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On Correctness of Cauchy problem for a Polynomial Difference Operator with Constant Coefficients *

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Abstract. The theory of linear difference equations is applied in various areas of mathematics and in the one-dimensional case is quite established. For $n > 1$, the situation is much more difficult and even for the constant coefficients a general description of the space of solutions of a difference equation is not available.

In the combinatorial analysis, difference equations combined with the method of generating functions produce a powerful tool for investigation of enumeration problems. Another instance when difference equations appear is the discretization of differential equations. In particular, the discretization of the Cauchy–Riemann equation led to the creation of the theory of discrete analytic functions which found applications in the theory of Riemann surfaces and the combinatorial analysis. The methods of discretization of a differential problem are an important part of the theory of difference schemes and also lead to difference equations. The existence and uniqueness of a solution is one of the main questions in the theory of difference schemes.

Another important question is the stability of a difference equation. For $n = 1$ and constant coefficients the stability is investigated in the framework of the theory of discrete dynamical systems and is completely defined by the roots of the characteristic polynomial, namely: they all lie in the unit disk.

In the present work, we give two easily verified sufficient conditions on the coefficients of a difference operator which guarantee the correctness of a Cauchy problem.

Keywords: polynomial difference operator, Cauchy problem, correctness.

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1. Introduction

The asymptotic behavior of solutions of a difference equation is studied in the framework of the theory of discrete dynamical systems and one of the important notions then is the stability of the system. There are several definitions of stability, but in the case of constant coefficients everything bowls down to the question if the zeros of the characteristic polynomial belong to the unit disk of the complex plane. When it comes to the multi-dimensional case, already the question on the form of additional (¡¡boundary¿¿, ¡¡initial¿¿) conditions for a solution of a difference equation which guarantee its existence and uniqueness is not trivial (cf., e.g., [2;3;5–7;18]). The difficulties of the formulation of the multi-dimensional version of the condition on the set of zeros of the characteristic polynomial guaranteeing the stability are explained, first of all, by the fact that this set is not discrete. It is an algebraic hyper-surface in the $n-$dimensional complex space.

A way to overcome these difficulties in the works [4; 10; 13] is based on the observation that the discrete Fourier transforms of some special solutions of a difference equation are rational functions with poles laying in the characteristic set of the difference equation. Some particular instances of such special solutions are: impulsive response in the theory of digital recursive filters (see., [1]), difference Green function of two-layered difference scheme with constant coefficients (see [10]), fundamental solution of a Cauchy problem for a polynomial difference operator [4]. In the present work, we consider the situation when the discrete Fourier transform of the fundamental solution of a difference equation is not a rational function. Such kind of problems arise, for example, in the theory of difference schemes in the case of implicit schemes. We investigate in this situation conditions of correctness for a Cauchy problem using the method of estimation of the norm of the inverse matrix [16;19].

2. Statement of the problem

For a complex-valued function $f(x)$ of integer variables $x=(x_1,...,x_n)$ define the shift operators δ_j in the variables x_j:

$$\delta_j f(x) = f(x_1,...,x_{j-1},...,x_j+1,x_{j+1},...,x_n)$$

and consider the polynomial difference operator of the order m

$$P(\delta) = \sum_{|\alpha|\leqslant m} c_\alpha \delta^\alpha,$$

where $\alpha=(\alpha_1,...,\alpha_n)$ is a multi-index, $|\alpha|=\alpha_1+...+\alpha_n$, $\delta^\alpha=\delta_1^{\alpha_1}...\delta_n^{\alpha_n}$, c_α are the constant coefficients of the difference operator. The relation of the form

$$P(\delta)f(x) = g(x), \ \ x \in \mathbb{Z}_+^n, \tag{2.1}$$

is called a difference equation, where $f(x)$ is the unknown function, and $g(x)$ is a function defined on $\mathbb{Z}_+^n = \mathbb{Z}_+ \times ... \times \mathbb{Z}_+$ and $\mathbb{Z}_+$ is the set of non-negative integers.

Conditions guaranteeing the existence and uniqueness of a solution can be stated in various ways (cf., e.g., [2; 18]). In the present work, we define them as follows [5; 7].

For two points x, y of the integer lattice $\mathbb{Z}^n$ the inequality $x \geqslant y$ means that $x_i \geqslant y_i$ for $i = 1, ..., n$, and the notation $x \ngeqslant y$ means that there exists $i_0 \in \{1, ..., n\}$ such that $x_{i_0} < y_{i_0}$. Fix a multi-index β such that

$$|\beta| = m \text{ and } c_\beta \neq 0, \tag{$*$}$$

denote $X_{0,\beta} = \{x \in \mathbb{Z}_+^n : x \ngeqslant \beta\}$ and state the problem:

find a solution $f(x)$ of Eq. (2.1) which for $x \in X_{0,\beta}$ coincides with a given function $\varphi(x)$, i.e. satisfies the condition

$$f(x) = \varphi(x), x \in X_{0,\beta}. \tag{2.2}$$

If $\beta = (m, 0, ..., 0)$ or $\beta = (0, ..., 0, m)$, then, from the point of view of the theory of difference schemes, we have an explisit difference scheme (cf., e.g., [9]). In this case, the solvability and uniqueness of Problem (2.1)–(2.2) are evident.

For other β such that $|\beta| = m$ it may happen that the solution is not unique or the problem does not have solutions. Consider, for example, Problem (2.1)–(2.2) for the difference equation $\left(\delta_1^2 - \delta_1\delta_2 + \delta_2^2\right) f(x, y) = g(x, y)$. As β we choose $\beta = (1, 1)$, take $g(x, y) = 0$ and "initial" data $f(0, k) = f(k, 0) = 0$, $k = 0, 1, ...,$ then for any constant T the function

$$f(x,y) = \begin{cases} T, \text{ if } (x,y) = (6k-5, 3p-1) \text{ or } (x,y) = (6k-4, 3p-2), \\ \ \text{for } k, p = 1, 2, ..., \\ -T, \text{ if } (x,y) = (6k-2, 3p-1) \text{ or } (x,y) = (6k-1, 3p-2), \\ \ \text{for } k, p = 1, 2, ..., \\ 0, \ \text{for other points } (x,y) \end{cases}$$

is a solution of Problem (2.1)–(2.2). Since T can be arbitrary, this means that the solution is not unique.

If we take $g(x, y) = 1$ with the same initial data, then substituting the values (x, y) equal $(1, 0)$ and $(0, 1)$ to the equation we obtain contradictory equalities $f(2, 1) - f(1, 2) = 1$ and $f(2, 1) - f(1, 2) = -1$, i.e. Problem (2.1)–(2.2) does not have solutions.

Therefore, the question on conditions for the coefficients c_α of the difference operator $P(\delta)$ which guarantee the existence and uniqueness of a solution to Problem (2.1)–(2.2) arises.

We call Problem (2.1)–(2.2) the Cauchy problem for the polynomial difference operator $P(\delta)$, and the function $\varphi(x)$ the initial data for this problem.

For a function $f: \mathbb{Z}_+^n \to \mathbb{C}$ denote $\|f\|_\infty = \sup\limits_{\mathbb{Z}_+^n} |f(x)|$.

We say (cf., e.g., [8], [10]) that *a problem of the form (2.1)–(2.2) for the polynomial difference operator $P(\delta)$ is correctly stated, if the following conditions hold:*

a) the problem is uniquely solvable, i.e. for any initial data $\varphi(x)$ and right-hand sides $g(x)$ there exists a unique solution;

b) there exists a constant $M > 0$ such that for any $\varphi(x)$ and $g(x)$ the following estimate is valid for the norm of the corresponding solution

$$\|f(x)\|_\infty \leqslant M\left(\|g(x)\|_\infty + \|\varphi(x)\|_\infty\right). \tag{2.3}$$

Note that when the condition (2.3) is satisfied the difference operator $P(\delta)$ is called stable.

Thus, the difference problem (2.1)–(2.2) *is correctly stated, if it is solvable and stable for any initial data $\varphi(x)$ and right-hand sides $g(x)$.*

Note that the condition

$$|c_\beta| > \sum_{|\alpha|=|\beta|, \alpha\neq\beta} |c_\alpha| \tag{2.4}$$

is sufficient (see [5], [7]) for the solvability of Problem (2.1)–(2.2). Its form is suggested by the work [11], where it was used to prove the solvability of a version of generalized Cauchy problem for a polynomial difference operator $P(D)$ with initial-boundary conditions of the Riquier type in the class of analytic functions. The coefficients of the power series expansion of the analytic solutions of this problem satisfy relations of the form (2.1)–(2.2).

For the polynomial difference operator $P(\delta_1, \delta_2) = -\delta_1^2 + 3\delta_1\delta_2 - \delta_2^2 - 1$, the initial data $\varphi(0, x_2) = 1$ and $\varphi(x_1, 0) = 1$, the right-hand side $g(x_1, x_2) = \begin{cases} 0, \; x_1 + x_2 = 2k+1, \\ 3, \; x_1 = x_2, \\ 1, \; x_1 + x_2 = 2k, \; x_1 \neq x_2, \end{cases}$ a solution of Problem (2.1)–(2.2) is the function $f(x_1, x_2) = \begin{cases} 1, \; x_1 + x_2 = 2k+1, \\ min(x_1, x_2) + 1, \; x_1 + x_2 = 2k. \end{cases}$ Thus, for bounded initial data and right-hand side we obtain an unbounded solution, i.e. *the condition (2.4) ensures the solvability of Problem (2.1)–(2.2), but the stability is absent in this case.*

For $n = 1$ the polynomial difference operator has the form $P(\delta) = c_\beta\delta^\beta + c_{\beta-1}\delta^{\beta-1} + \ldots + c_0$ and the solvability condition (2.4) means that $c_\beta \neq 0$. Note that *the condition $|c_\beta| > \sum\limits_{\alpha=0}^{\beta-1} |c_\alpha|$ is sufficient for the stability,* as from this inequality it follows, in view of Rouché's theorem (see [14]), that all the roots of the characteristic polynomial $P(z)$ are located inside the unit disk. We give an analogue of this condition for $n > 1$.

Theorem 1. *Let the coefficients of the polynomial difference operator* $P(\delta) = \sum\limits_{\|\alpha\|\leqslant m} c_\alpha \delta^\alpha$ *satisfy the condition (*) and the inequality*

$$|c_\beta| > \sum_{\alpha\neq\beta} |c_\alpha|, \tag{2.5}$$

then Problem (2.1)–(2.2) is correct.

Remark 1. Note that in the work [17] the statement of Theorem 1 is proved in the two-dimensional case, and the difference operator, in contrast with the present work, has the form $P(\delta_1, \delta_2) = \sum\limits_{j=0}^{m}\sum\limits_{i=0}^{k} c_{ij}\delta_1^i\delta_2^j$. In addition, the initial conditions for it are different from the conditions (2.2).

The geometric interpretation of the condition (2.5) in the theorem is as follows: *the points of intersection of the characteristic set*

$$V = \{z \in \mathbb{C}^n : P(z) = 0\}$$

of the difference equation (2.1) with the complex line $z_j = t$, $j = 1, ..., n$ lie in the unit semi-disk $U = \{x \in \mathbb{C}^n : |z_j| < 1, j = 1, ..., n\}$.

In fact, consider the restriction of the characteristic polynomial $P(z) = \sum\limits_{|\alpha|\leqslant|\beta|} c_\alpha z^\alpha$ to the complex line $z_j = t$, $j = 1, ..., n$, $t \in \mathbb{C}$:

$$\tilde{P}(t) = P(t, ..., t) = \sum_{|\alpha|\leqslant|\beta|} c_\alpha t^{|\alpha|} = \sum_{k=0}^{|\beta|}\left(\sum_{|\alpha|=k} c_\alpha\right) t^k.$$

From the condition (2.5) it follows, first, that the coefficient $\sum\limits_{|\alpha|=|\beta|} c_\alpha$ under the highest degree $|\beta|$ of the polynomial $\tilde{P}(t)$ is not equal to zero, as otherwise we would have the inequality $|c_\beta| \leqslant \sum_{|\alpha|=|\beta|,\alpha\neq\beta} |c_\alpha|$. Second, the following inequalities are true

$$\Big|\sum_{|\alpha|=|\beta|} c_\alpha\Big| \geqslant |c_\beta| - \sum_{|\alpha|=|\beta|,\alpha\neq\beta} |c_\alpha| \geqslant \sum_{k=0}^{|\beta|-1}\sum_{|\alpha|=k} |c_\alpha|,$$

from which, in view of Rouché's theorem, we infer that all the roots of the polynomial $\tilde{P}(t)$ lie inside the unit disk.

In the case of real coefficients of the difference operator $P(\delta)$, a sufficient condition for stability is also provided by the following theorem.

Theorem 2. *Let the coefficients c_α of the polynomial difference operator $P(\delta)$ be real and satisfy the conditions:*
1) $c_\beta > 0$ and $c_\alpha \leqslant 0$ for all $\alpha \neq \beta$,
2) for some point $\lambda = (\lambda_1, ..., \lambda_n)$ such that $0 < \lambda_j < 1$, $j = 1, ..., n$ the inequality $P(\lambda) > 0$ is valid,
then Problem (2.1)–(2.2) is correct.

Note that for the difference operator

$$P(\delta_1, \delta_2) = -2\delta_1^2 + 4\delta_1\delta_2 - 2\delta_2^2 - 1$$

the condition (2.5) is not satisfied, but the conditions of Theorem 2 are valid, and Problem (2.1)–(2.2) is correct.

3. Proofs of Theorems 1 and 2

The proof of solvability for Problem (2.1)–(2.2) in Theorems 1 and 2 is based on the fact (see [5], [7]) that we consider Eqs. (2.1)–(2.2) as an infinite system of linear equations with an infinite number of unknowns $f\,(y)$, $y \in \mathbb{Z}_+^n$. After the ordering, it will assume a specific form, namely: each equation of the system will have only a finite number of unknowns. Such a system is consistent if any finite number of equations from this system is consistent (see [12], Lemma 6.3.7). We construct the sequence of sub-systems of the system (2.1)–(2.2) which consist of a finite number of equations and each sub-system contains all the equations of the antecedent sub-system. The consistency of each sub-system from this sequence implies that any finite number of equations from (2.1)–(2.2) is consistent as well.

We order the set $\mathbb{Z}_+^n$ in the uniformly lexicographical way. Take an arbitrary $p \in \mathbb{Z}_+$ and construct a sub-system of the system (2.1)–(2.2) of the dimension $N_p \times N_p$, where $N_p = \frac{(n+p)!}{n!p!}$ is the number of elements of the set $J_p = \{y \in \mathbb{Z}_+^n : |y| \leqslant p\}$. We will "number" the unknowns $f(y)$ by elements of the set J_p. We "number" the equations by elements of two sets $I_p = \{x \in \mathbb{Z}_+^n : |x| \leqslant p - m\}$ and $I_{\beta,p} = \{\mu \in X_{0,\beta} : |\mu| \leqslant p\}$. If we denote by $\#M$ the number of elements of a finite set M, then it is not difficult to see that $\#I_p + \#I_{\beta,p} = \#J_p$, in addition we have $I_p \sqcup I_{\beta,p} = J_p$. Since $I_{\beta,p} \cup \{\beta + I_p\} = J_p$, we assign to elements of the set $I_{\beta,p}$ the same "numbers" with which they belong to the set J_p, and to elements x of the set I_p the "numbers" with which $\beta + x$ belong to J_p.

Consider a system of equations with a finite number of ordered unknowns $f(y)$, $y \in J_p$ of the form

$$\sum_{|\alpha| \leqslant m} c_\alpha f(x + \alpha) = g(x),\ x \in I_p, \tag{3.1}$$

$$f(\mu) = \varphi(\mu), \mu \in I_{\beta,p}. \tag{3.2}$$

Denote by A_p the matrices of the system of equations (3.1)–(3.2) and by $detA_p$ their determinants.

Example 1. For $n = 2$, consider the difference operator

$$P(\delta_1, \delta_2) = c_{2,0}\delta_1^2 + c_{1,1}\delta_1\delta_2 + c_{0,2}\delta_2^2 + c_{1,0}\delta_1 + c_{0,1}\delta_2 + c_{0,0},$$

where $m = 2$, $\beta = (1, 1)$. For $p = 2$ the system of equations (3.1)–(3.2) will have the form

$$\begin{gathered} c_{2,0}f(x+2,y) + c_{1,1}f(x+1,y+1) + c_{0,2}f(x,y+2)+ \\ +c_{1,0}f(x+1,y) + c_{0,1}f(x,y+1) + c_{0,0}f(x,y) = g(x,y), \\ (x,y) \in I_2, \end{gathered} \tag{3.3}$$

$$f(x,y) = \varphi(x,y), \ \ (x,y) \in I_{(1,1),2}. \tag{3.4}$$

It has six unknowns $f(y_1, y_2)$, $(y_1, y_2) \in J_2 = \{(0, 0), (1, 0), (0, 1), (2,0), (1,1), (0,2)\}$. Eqs. (3.3) are numbered by elements of the set $I_2 = \{(\bar{0},\bar{0}), (\bar{1},\bar{0}) (\bar{0},\bar{1})\}$ and Eqs. (3.4) by elements of the set

$$I_{(1,1),2} = \{(0,0), (1,0), (0,1)\}.$$

Since the union $I_2 \sqcup I_{(1,1),2}$ is disjoint, the points with the coordinates (x, y) and $(\bar{x}, \bar{y})$ are considered to be different.

The determinant of the system of equations (3.3)–(3.4) has the form

$$detA_2 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ c_{0,0} & c_{1,0} & c_{0,1} & c_{2,0} & c_{1,1} & c_{0,2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

Let $\|A\|_\infty = \underset{i}{max} \sum\limits_{j=1}^{n} |a_{ij}|$ be the maximum-norm of a matrix A and denote (see [16], [19]) by $R_i(A) = |a_{ii}| - \sum\limits_{j \neq i} |a_{ij}|$, $i = 1, 2, ...n$, the magnitude of diagonal dominance in each row, also we set $R_*(A) = \underset{1\leqslant i\leqslant n}{min} R_i(A)$. If $R_*(A) \geqslant 0$, then A is a diagonally dominant matrix.

Proof of Theorem 1. Under assumptions of Theorem 1, the main diagonal of the determinants $detA_p$ of the matrices A_p of the system of equations (3.1)–(3.2) contains units and an allocated coefficient c_β. If the condition (2.5) is fulfilled, for any $p \in \mathbb{Z}_+$ the magnitudes of diagonal dominance in the rows with "numbers" $\mu \in I_{\beta,p}$ are equal to $R_\mu(A_p) = 1$ and in the rows with "numbers" $x \in I_p$ are equal to $R_x(A_p) = |c_\beta| - \sum\limits_{\alpha \neq \beta} |c_\alpha|$. These magnitudes do not depend on p. Therefore,

$$R_*(A_p) = min \left\{ 1,\ |c_\beta| - \sum_{\alpha \neq \beta} |c_\alpha| \right\} \neq 0 = R_*,$$

where R_* *does not depend on* p. Since A_p are diagonally dominant matrices, we see that $detA_p \neq 0$ for any p and Problem (2.1)–(2.2) has a unique solution.

To prove the stability, we need an estimate for the norm of the matrices A_p^{-1} which are inverses for the matrices A_p of the sub-systems (3.1)–(3.2). For the diagonally dominant matrices A_p the following estimate is valid (see [16], [19])

$$\left\| A_p^{-1} \right\|_\infty \leqslant \frac{1}{R_*}. \tag{3.5}$$

Consider the vector f_p with the coordinates $f(y)$, $y \in J_p$ and the vector h_p with the coordinates $g(x)$, $\varphi(\mu)$, where $x \in I_p$, $\mu \in I_{\beta,p}$, $\|h_p\|_\infty \leqslant \|g(x)\|_\infty + \|\varphi(x)\|_\infty$ for all p. We write the systems (3.1)–(3.2) in the form $A_p f_p = h_p$, where $detA_p \neq 0$, and find $f_p = A_p^{-1} h_p$. Taking into account (3.5) we estimate the norm f_p:

$$\begin{gathered} \|f_p\|_\infty = \left\| A_p^{-1} h_p \right\|_\infty \leqslant \left\| A_p^{-1} \right\|_\infty \|h_p\|_\infty \leqslant \tfrac{1}{R_*} \|h_p\|_\infty \leqslant \\ \leqslant M \left(\|g(x)\|_\infty + \|\varphi(x)\|_\infty \right), \end{gathered}$$

where $M = \frac{1}{R_*}$. Since the last inequality is valid for any p, we have $\|f\|_\infty \leqslant M\left(\|g(x)\|_\infty + \|\varphi(x)\|_\infty\right)$ and, consequently, Problem (2.1)–(2.2) is stable.

□

Proof of Theorem 2. One can directly verify that $f(x)$ is a solution of Problem (2.1)–(2.2) if and only if $\tilde{f}(x) = \frac{f(x)}{\lambda^x}$, $\lambda = (\lambda_1, ..., \lambda_n)$, $\lambda_j \neq 0$, $j = 1, ..., n$ is a solution of the problem

$$\sum_{|\alpha| \leqslant m} c_\alpha \lambda^\alpha \tilde{f}(x + \alpha) = \tilde{g}(x),\ x \in \mathbb{Z}_+^n, \tag{3.6}$$

$$\tilde{f}(x) = \tilde{\varphi}(x),\ x \in X_{0,\beta}, \tag{3.7}$$

where $\tilde{g}(x) = \frac{g(x)}{\lambda^x}$, $\tilde{\varphi}(x) = \frac{\varphi(x)}{\lambda^x}$.

The sequence of sub-systems of the infinite system of equations (3.6)–(3.7) constructed as in Theorem 1 for $p \in \mathbb{Z}_+$ has the form

$$\sum_{|\alpha| \leqslant m} c_\alpha \lambda^x \tilde{f}(x + \alpha) = \tilde{g}(x),\ x \in I_p, \tag{3.8}$$

$$\tilde{f}(\mu) = \tilde{\varphi}(\mu),\ \mu \in I_{\beta,p}. \tag{3.9}$$

The magnitudes of diagonal dominance of the matrices $\tilde{A}_p$ of the system (3.8)–(3.9) in the rows with "numbers" $\mu \in I_{\beta,p}$ are equal to $R_\mu\left(\tilde{A}_p\right) = 1$ and in the rows with "numbers" $x \in I_p$ are equal to

$$R_x\left(\tilde{A}_p\right) = |c_\beta \lambda^\beta| - \sum_{\alpha\neq\beta} |c_\alpha \lambda^\alpha| = c_\beta \lambda^\beta + \sum_{\alpha\neq\beta} c_\alpha \lambda^\alpha = P\left(\lambda\right) > 0.$$

Hence, $R_*\left(\tilde{A}_p\right) = min\left\{1, P\left(\lambda\right)\right\} = R_* \neq 0$ and R_* *does not depend on* p, $\tilde{A}_p$ are diagonally dominant matrices. Similarly to the proof of Theorem 1, this implies that Problem (2.1)–(2.2) is solvable.

To prove the stability of Problem (2.1)–(2.2) we invoke one of the results of the work [4], namely: the stability of Problem (2.1)–(2.2) is equivalent to the absolute summability of the fundamental solution, i.e. to the convergence of the series $\sum\limits_{x\in \mathbb{Z}_+^n} |\mathcal{P}_\beta\left(x\right)|$.

The solution $\mathcal{P}_\beta\left(x\right)$ *of Problem (2.1)–(2.2) with the initial data* $\varphi\left(x\right) \equiv 0$ *and the right-hand side* $g\left(x\right) = \delta_0\left(x\right) = \begin{cases} 1,\ x = 0, \\ 0,\ x \neq 0 \end{cases}$ *is called fundamental.*

The stability of Problem (3.6)–(3.7) implies the absolute summability of the fundamental solution $\tilde{\mathcal{P}}_\beta\left(x\right)$. Since $\tilde{\mathcal{P}}_\beta\left(x\right) = \frac{\mathcal{P}_\beta(x)}{\lambda^x}$, the series $\sum\limits_{x\geqslant 0} \frac{\mathcal{P}_\beta(x)}{\lambda^x}$ absolutely converges. In view of Abel's lemma ([15]), the absolute convergence of the series $\sum\limits_{x\geqslant 0} \mathcal{P}_\beta\left(x\right) z^x$ at a point z^0 implies its absolute convergence in the polydisk $\left\{z : |z_j| < \left|z_j^0\right|, j = 1, ..., n\right\}$. By the assumption of the theorem we have $0 < \lambda_j \leqslant 1$. Therefore, the series $\sum\limits_{x\geqslant 0} \mathcal{P}_\beta\left(x\right) z^x$ absolutely converges at the point $z^0 = (\frac{1}{\lambda_1}, ..., \frac{1}{\lambda_n})$, and hence at the point $z = (1, ..., 1)$. Thus, the fundamental solution $\mathcal{P}_\beta\left(x\right)$ of Problem (2.1)–(2.2) is absolutely summable, which implies the stability of Problem (2.1)–(2.2).

□

4. Conclusion

In the present work, sufficient conditions (Theorems 1 and 2) for the correctness of a Cauchy problem for a polynomial difference operator with constant coefficients are proven.

The proof of solvability of the Cauchy problem is based on the property of diagonal dominance of matrices. The stability of the Cauchy problem is proved using the method of estimation of the norm of the inverse matrix. The assumptions of Theorem 2 do not allow to obtain estimates for the norm of the inverse matrix. Hence, in Theorem 2 one employs methods of the discrete Fourier transform of the fundamental solution of the difference operator.

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О корректности задачи Коши для полиномиального разностного оператора с постоянными коэффициентами

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Аннотация. Теория линейных разностных уравнений применяется в различных областях математики и в одномерном случае имеет вполне завершенный вид. Для $n > 1$ ситуация значительно сложнее и даже для постоянных коэффициентов общего описания пространства решений разностного уравнения нет.

В комбинаторном анализе разностные уравнения в сочетании с методом производящих функций дают мощный аппарат исследования перечислительных задач.

Другой источник появления разностных уравнений – дискретизация дифференциальных уравнений. Так, дискретизация уравнения Коши – Римана привела к созданию теории дискретных аналитических функций, которая нашла применение в теории римановых поверхностей и комбинаторном анализе. Методы дискретизации дифференциальной задачи являются важной составной частью теории разностных схем и также приводят к разностным уравнениям. Вопрос о существовании и единственности решения относится к числу основных в теории разностных схем.

Другим важнейшим вопросом является вопрос об устойчивости разностного уравнения. Для $n = 1$ и постоянных коэффициентов устойчивость исследуется в рамках теории дискретных динамических систем и полностью определяется корнями характеристического многочлена, а именно: все они лежат в единичном круге.

В данной работе приведены два просто проверяемых достаточных условия на коэффициенты разностного оператора, обеспечивающие корректность задачи Коши.

Ключевые слова: полиномиальный разностный оператор, задача Коши, корректность.

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