2}$ is the Euclidean norm in $\mathbb{R}^n$. In this case, the Helmholz equation (3.1) is transformed into the Bessel equation

$$\frac{d^2 f}{dr^2} + \frac{n-1}{r}\frac{df}{dr} + \frac{\alpha}{K} f = 0, \quad n \in \mathbb{N}, \quad n \geq 2.$$

The general solution to this ODE has the form

$$f(r) = r^{1-\frac{n}{2}} \left[C_1 J_{\frac{n}{2}-1}\left(\frac{\alpha}{K} r\right) + C_2 Y_{\frac{n}{2}-1}\left(\frac{\alpha}{K} r\right) \right]. \tag{4.1}$$

Here $J_{\frac{n}{2}-1}(\cdot)$ and $Y_{\frac{n}{2}-1}(\cdot)$ are the Bessel functions, C_1, C_2 are arbitrary constants. Thus, in the three-dimensional case $n = 3$ the solution (4.1) is expressed either through trigonometric or hyperbolic functions, according to the sign of the parameter α.

Example 1. Consider a particular case of (2.1) in the three-dimensional coordinate space of the following form

$$\begin{aligned} \frac{\partial U_1}{\partial t} &= K\Delta U_1 + \alpha U_1 + a(x_1, x_2, x_3)\, U_1^2 \left(\lambda U_2 - \mu U_3\right), \\ \frac{\partial U_2}{\partial t} &= K\Delta U_2 + \alpha U_2 + a(x_1, x_2, x_3)\, U_2^2 \left(-\lambda U_1 + \sigma U_3\right), \\ \frac{\partial U_3}{\partial t} &= K\Delta U_3 + \alpha U_3 + a(x_1, x_2, x_3)\, U_3^2 \left(\mu U_1 - \sigma U_2\right), \end{aligned} \tag{4.2}$$

where Δ is the Laplace operator in the three-dimensional coordinate space, $\alpha > 0$, and the function $a(x_1, x_2, x_3)$ is defined by the formula

$$a(x_1, x_2, x_3) = \frac{x_1^2 + x_2^2 + x_3^2}{(C_1 \sin Q + C_2 \cos Q)^2}, \quad Q = \sqrt{\frac{\alpha}{K}} \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{4.3}$$

Here C_1, C_2 are arbitrary constants such that $C_1 C_2 \neq 0$. In this case, the reaction-diffusion system (4.2) has an exact solution if considered in the area where the denominator of the expression that defines the function $a(x_1, x_2, x_3)$ is non-vanishing. The solution is time-periodic and radially symmetric with respect to the spatial variables:

$$U_i(x_1, x_2, x_3, t) = \frac{C_1 \sin Q + C_2 \cos Q}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\, X_i(t), \tag{4.4}$$

where the functions $X_i(t)$, $i = \overline{1,3}$ are defined by the formulas (2.5)–(2.7).

Example 2. The reaction-diffusion system (4.2) with the function $a(x_1, x_2, x_3)$ of the form (4.3) and the parameter $\alpha > 0$ has an exact solution (4.4) if considered in the area where the denominator of the expression that defines the function $a(x_1, x_2, x_3)$ is non-vanishing. The solution (4.4) blows up and radially symmetric with respect to the spatial variables and the functions $X_i(t)$, $i = \overline{1,3}$ are defined by the formulas (3.3).

5. Conclusion

This paper studies time-periodic exact solutions to a reaction-diffusion system of a special type. These solutions have been expressed through the known solutions to the Helmholz equation with respect to the spatial variables and through the solutions to the polynomial systems of ODEs with respect to time. It should be noted that in the paper [5] the authors showed that the solutions (2.5)–(2.7) to the autonomous system of ODEs (2.2) satisfy some nonlinear differential equations with a delay (advanced) argument, which value depends on the choice of initial conditions. Here it has been revealed that the periodic solutions (2.5)–(2.7) are analytic functions which can be represented in the neighborhood of every point on the period by convergent power series and expressed via special Jacobi functions. Therefore, the present paper establishes a deep connection between the various classes of differential equations (partial differential equations, ordinary differential equations, equations with a deviating argument). We have found various interesting properties of the solutions: the presence of families of periodic solutions, their representations via special functions, analyticity etc. The approach proposed in this paper can be used to study other classes of nonlinear reaction-diffusion problems and to analyze properties of their solutions.

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Received 15.08.18

О периодических решениях одной нелинейной системы реакции-диффузии

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Аннотация. Рассматривается система трех параболических уравнений в частных производных специального вида, относящаяся к типу уравнений реакции-диффузии. В этой системе слагаемые, описывающие диффузию, являются одинаковыми и линейными с постоянными коэффициентами, а реакции описываются однородными полиномами третьей степени, зависящими от трех параметров. Искомые функции считаются зависящими от времени и произвольного количества пространственных переменных (многомерный случай). Показано, что рассматриваемая система реакции-диффузии имеет целое семейство точных решений, выражаемых через произведение решения уравнения Гельмгольца и решения системы обыкновенных дифференциальных уравнений с однородными полиномами в правых частях, взятыми из исходной системы. Приведены два первых интеграла и построено общее решение упомянутой системы трех обыкновенных дифференциальных уравнений, представимое эллиптическими функциями Якоби. Установлено, что все частные решения, получаемые из общего решения системы обыкновенных дифференциальных уравнений, являются периодическими функциями времени с периодами, зависящими от выбора начальных условий. Кроме того, показано также наличие у данной системы обыкновенных дифференциальных уравнений “взрывающихся” по времени решений, существующих лишь на конечном интервале времени. Соответствующие им значения первых интегралов и начальные данные выделяются условиями типа равенства. Отдельно рассмотрен класс радиально симметричных по пространственным переменным решений. В этом случае уравнение Гельмгольца вырождается в неавтономное линейное обыкновенное дифференциальное уравнение второго порядка, общее решение которого выражается через комбинацию степенных функций и функций Бесселя. В частном случае трех пространственных переменных общее решение выражается через тригонометрические либо гиперболические функции.

Ключевые слова: система реакция – диффузия, точные решения, редукция к системе оду, периодические решения, эллиптические функции Якоби.

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Поступила в редакцию 15.08.18