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On the divergence stability loss of elongated plate in supersonic gas flow subjected to compressing or extending stresses

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Abstract. Buckling of a thin flexible elongated plate subjected to supersonic flow of a gas along the Ox-axis and compressed or extended by external boundary stresses at the edges $x = 0$ and $x = 1$ is investigated. This problem is described by a nonlinear ordinary differential equation in dimensionless variables with two bifurcation parameters one of which characterizes the compression (extension) of the plate orthogonally to Oy-axis and the other is the Mach number. Six types of boundary conditions are considered according to different fixing conditions of the edges $x = 0$ and $x = 1$. In the case of unsymmetrical boundary conditions four possible variants of them are considered. The Lyapounov-Schmidt method of bifurcation theory is applied. In a neighborhood of each point of bifurcation curve small solutions asymptotics in form of convergent series of two small parameters are computed. In comparison with our previous results the integral term is introduced in the nonlinear equation taking into account complementary forces in the middle surface of the buckled plate. The main difficulties have arisen in the investigation of relevant two-parametric eigenvalue problems and were overcome with the aid of the bifurcation curves representation through the roots of the corresponding characteristic equation.

Keywords: bifurcation theory, boundary value problems, stability.

1. Introduction

The problem of thin flexible elongated plate (strip-plate) buckling in supersonic gas flow which is compressed or extended by external boundary

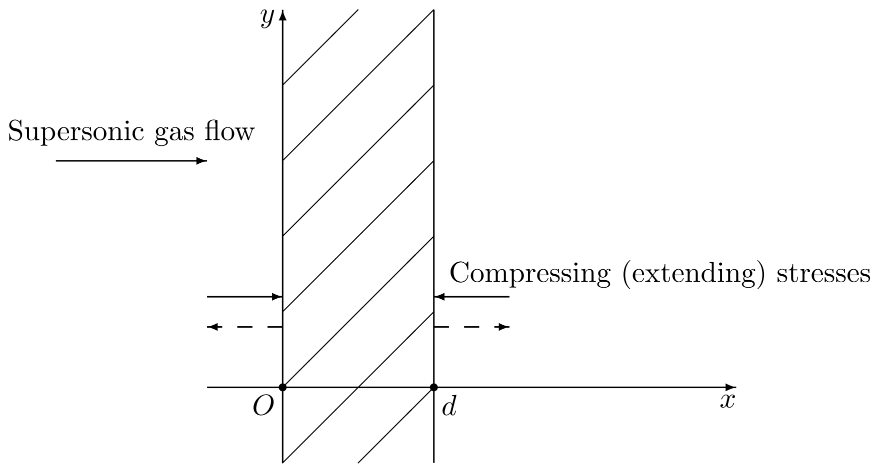


Figure 1. The problem of strip-plate buckling

stresses along the Ox-axis is investigated (see Fig. 1). Six types of boundary conditions are considered:

A. both edges are hingely fastened, $w(0) = 0$, $w''(0) = 0$; $w(1) = 0$, $w''(1) = 0$;

B. the left edge is free, the right one is rigidly fixed, $w''(0) = 0$, $w'''(0) = 0$; $w(1) = 0$, $w'(1) = 0$;

B$'$. the right edge is free, the left one is rigidly fixed, $w(0) = 0$, $w'(0) = 0$; $w''(1) = 0$, $w'''(1) = 0$;

C. both edges are rigidly fixed, $w(0) = 0$, $w'(0) = 0$; $w(1) = 0$, $w'(1) = 0$;

D. the left edge is fixed, the right one is rigidly fixed, $w'(0) = 0$, $w'''(0) = 0$; $w(1) = 0$, $w'(1) = 0$;

D$'$. the right edge is fixed, the left one is rigidly fixed, $w(0) = 0$, $w'(0) = 0$; $w'(1) = 0$, $w'''(1) = 0$.

In dimensionless variables the problem is described by the following equation

$$\chi^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{w''}{(1+w'^2)^{3/2}}\right) - T\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = kK\Big(\frac{\mathrm{d}w}{\mathrm{d}x}, M, \kappa\Big) + \theta w'' \int_0^1 \Big[\big(1+w'^2\big)^{1/2} - 1\Big]\mathrm{d}x, \tag{1.1}$$

where $K(w'_x, M, \kappa) = [1 - (1 + \frac{\kappa-1}{2} M w'_x)^{\frac{2\kappa}{\kappa-1}}]$ for one-sided flow around and $K(w'_x, M, \kappa) = [(1 - \frac{\kappa-1}{2} M w'_x)^{\frac{2\kappa}{\kappa-1}} - (1 + \frac{\kappa-1}{2} M w'_x)^{\frac{2\kappa}{\kappa-1}}]$ for two-sided flow around by supersonic gas flow along the Ox-axis [1]–[3]. Here $w = w(x)$ is the plate deflection, $0 < x < 1$; $x = \frac{x_1}{d}$, $0 \le x_1 \le d$, $-\infty < y_1 < \infty$ are rectangular coordinates; $\chi^2 = \frac{h^2}{12(1-\mu^2)d^2}$, $T = \frac{qd}{Eh}$ and $k = \frac{p_0 d}{Eh}$; d is the width of the plate, h is its thickness; E is the Young module;

μ is the Poisson coefficient; $q < 0$ ($q > 0$) is the compressing (extending) stress; M is the Mach number, p_0 is the pressure and κ is the polytropic exponent; the integral term takes into account the complementary force in the middle surface of the buckled plate, $\theta = \frac{1}{1-\mu^2}$. In our previous article [4] this term was not included in the equation. In the books [1], [2] and in the article [3] the problem of rectangular plate divergence is investigated, not subjected to the compression/extension conditions.

For the computation of buckling forms in neighborhoods of parameter critical values bifurcation theory methods [5] are applied. Everywhere below the terminology and notations of the monograph [5] are used.

Let E_1 and E_2 be Banach spaces. The nonlinear equation

$$Bx = R(x, \lambda), \quad R(0,0) = 0,\ R_x(0,0) = 0 \tag{1.2}$$

is considered. Here $B : E_1 \to E_2$ is a closed linear Fredholm operator ($R(B) = \overline{R(B)}$, $R(B)$ is the range of B) with dense in E_1 domain $D(B)$, $N(B) = \operatorname{span}\{\varphi_1, \dots, \varphi_n\}$ is its zero-subspace, $N^*(B) = \operatorname{span}\{\psi_1, \dots, \psi_n\} \subset E_2^*$ is its defect-subspace. The nonlinear operator $R(x, \lambda)$ is supposed to be defined and sufficiently smooth by x and λ in a neighborhood of $(0,0) \in E_1 \dot{+} \Lambda$, Λ is the parameter space. According to Hahn-Banach theorem there exist biorthogonal systems $\{\gamma_j\}_1^n \in E_1$, $\langle \varphi_i, \gamma_j \rangle = \delta_{ij}$ and $\{z_k\}_1^n \in E_2$, $\langle z_k, \psi_l \rangle = \delta_{kl}$, generating the projectors $P = \sum\limits_{j=1}^{n} \langle \cdot, \gamma_j \rangle \varphi_j : E_1 \to N(B)$, $Q = \sum\limits_{j=1}^{n} \langle \cdot, \psi_j \rangle z_j : E_2 \to E_{2,n} = \operatorname{span}\{z_1, \dots, z_n\}$ and the following direct sum expansions $E_1 = E_1^n \dot{+} E_1^{\infty-n}$, $E_1^n = N(B)$, $E_2 = E_{2,n} \dot{+} E_{2,\infty-n}$, $E_{2,\infty-n} = R(B)$. Then the Lyapounov-Schmidt method [5] allows to reduce the problem (1.2) of construction of small by norm solutions to nonlinear finite-dimensional equations system that is bifurcation equation. According to E. Schmidt lemma the operator $\tilde{B} = B + \sum\limits_{k=1}^{n} \langle \cdot, \gamma_k \rangle z_k$ is continuously invertible, and the equation (1.2) can be rewritten in the form of the system

$$\tilde{B}x = R(x, \lambda) + \sum_{i=1}^{n} \xi_i z_i, \quad \xi_i = \langle x, \gamma_i \rangle, \quad i = \overline{1, n}. \tag{1.3}$$

On the implicit operators theorem the first equation (1.3) has a unique solution $x = x(\xi, \lambda)$. Its substitution in the second one gives the bifurcation equation (BEq)

$$f(\xi, \lambda) \equiv \xi_i - \langle x(\xi, \lambda), \gamma_i \rangle = 0, \quad i = 1, \dots, n. \tag{1.4}$$

The system (1.4) relative to vector $\xi = (\xi_1, \dots, \xi_n)$ is equivalent to the equation (1.2) in Banach spaces [5] in the sense that the equations (1.2) and (1.4) have the same number of small solutions. They are represented in the form of series on equal fractional degrees of parameters.

In this article the Lyapounov-Schmidt method is applied to the nonlinear equation (1.1) in neighborhoods of parameters T and M critical values (in the points of bifurcation curves). Here the nonlinear operator R is analytic, small solutions of (1.1) are presented in the form of convergent series of two small parameters in a small neighborhood of the bifurcation point. Naturally, the most difficulties arise in the investigation of the relevant linearized problems (according to the boundary conditions A–D′, see Fig. 1−18).

The obtained results are enclosed in our grant application to RFBR, project 07–01–00197.

2. Computation of Bifurcational Solutions

The linearized equation (1.1)

$$\chi^2 w^{(4)}_{x^4} - T w^{(2)}_{x^2} + \sigma w^{(1)}_{x} = 0, \quad \sigma = 1(2)k\kappa M \tag{2.1}$$

and six types of the boundary conditions are two-parametric spectral problems, i. e. spectral two-point boundary value problems. Here the factor 1(2) in the parameter σ corresponds to one-sided (two-sided) flow around of strip-plate by supersonic gas flow.

At the investigation of these two-point boundary value problems the following possibilities arise 1) $4T^3 - 27\sigma^2\chi^2 > 0$, 2) $4T^3 - 27\sigma^2\chi^2 = 0$, 3) $4T^3 - 27\sigma^2\chi^2 < 0$, where $\sigma \geq 0$, $T < 0$ is the compressing stress, $T > 0$ is the extension stress, $T = 0$ corresponds only to flow around.

In the first case T is necessarily greater then 0. The characteristic equation

$$f_0(\lambda) = \chi^2\lambda^4 - T\lambda^2 + \sigma\lambda = 0 \tag{2.2}$$

has one negative root $-\alpha$ and two positive roots $\beta_2 > \beta_1 > 0$ ($\alpha = \beta_1 + \beta_2$). Again T is greater than 0 for $4T^3 - 27\sigma^2\chi^2 = 0$ and (2.2) has two equal roots $\beta_1 = \beta_2 = \beta > 0$ and one negative root $-\alpha$. It will be useful to indicate here some relations between roots $\beta_2 > \beta_1$ and parameters σ and T

$$\frac{\sigma}{T} < \beta_1, \beta_2 < \frac{3}{2}\frac{\sigma}{T}, \quad \beta_2 = \beta_1 + \frac{1}{2\chi\beta_1^{1/2}}\Big[\sqrt{T\beta_1 + 3T} - 3\chi\beta_1^{3/2}\Big],$$

which follow from the known Vieta formulae. In the third case which is possible at both extension ($T > 0$) and compression ($T < 0$) of the plate the roots of (2.2) are $\gamma \pm \delta\mathrm{i}$ ($\gamma, \delta > 0$) and $-\alpha < 0$ ($\alpha = 2\gamma$). Here for the buckling investigation it is convenient to introduce the following designations $\delta = \gamma u$, $u = \sqrt{3 - \frac{T}{\gamma^2\chi^2}}$, $\sigma = 2\gamma\chi^2(\gamma^2 + \delta^2) = 2\gamma^3\chi^2(1 + u^2)$. It is not difficult to see that the values

$$0 < u < \sqrt{3} \Rightarrow 2\gamma^3\chi^2 \leq \sigma < 8\gamma^3\chi^2 \tag{2.3}$$

respond to the plate extension, and the values

$$u > \sqrt{3} \Rightarrow \sigma > 8\gamma^3\chi 2 \tag{2.4}$$

respond to the plate compression. The value $u = \sqrt{3}$ implies $T = 0$, i.e. the extension/compression absence. The value $u = 0$ corresponds to $4T^3 - 27\sigma^2\chi^2 = 0$.

Note, that in the investigation of algebraic equation (2.2) with two parameters T and σ Sturm method for roots separation was used.

Asymptotics of bifurcating solutions (buckling forms) on small parameters $\varepsilon_1, \varepsilon_2, T = T_0+\varepsilon_1, M = M_0+\varepsilon_2$, in bifurcation point (T_0, M_0) are computed for all cases of bifurcation curves existence. Linearized in bifurcation point equation determines the Fredholm operator $B : C^{4+\alpha}[0,1] \to C^{\alpha}[0,1]$ with one-dimensional zero-subspace $N(B) = \mathrm{span}\{\varphi(x)\}$ and defect-subspace $N^*(B) = \mathrm{span}\{\psi(x)\}$. For one-sided flow around the asymptotics of bifurcating solutions takes the form

$$X(x) = -\frac{L^1_{110}\varepsilon_1 + L^1_{101}\varepsilon_2}{L^1_{200}}\,\varphi(x) + o(|\varepsilon|),$$

and for two-sided flow around it is

$$X(x) = \pm\sqrt{-\frac{L^2_{110}\varepsilon_1 + L^2_{101}\varepsilon_2}{L^2_{300}}}\,\varphi(x) + O(|\varepsilon|),$$

where signs of ε_1, ε_2 are determined by the radicand nonnegativity constraint. Bifurcation equation coefficients are computed according to [5] by Nekrasov-Nazarov indeterminate coefficients method

$$\begin{gathered} L^1_{110} = \int_0^1 \varphi''\psi dx,\; L^1_{101} = -k\kappa\int_0^1 \varphi'\psi dx, \\ L^1_{200} = -\frac{k\kappa(\kappa+1)M_0^2}{4}\int_0^1 \varphi'^2\psi dx,\; \ldots \end{gathered} \tag{2.5}$$

for one-sided flow around, or

$$\begin{aligned} L^2_{110} &= \int_0^1 \varphi''\psi dx = L^1_{110},\; L^2_{101} = -2k\kappa\int_0^1 \varphi'\psi dx = 2L^1_{101},\; L^2_{200} = 0, \\ L^2_{300} &= 3\chi^2\int_0^1 \varphi''^3\psi dx + \frac{3}{2}\chi^2\int_0^1 \varphi^{(4)}\varphi'^2\psi dx + 9\chi^2\int_0^1 \varphi'\varphi''\varphi'''\psi dx \\ &\quad - \frac{1}{6}\,k\kappa(\kappa+1)M_0^3\int_0^1 \varphi'^3\psi dx + \frac{1}{2}\theta\int_0^1 \varphi'^2 \mathrm{d}x\int_0^1 \varphi''\psi \mathrm{d}x,\; \ldots \end{aligned} \tag{2.6}$$

for two-sided flow around. By virtue of the limited size of the article the values of the BEq coefficients L will be given everywhere below only if they have sufficiently compact form. These coefficients were computed by the usage of Maple 9.

More interesting cases B, B′ and D, D′ which have more degrees of freedom at the edge $x = 0$ of the plate and are similar from computational point of view will be investigated first.

2.1. Boundary Conditions B

1) If $4T^3 - 27\sigma^2\chi^2 > 0 \Rightarrow T > 0$, then using the boundary conditions we obtain the equation

$$\Delta_B = \beta_2(\beta_1+\beta_2)(\beta_1+2\beta_2)\mathrm{e}^{\beta_1} - \beta_1(\beta_1+\beta_2)(\beta_2+2\beta_1)\mathrm{e}^{\beta_2} + \\ + \beta_1\beta_2(\beta_2-\beta_1)\mathrm{e}^{-(\beta_1+\beta_2)} = 0$$

which defines possible bifurcation points. Here the symbol Δ denotes the determinant of the boundary conditions system matrix. Computational experiment shows that due to exponential decreasing of the third summand this equation implicitly defines the unique curve in the region $\beta_2 > \beta_1$ only if $\beta_1 < \beta_0$ where $\beta_0 \approx 1.336358362$. (See Fig. 2).

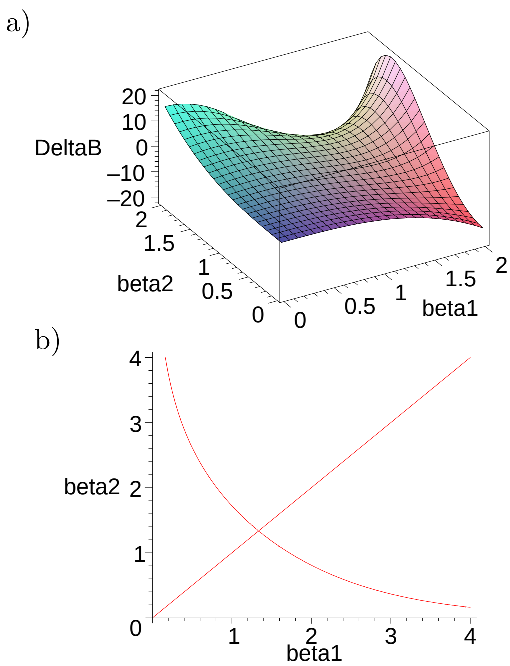


Figure 2. Boundary conditions B, $4T^3 - 27\sigma^2\chi^2 > 0$. (**a**) The 3D plot of Δ_B. (**b**) The plot of $\Delta_B = 0$ solution

In this case

$$
\begin{aligned}
\varphi(x) = {} & \frac{1}{\beta_1^2\beta_2^2(\beta_2-\beta_1)}\Big[(\beta_1+\beta_2)^2\{-\beta_2^2(\beta_1+2\beta_2)\mathrm{e}^{\beta_1 x}+\beta_1^2(2\beta_1+\beta_2)\mathrm{e}^{\beta_2 x}\} \\
& + \beta_1^2\beta_2^2(\beta_2-\beta_1)\mathrm{e}^{-(\beta_1+\beta_2)x} + \\
& + (\beta_1+\beta_2)(\beta_1+2\beta_2)(2\beta_1+\beta_2)\{\beta_2^2\mathrm{e}^{\beta_1}-\beta_1^2\mathrm{e}^{\beta_2}\}\Big], \\
\psi(x) = {} & \frac{1}{\beta_2-\beta_1}\Big[\beta_1\beta_2(2\beta_1+\beta_2)\mathrm{e}^{-\beta_1-\beta_2 x}-\beta_1\beta_2(\beta_1+2\beta_2)\mathrm{e}^{-\beta_2-\beta_1 x} \\
& - (\beta_1+\beta_2)\{\beta_2(\beta_1+2\beta_2)\mathrm{e}^{-\beta_2}-\beta_1(2\beta_1+\beta_2)\mathrm{e}^{-\beta_1}\}\mathrm{e}^{(\beta_1+\beta_2)x}\Big].
\end{aligned}
\tag{2.7}
$$

Here it should be taken into account the relation $\Delta_B = 0$. BEq coefficients are as follows

$$
\begin{aligned}
L_{110}^1 = {} & \frac{(\beta_1+\beta_2)^2\mathrm{e}^{-2\beta_1-2\beta_2}}{(2\beta_1+\beta_2)(\beta_1+2\beta_2)(-\beta_2+\beta_1)^3}\cdot \\
& \cdot\Big[\beta_2(-\beta_2+\beta_1)(\beta_1+\beta_2)(\beta_1+2\beta_2)^3\mathrm{e}^{4\beta_1+2\beta_2} \\
& + \beta_1(-\beta_2+\beta_1)(\beta_1+\beta_2)(2\beta_1+\beta_2)^3\mathrm{e}^{4\beta_2+2\beta_1} \\
& - (-\beta_2+\beta_1)(2\beta_1+\beta_2)(\beta_1+2\beta_2)(\beta_1+\beta_2)(\beta_1^2+4\beta_1\beta_2+\beta_2^2)\mathrm{e}^{3\beta_1+3\beta_2} \\
& - \beta_1(\beta_1+2\beta_2)(2\beta_1+\beta_2)(2\beta_2^4-2\beta_2^3+4\beta_1\beta_2^3-11\beta_1\beta_2^2-3\beta_1^2\beta_2^2 \\
& - 8\beta_1^2\beta_2-5\beta_1^3\beta_2+3\beta_1^3+2\beta_1^4)\mathrm{e}^{\beta_1+2\beta_2}+\beta_2(\beta_1+2\beta_2)(2\beta_1+\beta_2)(2\beta_2^4 \\
& + 3\beta_2^3-5\beta_1\beta_2^3-8\beta_1\beta_2^2-3\beta_1^2\beta_2^2-11\beta_1^2\beta_2+4\beta_1^3\beta_2+2\beta_1^4-2\beta_1^3) \\
& \times\mathrm{e}^{2\beta_1+\beta_2}+3\beta_2\beta_1(\beta_1+\beta_2)(-\beta_2+\beta_1)^3\Big],
\end{aligned}
$$

$$
\begin{aligned}
L_{101}^1 = {} & -\frac{k\kappa(\beta_1+\beta_2)\mathrm{e}^{-2\beta_1-2\beta_2}}{\beta_1\beta_2(\beta_1+2\beta_2)(2\beta_1+\beta_2)(-\beta_2+\beta_1)^3} \\
& \times\Big[\beta_2^2(-\beta_2+\beta_1)(\beta_1+\beta_2)^2(\beta_1+2\beta_2)^3\mathrm{e}^{4\beta_1+2\beta_2} \\
& + \beta_1^2(-\beta_2+\beta_1)(\beta_1+\beta_2)^2(2\beta_1+\beta_2)^3\mathrm{e}^{4\beta_2+2\beta_1} \\
& - 3\beta_1\beta_2(-\beta_2+\beta_1)(\beta_1+2\beta_2)(2\beta_1+\beta_2)(\beta_1+\beta_2)^3\mathrm{e}^{3\beta_1+3\beta_2} \\
& + \beta_1^2(\beta_1+2\beta_2)(2\beta_1+\beta_2)(6\beta_1^4\beta_2-4\beta_1^4+3\beta_1^3\beta_2+3\beta_1^3\beta_2^2+13\beta_1^2\beta_2^2 \\
& - 6\beta_1^2\beta_2^3-3\beta_1\beta_2^4+17\beta_1\beta_2^3+7\beta_2^4)\mathrm{e}^{\beta_1+2\beta_2}+\beta_2^2(2\beta_1+\beta_2)(\beta_1+2\beta_2) \\
& \times(3\beta_1^4\beta_2-7\beta_1^4-17\beta_1^3\beta_2+6\beta_1^3\beta_2^2-3\beta_1^2\beta_2^3-13\beta_1^2\beta_2^2-6\beta_1\beta_2^4-3\beta_1\beta_2^3 \\
& + 4\beta_2^4)\mathrm{e}^{2\beta_1+\beta_2}-3\beta_1^2\beta_2^2(\beta_1+\beta_2)(-\beta_2+\beta_1)^3\Big],
\end{aligned}
$$

L_{200}^2 is omitted

for one-sided flow around, and

$$L_{110}^2 = L_{110}^1, \quad L_{101}^2 = 2L_{101}^1, \quad L_{300}^2 \text{ is omitted}$$

for two-sided flow around.

2)If $4T^3 - 27\sigma^2\chi^2 = 0 \Rightarrow T > 0$, the equation

$$\Delta_B = e^{-3\,\beta} + 8 - 6\,\beta = 0$$

defines only one bifurcation point $\beta \approx 1.336358362$ which naturally coincides with the critical value β_0 in the previous case. (See Fig. 3).

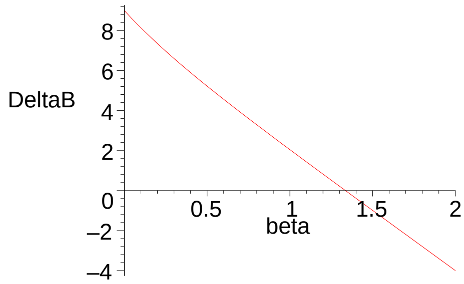


Figure 3. Boundary conditions B, $4T^3 - 27\sigma^2\chi^2 = 0$. The plot of Δ_B

Here

$$\varphi(x) = \mathrm{e}^{-2\beta x} + 4\mathrm{e}^{\beta x}(3\beta x - 7) + 18\mathrm{e}^{\beta}(2 - \beta),$$
$$\psi(x) = \mathrm{e}^{2\beta x}(6\beta - 8) - \mathrm{e}^{-\beta x}(3\beta x - 3\beta + 1)$$

may be computed both directly with the aid of the boundary conditions and by means of limit passage at $\beta_2 \to \beta_1 = \beta$ for the case 1) when $\psi(x)$ gains the additional cofactor $\beta^2\mathrm{e}^{-\beta}$ or also at $\delta \to 0$ ($u \to 0$) for the case 3) with cofactor (-1) for $\psi(x)$.

BEq coefficients have the form

$$L^1_{110} = \frac{2}{3}\,\beta\Big[(-12\beta - 36 + 18\beta^2 + 9\beta^3)\mathrm{e}^{3\beta} + 4 + (-72\beta + 32 + 36\beta^2)\mathrm{e}^{6\beta}\Big]\mathrm{e}^{-3\beta},$$
$$L^1_{101} = -\frac{2}{3}\,k\kappa\Big[(-63\beta^2 + 105\beta + 9\beta^3 - 78)\mathrm{e}^{3\beta} - 2 + (-108\beta + 80 + 36\beta^2)\mathrm{e}^{6\beta}\Big]\mathrm{e}^{-3\beta},$$
$$L^1_{200} = -\frac{1}{100}\,k\kappa(\kappa+1)M_0^2\beta\Big[(5400\beta^3 - 24300\beta^2 + 36675\beta - 18500)\mathrm{e}^{9\beta} + (-28800\beta + 96000)\mathrm{e}^{6\beta} + (-98415\beta - 78732)\mathrm{e}^{5\beta} + (-1800\beta + 1200)\mathrm{e}^{3\beta} + 32\Big]\mathrm{e}^{-5\beta}$$

for one-sided flow around, and

$$L^2_{110} = L^1_{110}, \quad L^2_{101} = 2L^1_{101},$$

$$
\begin{aligned}
L_{300}^{2} = {} & \frac{1}{367500}\,\beta^{2}\Big[-k\kappa(\kappa+1) \\
& \times\{(127008000\beta^{4}-753580800\beta^{3}+1690053120\beta^{2}-1698031104\beta \\
& +644360192)\mathrm{e}^{12\beta}+(2090688894\beta-19663317)\mathrm{e}^{7\beta} \\
& +(-132300000\beta^{3}+529200000\beta^{2}-507150000\beta-149450000)\mathrm{e}^{9\beta} \\
& +(4593750\beta-4746875)\mathrm{e}^{3\beta}-100000 \\
& +(-264600000\beta^{2}-88200000\beta-470400000)\mathrm{e}^{6\beta}\}M_{0}^{3} \\
& +81\beta^{3}\{(127008000\beta^{4}-372556800\beta^{3}+267563520\beta^{2} \\
& +71049216\beta-88711168)\mathrm{e}^{12\beta} \\
& +(88577874\beta+220425543)\mathrm{e}^{7\beta} \\
& +(26460000\beta^{3}-185220000\beta^{2}+366030000\beta-242060000)\mathrm{e}^{9\beta} \\
& +(-9738750\beta+15465625)\mathrm{e}^{3\beta}+800000 \\
& +(52920000\beta^{2}-141120000\beta+94080000)\mathrm{e}^{6\beta}\}\chi^{2}\Big]\mathrm{e}^{-7\beta} \\
& +\frac{1}{3}\,\theta\beta^{2}\Big[-\mathrm{e}^{-4\beta}+(48\beta-16)\mathrm{e}^{-\beta}+(72\beta^{2}-264\beta+260)\mathrm{e}^{2\beta}-243\Big] \\
& \times\Big[9\beta^{3}+18\beta^{2}+12\beta-68+(36\beta^{2}-72\beta+32)\mathrm{e}^{3\beta}\Big]
\end{aligned}
$$

for two-sided flow around, where β satisfies the equation $\Delta_B = 0$. According to formulae (2.5) and (2.6) at the limit passage $\beta_2 \to \beta_1 = \beta$ for the case 1) ($\delta \to 0 \Rightarrow u \to 0$ for the case 3)) in the corresponding BEq coefficients all of them evaluated for $4T^3 - 27\sigma^2\chi^2 = 0$ are multiplied by the cofactor $\beta^2\mathrm{e}^{-\beta}$ (cofactor (-1)). Thus we have here the additional possibility to verify the performed computations.

3) When $4T^3 - 27\sigma^2\chi^2 < 0$, that is possible at both extension ($T > 0$) and compression ($T < 0$) of the plate we obtain the following equation

$$\Delta_B = 2(u^2-3)\sin(\gamma u) + 8u\cos(\gamma u) + u(1+u^2)\mathrm{e}^{-3\gamma} = 0$$

which implicitly defines the bifurcation curve. (See Fig. 4).

Here

$$
\begin{aligned}
\varphi(x) = {} & \frac{1}{u(1+u^2)^2}\Big[u(1+u^2)^2\mathrm{e}^{-2\gamma x}+4u(u^2-7)\mathrm{e}^{\gamma x}\cos(\gamma u x)+4(3-5u^2) \\
& \times \mathrm{e}^{\gamma x}\sin(\gamma u x)+2\mathrm{e}^{\gamma}(u^2+9)\{2u\cos(\gamma u)+(u^2-1)\sin(\gamma u)\}\Big], \\
\psi(x) = {} & \frac{1}{u}\Big[\{3(1+u^2)\sin(u\gamma(x-1))+u(1+u^2)\cos(u\gamma(x-1))\}\mathrm{e}^{-\gamma x} \\
& +\{2(u^2-3)\sin(\gamma u)+8u\cos(\gamma u)\}\mathrm{e}^{2\gamma x}\Big]
\end{aligned}
$$

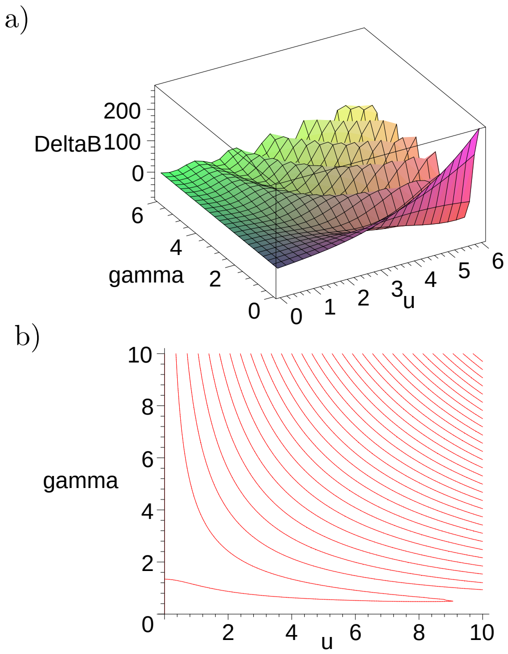


Figure 4. Boundary conditions B, $4T^3 - 27\sigma^2\chi^2 < 0$. (**a**) The 3D plot of Δ_B. (**b**) The plot of $\Delta_B = 0$ solution

and BEq coefficients have the form

$$
\begin{aligned}
L^1_{110} = {} & -\frac{2\gamma}{u^3(u^2+9)}\Big[\{4u^2(-27+5u^2)\sin(2\gamma u) - 2u(27-36u^2 \\
& + u^4)\cos(2\gamma u) + 24u^3 + 54u + 2u^5\}\mathrm{e}^{3\gamma} + 12u^3(1+u^2)\mathrm{e}^{-3\gamma} \\
& - (u^2+9)(10\gamma u^4 + u^4 - 6\gamma u^2 - 20u^2 - 9)\sin(\gamma u) \\
& + (u^2+9)u(\gamma u^4 - 24\gamma u^2 - 12u^2 - 9\gamma)\cos(\gamma u)\Big],
\end{aligned}
$$

$$
\begin{aligned}
L^1_{101} = {} & \frac{2k\kappa}{u^3(1+u^2)(u^2+9)}\Big[\{-2u^2(u^2-8u+9)(u^2+8u+9)\sin(2\gamma u) \\
& - 2u(-90u^2 + 27 + 11u^4)\cos(2\gamma u) + 60u^3 + 6u^5 + 54u\}\mathrm{e}^{3\gamma} \\
& - 6u^3(1+u^2)^2\mathrm{e}^{-3\gamma} + (u^2+9)(3u^6\gamma + u^6 - 18\gamma u^4 - 11u^4 - 21\gamma u^2 + 29u^2 \\
& + 9)\sin(\gamma u) + (u^2+9)u(15\gamma u^4 + 6u^4 + 6\gamma u^2 - 26u^2 - 9\gamma)\cos(\gamma u)\Big],
\end{aligned}
$$

$$
\begin{aligned}
L^1_{200} = {} & \frac{k\kappa(\kappa+1)M_0^2\gamma}{u^3(1+9u^2)(4+u^2)(25+u^2)(1+u^2)^2}\Big[16u^2(4+u^2)(25+u^2) \\
& \times \{(3u^6 - 99u^4 + 145u^2 - 9)\sin(2\gamma u) + 4u(7u^4 - 42u^2 + 15) \\
& \times \cos(2\gamma u)\}\mathrm{e}^{\gamma} - (25+u^2)(1+9u^2)\{(u^2-2)(1+u^2)(u^2+9)^2 \\
& \times \sin(\gamma u) + (-14u^6 + 54 - 450u^2 + 250u^4)\sin(3\gamma u) + 3u(1+u^2) \\
& \times (u^2+9)^2\cos(\gamma u) + u(u^6 - 81u^4 + 443u^2 - 243)\cos(3\gamma u)\}\mathrm{e}^{4\gamma} \\
& + 8u^3(4+u^2)(1+9u^2)(1+u^2)^3\mathrm{e}^{-5\gamma} + 2u^2(1+9u^2)(4+u^2) \\
& + (25+u^2)(1+u^2)^2\{(u^2-9)\sin(\gamma u) + 6u\cos(\gamma u)\})\mathrm{e}^{-2\gamma} - 9u^2(-3+u^2) \\
& \times (u^2+9)^3\{(3u^4 - 26u^2 - 5)\sin(\gamma u) + 4u(5u^2-1)\cos(\gamma u)\}\Big]
\end{aligned}
$$

for one-sided flow around, and

$$L^2_{110} = L^1_{110}, \quad L^2_{101} = 2L^1_{101}, \quad L^2_{300} \text{ is omitted}$$

for two-sided flow around. Again the parameters u and γ are connected by the corresponding relation $\Delta_B = 0$.

Note also, that Δ_B written out in the following form

$$\Delta_B = u\Big[2(u^2-3)\frac{\sin(\gamma u)}{u} + 8\cos(\gamma u) + (1+u^2)\mathrm{e}^{-3\gamma}\Big]$$

gives us Δ_B for the case 2) when $u \to 0$.

2.2. Boundary Conditions B′

1) If $4T^3 - 27\sigma^2\chi^2 > 0$, then using the boundary conditions we obtain the following equation

$$\Delta_{B'} = -\beta_2(\beta_1+\beta_2)(\beta_1+2\beta_2)\mathrm{e}^{-\beta_1} + \beta_1(\beta_2+2\beta_1)(\beta_1+\beta_2)\mathrm{e}^{-\beta_2} + \beta_2\beta_1(\beta_1-\beta_2)\mathrm{e}^{\beta_1+\beta_2}.$$

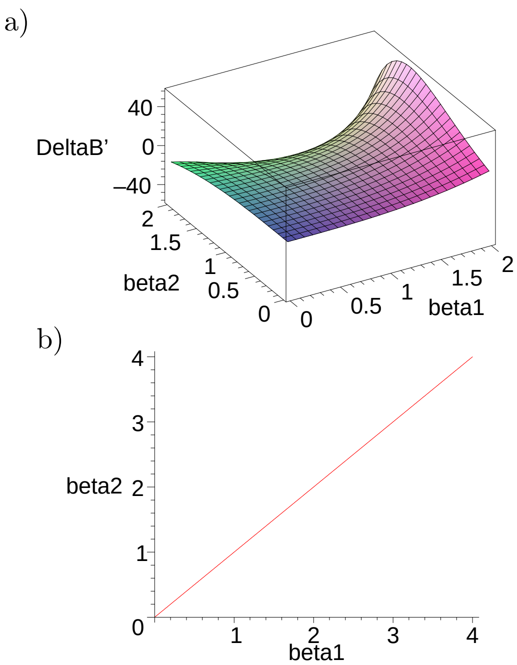


Figure 5. Boundary conditions B′, $4T^3 - 27\sigma^2\chi^2 > 0$. (**a**) The 3D plot of $\Delta_{B'}$. (**b**) The plot of $\Delta_{B'} = 0$ solution

The substitution $\beta_1 = \beta_2 + \epsilon$ implies

$$\begin{aligned}\Delta_{B'} &= -(6\beta_2^3 + 5\beta_2^2\epsilon + \beta_2\epsilon^2)\mathrm{e}^{-\beta_2-\epsilon} \\ &\quad + (6\beta_2^3 + 13\beta_2^2\epsilon + 9\beta_2\epsilon^2 + 2\epsilon^3)\mathrm{e}^{-\beta_2} + (\beta_2^2\epsilon + \beta_2\epsilon^2)\mathrm{e}^{2\beta_2+\epsilon}\end{aligned}$$

whence regrouping the summands as follows

$$\begin{aligned}\Delta_{B'} &= 6\beta_2^3(\mathrm{e}^{-\beta_2} - \mathrm{e}^{-\beta_2-\epsilon}) + \beta_2^2\epsilon(13\mathrm{e}^{-\beta_2} - 5\mathrm{e}^{-\beta_2-\epsilon})\\ &+ \beta_2\epsilon^2(9\mathrm{e}^{-\beta_2} - \mathrm{e}^{-\beta_2-\epsilon}) + 2\epsilon^3\mathrm{e}^{-\beta_2} + (\beta_2^2\epsilon + \beta_2\epsilon^2)\mathrm{e}^{2\beta_2+\epsilon}\end{aligned}$$

it can be seen that $\Delta_{B'} > 0$ in the region $\beta_1 > \beta_2$ ($\Delta_{B'} < 0$ for $\beta_2 > \beta_1$). Consequently, the divergence is absent.

2) If $4T^3 - 27\sigma^2\chi^2 = 0$, then $\Delta_{B'} = 1 + (8 + 6\beta)\mathrm{e}^{-3\beta} > 0$ (see Fig. 6) and the divergence is absent.

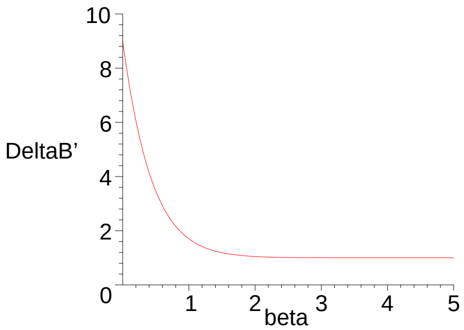


Figure 6. Boundary conditions B′, $4T^3 - 27\sigma^2\chi^2 = 0$. The plot of $\Delta_{B'}$

3) When $4T^3 - 27\sigma^2\chi^2 < 0$, then

$$\Delta_{B'} = 2(3 - u^2)\sin(\gamma u) + 8u\cos(\gamma u) + u(1 + u^2)\mathrm{e}^{3\gamma}.$$

As for the case B, the transformation of $\Delta_{B'}$

$$\Delta_{B'} = u\mathrm{e}^{3\gamma}\Big[\Big(2(3-u^2)\frac{\sin(\gamma u)}{u} + 8\cos(\gamma u)\Big)\mathrm{e}^{-3\gamma} + (1+u^2)\Big]$$

shows its positivity for small u. For large u and γ the third summand grows faster than two first ones (see Fig. 7). Consequently, the divergence is absent again.

2.3. Boundary Conditions D

1) If $4T^3 - 27\sigma^2\chi^2 > 0 \Rightarrow T > 0$, the boundary conditions give

$$\begin{aligned}\Delta_D &= \beta_1(\beta_1+2\beta_2)\mathrm{e}^{\beta_1} - \beta_2(2\beta_1+\beta_2)\mathrm{e}^{\beta_2} + (\beta_2^2-\beta_1^2)\mathrm{e}^{-(\beta_1+\beta_2)} = 2(\beta_1^3-\beta_2^3)\\ &+ 3\beta_1\beta_2(\beta_1-\beta_2) + \sum_{k=1}^{\infty}\frac{1}{(2k+1)!}\Big\{[(2k+1)\beta_1^{2k+1} + \beta_1^{2k+2}](\beta_1+2\beta_2)\\ &- [(2k+1)\beta_2^{2k+1} + \beta_2^{2k+2}](2\beta_1+\beta_2) + [(2k+1)(\beta_1+\beta_2)^{2k+1}\\ &- (\beta_1+\beta_2)^{2k+2}](\beta_2-\beta_1)\Big\}.\end{aligned}$$

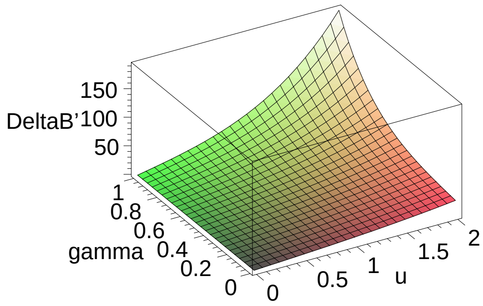


Figure 7. Boundary conditions B′, $4T^3 - 27\sigma^2\chi^2 > 0$. The 3D plot of $\Delta_{B'}$.

By direct computations it can be verified that all expressions in braces are negative for $\beta_2 > \beta_1$. Consequently, the divergence of the plate don't take place.

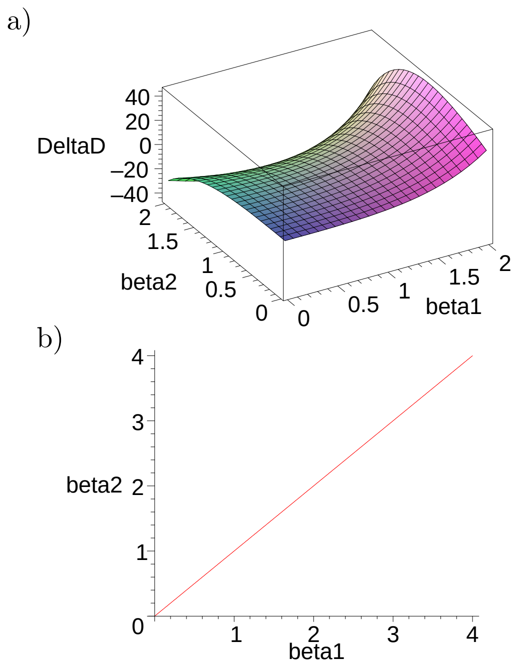


Figure 8. Boundary conditions D, $4T^3 - 27\sigma^2\chi^2 > 0$. (**a**) The 3D plot of Δ_D. (**b**) The plot of $\Delta_D = 0$ solution

2) For $4T^3 - 27\sigma^2\chi^2 = 0$ the boundary conditions system determinant is the following: $\Delta_D = 2 + 3\beta - 2\mathrm{e}^{-3\beta}$. Since $\frac{\mathrm{d}\Delta_D}{\mathrm{d}\beta} = 3 + 6\mathrm{e}^{-3\beta}$ and $\Delta_D(0) = 0$, the equation $\Delta_D = 0$ has no positive roots (see Fig. 9) and the divergence

of the plate is absent again.

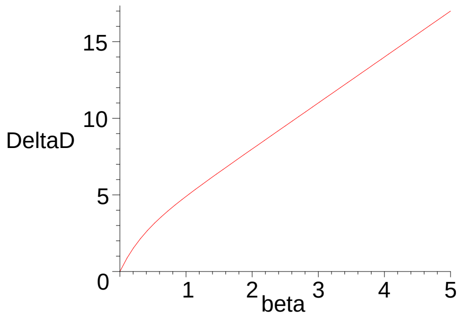


Figure 9. Boundary conditions D, $4T^3 - 27\sigma^2\chi^2 = 0$. The plot of Δ_D

3) Let now $4T^3 - 27\sigma^2\chi^2 < 0$. Then the following equation

$$\Delta_D = -2ue^{-3\gamma} + 2u\cos(\gamma u) + (u^2 + 3)\sin(\gamma u) = 0$$

arises, which determinates the bifurcation curve (see Fig. 10).

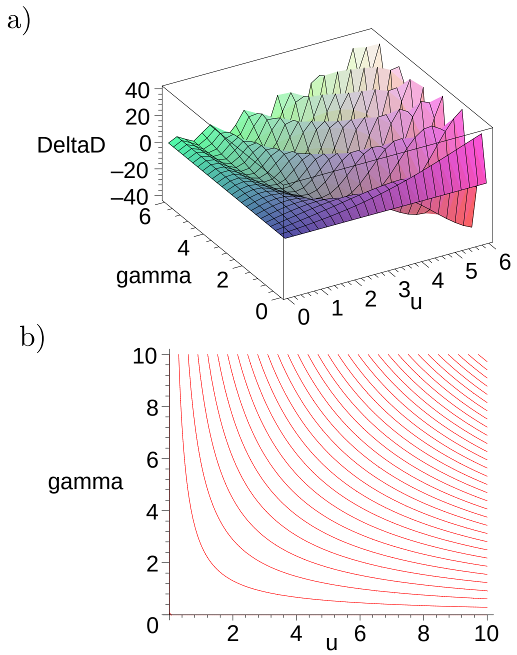


Figure 10. Boundary conditions D, $4T^3 - 27\sigma^2\chi^2 < 0$. (**a**) The 3D plot of Δ_D. (**b**) The plot of $\Delta_D = 0$ solution

The basis elements of zero and defect subspaces are

$$\varphi = \frac{1}{u}\Big[e^{\gamma}(u^2+9)\sin(\gamma u) - 2ue^{-2\gamma x} + 2e^{\gamma x}\{u\cos(\gamma ux) - 3\sin(\gamma ux)\}\Big],$$
$$\psi = \frac{1}{u}\Big[e^{2\gamma x}\{2u\cos(\gamma u) + (u^2+3)\sin(\gamma u)\} - 2e^{-\gamma x}\{3\sin(\gamma u(x-1)) + u\cos(\gamma u(x-1))\}\Big].$$

The buckling forms are determined by the following BEq coefficients

$$L^1_{110} = \frac{\gamma}{u^3(u^2+9)}\Big[\{-2u^2(14u^2+3u^4+27)\sin(2\gamma u) + u(u^2+3)(u^2-3)^2 \times \cos(2\gamma u) - u(u^2+9)(u^2+3)(1+u^2)\}e^{3\gamma} - 96u^3e^{-3\gamma} + (-162 + (-16\gamma+10)u^6 + (-192\gamma+98)u^4 + (-432\gamma+54)u^2) \times \sin(\gamma u) + (2\gamma u^7 + (6\gamma+16)u^5 + (-90\gamma+144)u^3 + 162\gamma u)\cos(\gamma u)\Big],$$

$$L^1_{101} = -\frac{k\kappa}{u^3(u^2+9)}\Big[\{u^2(u^4-10u^2-27)\sin(2\gamma u) + u(7u^4+27+18u^2) \times \cos(2\gamma u) - 3u(u^2+9)(1+u^2)\}e^{3\gamma} + 48u^3e^{-3\gamma} + (-162 + (-2+6\gamma)u^6 + (-10+60\gamma)u^4 + (54\gamma+54)u^2) \times \sin(\gamma u) + ((18\gamma-4)u^5 + (-36+180\gamma)u^3 + 162\gamma u)\cos(\gamma u)\Big],$$

$$L^1_{200} = -\frac{k\kappa(\kappa+1){M_0}^2\gamma}{u^3(1+9u^2)(4+u^2)(25+u^2)(1+u^2)}\Big[64u^2(25+u^2)(4+u^2) \times (1+u^2)\{(3u^4-14u^2-9)\sin(2\gamma u) + (16u^3+24u)\cos(2\gamma u)\}e^{\gamma} - 32u^2(1+9u^2)(4+u^2)(25+u^2)\{(u^4+2u^2+9)\sin(\gamma u) + (4u^3+12u) \times \cos(\gamma u)\}e^{-2\gamma} + 512u^3(1+9u^2)(4+u^2)(1+u^2)e^{-5\gamma} + \{(25+u^2) \times (1+9u^2)(u^2+2)(u^2+9)^2(1+u^2)^2\sin(\gamma u) - 2(1+9u^2)(25+u^2) \times (1+u^2)(4u^6+11u^4+18u^2+27)\sin(3\gamma u) + u(25+u^2)(1+9u^2) \times (u^2+9)^2(1+u^2)^2\cos(\gamma u) + u(1+9u^2)(25+u^2)(1+u^2) \times (u^3-5u^2+5u-9)(u^3+5u^2+5u+9)\cos(3\gamma u)\}e^{4\gamma} - u^2(9u^6+133u^4 + 527u^2+115)(u^2+9)^3\sin(\gamma u) - 18u^3(u^2-3)^2(u^2+9)^3\cos(\gamma u)\Big]$$

for one-sided flow around, and

$$L^2_{110} = L^1_{110}, \quad L^2_{101} = 2L^1_{101}, \quad L^2_{300} \text{ is omitted}$$

for two-sided flow around. Here the parameters u and γ are bound by the corresponding relation $\Delta_D = 0$.

2.4. Boundary Conditions D′

1) If $4T^3 - 27\sigma^2\chi^2 > 0$, then using the boundary conditions we obtain the equation

$$\Delta_{D'} = -\beta_2(2\beta_1 + \beta_2)\mathrm{e}^{-\beta_2} + \beta_1(\beta_1 + 2\beta_2)\mathrm{e}^{-\beta_1} - (\beta_1^2 - \beta_2^2)\mathrm{e}^{\beta_1+\beta_2}.$$

Transforming it as follows

$$\Delta_{D'} = 2\beta_1\beta_2\frac{\mathrm{e}^{\beta_2} - \mathrm{e}^{\beta_1}}{\mathrm{e}^{\beta_1+\beta_2}} + \frac{(\beta_2^2 - \beta_1^2)\mathrm{e}^{2(\beta_1+\beta_2)} - \beta_2^2\mathrm{e}^{\beta_1} + \beta_1^2\mathrm{e}^{\beta_2}}{\mathrm{e}^{\beta_1+\beta_2}}$$

we can see that $\Delta_{D'} > 0$ for $\beta_2 > \beta_1$ ($\Delta_{D'} < 0$ for $\beta_1 > \beta_2$), i. e. the divergence is absent.

a)

b)

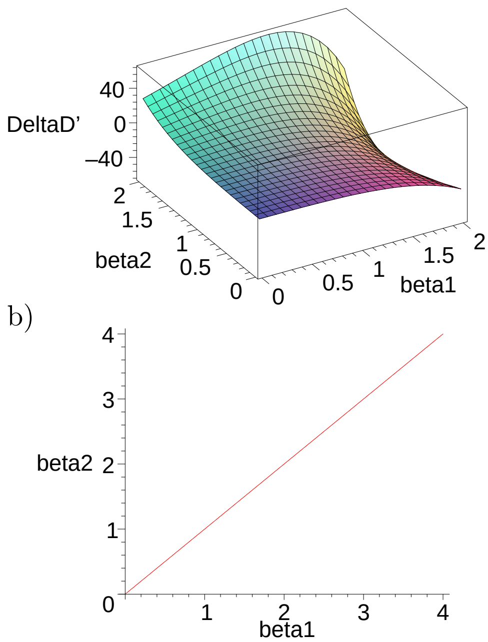


Figure 11. Boundary conditions D′, $4T^3 - 27\sigma^2\chi^2 > 0$. (**a**) The 3D plot of $\Delta_{D'}$. (**b**) The plot of $\Delta_{D'} = 0$ solution

2) If $4T^3 - 27\sigma^2\chi^2 = 0$, then $\Delta_{D'} = 2 + (3\beta - 2)\mathrm{e}^{-3\beta} > 0$ (see Fig. 12) and the divergence is absent.

3) When $4T^3 - 27\sigma^2\chi^2 < 0$, then the equation

$$\Delta_{D'} = (u^2 + 3)\sin(\gamma u) - 2u\cos(\gamma u) + 2u\mathrm{e}^{3\gamma}$$

determines the bifurcation curves (see Fig. 13).

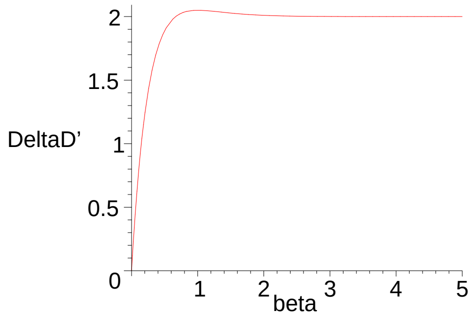


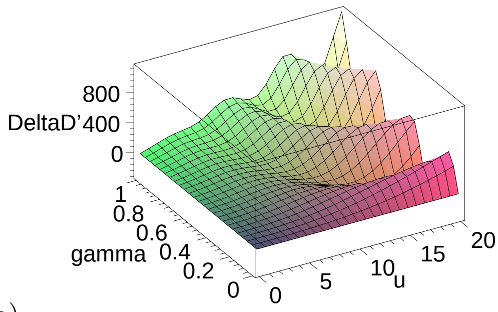
Figure 12. Boundary conditions D′, $4T^3 - 27\sigma^2\chi^2 = 0$. The plot of $\Delta_{D'}$

The basis elements of zero and defect subspaces are

$$\varphi = \frac{1}{u}\Big[u - u\mathrm{e}^{-2\gamma x} - \{u\cos(\gamma u) + 3\sin(\gamma u)\}\mathrm{e}^{-3\gamma} + \{u\cos(\gamma u(x-1)) - 3\sin(\gamma u(x-1))\}\mathrm{e}^{-\gamma(3-x)}\Big],$$

$$\psi = \frac{1}{u}\Big[\{u\cos(\gamma ux) + 3\sin(\gamma ux)\}\mathrm{e}^{-\gamma x} - u\mathrm{e}^{2\gamma x}\Big].$$

a)



b)

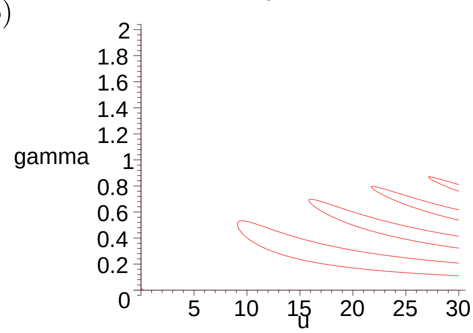


Figure 13. Boundary conditions D′, $4T^3 - 27\sigma^2\chi^2 > 0$. (**a**) The 3D plot of $\Delta_{D'}$. (**b**) The plot of $\Delta_{D'} = 0$ solution

The buckling forms are determined by the following BEq coefficients

$$
\begin{aligned}
L^1_{110} &= -\frac{\gamma \mathrm{e}^{-3\gamma}}{2u^3(u^2+9)}\Big[\{(-8\gamma-8)u^5+(24-72\gamma)u^3\}\mathrm{e}^{3\gamma}\\
&+\{(4\gamma-1)u^6+(21+48\gamma)u^4+(108\gamma-27)u^2-81\}\sin(\gamma u)\\
&+\{\gamma u^7+(8+11\gamma)u^5+(-24+27\gamma)u^3+81\gamma u\}\cos(\gamma u)\Big],\\
L^1_{101} &= \frac{k\kappa \mathrm{e}^{-3\gamma}}{2u^3(u^2+9)}\Big[\{(2+4\gamma)u^5+(-30+36\gamma)u^3\}\mathrm{e}^{3\gamma}\\
&+\{-\gamma u^6+(-6\gamma-15)u^4+27\gamma u^2-81\}\sin(\gamma u)\\
&+\{(5\gamma-2)u^5+(30+54\gamma)u^3+81\gamma u\}\cos(\gamma u)\Big],
\end{aligned}
$$

$$
\begin{aligned}
L^1_{200} &= -\frac{k\kappa(\kappa+1)M_0^2\beta(u^2+9)}{32u^2(1+9u^2)(4+u^2)(25+u^2)(1+u^2)}\\
&\times\Big[\{(-64u^7+2356u^5+5264u^3-6u^9+2850u)\sin(2\beta u)\\
&+(44u^8-900+1136u^6+856u^4-1136u^2)\cos(2\beta u)\\
&+9361u^2+3630u^6+900+352u^8+11748u^4+9u^{10}\}\mathrm{e}^{-6\beta}\\
&+\{-9u^{10}-2366u^6-2284u^4-225u^2-316u^8\}\mathrm{e}^{-2\beta}\\
&+\{(5496u^5+18840u^3-24u^9+4320u+168u^7)\sin(\beta u)\\
&+(208u^8-6336u^2+5200u^4+2528u^6)\cos(\beta u)\}\mathrm{e}^{-5\beta}\\
&+\{(23096u^3+2312u^7+2400u+13720u^5+72u^9)\sin(\beta u)\\
&+(-144u^8-14864u^4-4192u^6-1600u^2)\cos(\beta u)\}\mathrm{e}^{-3\beta}\\
&-656u^4-736u^6-144u^8-64u^2\Big]
\end{aligned}
$$

for one-sided flow around, and

$$L^2_{110}=L^1_{110},\quad L^2_{101}=2L^1_{101},\quad L^2_{300}\text{ is omitted}$$

for two-sided flow around. The parameters u and γ are bound by the corresponding relation $\Delta_{D'}=0$.

2.5. Boundary Conditions A and C

The boundary conditions here are symmetric. Then according to [6] for $T=0$ the plate divergence is absent and in dynamic situation the flutter takes place. However we consider stationary bifurcation and the additional extension $T>0$ for the cases 1) and 2) can't lead to the plate buckling.

These results can be verified analytically similarly 1) and 2) of subsection 2.2 expanding the relevant boundary conditions system determinants into series

$$\begin{aligned}\Delta_A &= \beta_2^3(2\beta_1+\beta_2)\cosh\beta_2 - \beta_1^3(\beta_1+2\beta_2)\cosh\beta_1\\ &\quad - (\beta_2-\beta_1)(\beta_1+\beta_2)^3\cosh(\beta_1+\beta_2),\\ \Delta_C &= \beta_1(\beta_1+2\beta_2)\cosh\beta_1 - \beta_2(2\beta_1+\beta_2)\cosh\beta_2 + (\beta_2^2-\beta_1^2)\cosh(\beta_1+\beta_2),\end{aligned}$$

i. e. by the proof of $\Delta_A, \Delta_C \neq 0$ for $\beta_2 > \beta_1$. More easily it can be checked for the case $4T^3 - 27\sigma^2\chi^2 = 0$, where $\Delta_A = 16\cosh 2\beta - 16\cosh\beta - 3\beta\sinh\beta$, $\Delta_C = 2\cosh 2\beta - 2\cosh\beta - 3\beta\sinh\beta$.

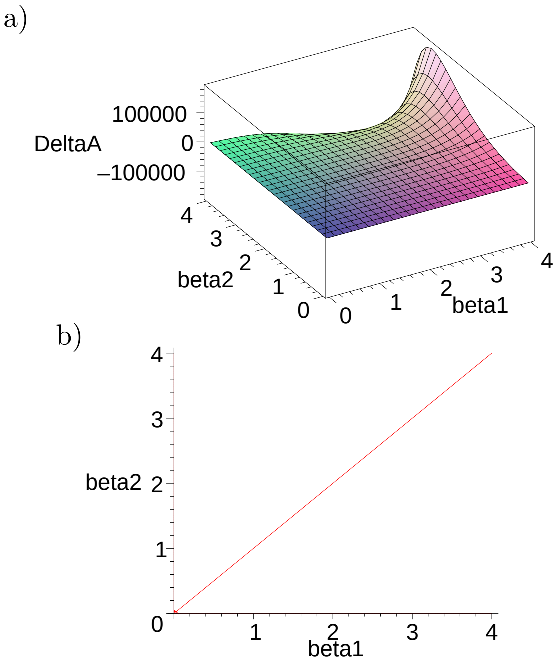


Figure 14. Boundary conditions A, $4T^3 - 27\sigma^2\chi^2 > 0$. (**a**) The 3D plot of Δ_A. (**b**) The plot of $\Delta_A = 0$ solution

Let now $4T^3 - 27\sigma^2\chi^2 < 0$. It should be noted that the values $0 < u < \sqrt{3}$ respond to the plate extension, and the values $u > \sqrt{3}$ to the plate compression (see (2.3), (2.4)). The mechanical considerations mentioned above show the divergence absence for $0 < u < \sqrt{3}$. Therefore the condition $u > \sqrt{3}$ will be investigated further. Note by the way, that here for $\sigma = 0$ the plate buckling takes place ([1], [2], [7]). The relevant bifurcation curves are determined by the equations

$$\Delta_A = (3 - 6u^2 - u^4)\sinh(\gamma)\sin(\gamma u) - 8u(\cosh(2\gamma) - \cosh(\gamma)\cos(\gamma u)) = 0, \tag{2.8}$$

$$\Delta_C = 2u(\cosh(2\gamma) - \cosh(\gamma)\cos(\gamma u)) - (u^2+3)\sinh(\gamma)\sin(\gamma u) = 0 \tag{2.9}$$

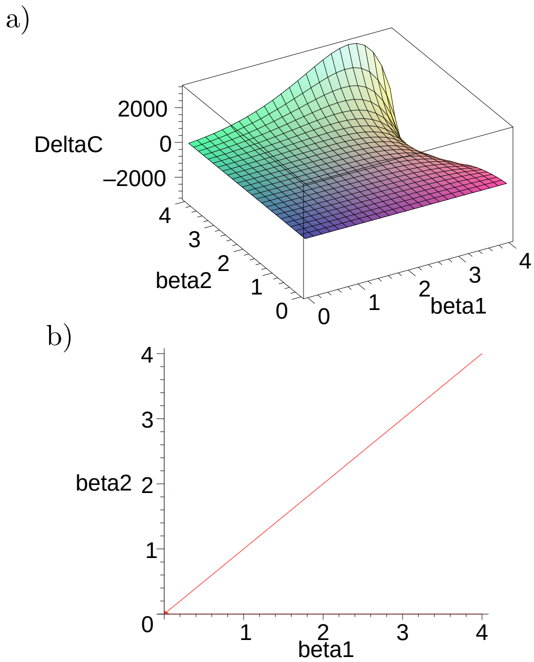


Figure 15. Boundary conditions C, $4T^3 - 27\sigma^2\chi^2 > 0$. (**a**) The 3D plot of Δ_C. (**b**) The plot of $\Delta_C = 0$ solution

(see Figs. 17 and 18).

The bases of zero and defect-subspaces are the following

$$\begin{aligned}
\varphi_A = \psi_A = \frac{1}{\sin(\gamma u)}\Big[&(u^2+1)^2\sin(\gamma u)\mathrm{e}^{-2\gamma x}+4\mathrm{e}^{\gamma(x-3)}(2u\cos(\gamma u x)+(u^2-1)\\
&\times\sin(\gamma u x))-4\mathrm{e}^{\gamma x}((u^2-1)\sin(\gamma u(x-1))+2u\cos(\gamma u(x-1)))\\
&+8\mathrm{e}^{\gamma}u+\mathrm{e}^{-2\gamma}((3-6u^2-u^4)\sin(\gamma u)-8u\cos(\gamma u))\Big],\\
\varphi_C = \psi_C = \frac{1}{\sin(\gamma u)}\Big[&(u^2+1)\sin(\gamma u)\mathrm{e}^{-2\gamma x}+2\mathrm{e}^{\gamma(x-3)}\{\sin(\gamma u x)\\
&+2e^{\gamma x}\{u\cos(\gamma u(x-1))-\sin(\gamma u(x-1))\}-2\mathrm{e}^{\gamma}u\\
&-u\cos(\gamma u x)\}-\mathrm{e}^{-2\gamma}\{\sin(\gamma u)(u^2+3)-2u\cos(\gamma u)\}\Big].
\end{aligned}$$

The denominator $\sin(\gamma u)$ is equal to zero only at $u = 0$, since the other roots don't satisfy the equations (2.8), (2.9), i.e. on the bifurcation curves (2.8), (2.9) $\sin(\gamma u) \neq 0$.

The BEq coefficients determining the solutions asymptotics have the forms (where the connections $\Delta_A = 0$ and $\Delta_C = 0$ should be taken into account)

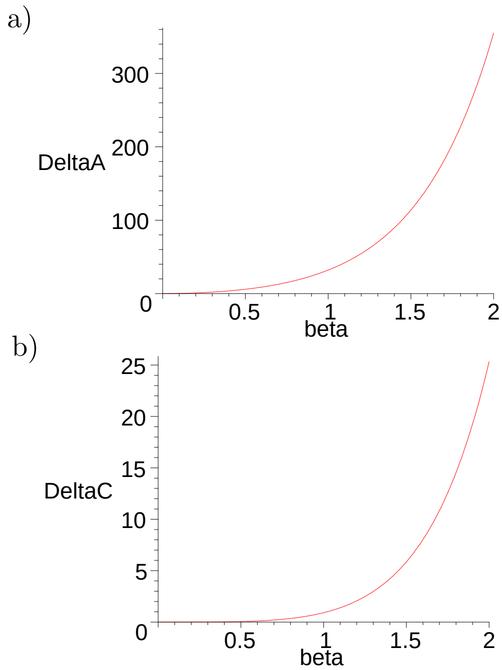


Figure 16. Boundary conditions A and C, $4T^3 - 27\sigma^2\chi^2 = 0$. (**a**) The plot of Δ_A. (**b**) The plot of Δ_C

A.

$$\begin{aligned}
L^1_{110} = {}& -\frac{\gamma(u^2+1)\mathrm{e}^{-6\gamma}}{\sin^2(\gamma u)}\Big[\{(-18u^4+18+2u^2-2u^6)\mathrm{e}^{4\gamma}+(u^6+59u^2+19u^4\\
&+9)\mathrm{e}^{2\gamma}+(u^6-u^4+3u^2-27)\mathrm{e}^{6\gamma}\}\cos^2(\gamma u)-4u\{((-9-u^4-10u^2)\mathrm{e}^{4\gamma}\\
&+\mathrm{e}^{6\gamma}(u^2-3)(u^2+3)-2\mathrm{e}^{2\gamma}(u^2-3))\sin(\gamma u)+2((-u^2+3)\mathrm{e}^{3\gamma}\\
&+\mathrm{e}^{5\gamma}(u^2+5)+4\mathrm{e}^{\gamma}+4\mathrm{e}^{7\gamma})u\}\cos(\gamma u)-4u\{(9-u^4)\mathrm{e}^{3\gamma}\\
&+\mathrm{e}^{\gamma}(-3+u^4+6u^2)+2\mathrm{e}^{5\gamma}(u^2-3)+4\mathrm{e}^{7\gamma}(u^2+3)\}\sin(\gamma u)\\
&+(2u^6+18u^4-18+62u^2)\mathrm{e}^{4\gamma}+(-15u^4-39u^2-u^6-9)\mathrm{e}^{2\gamma}\\
&+(-u^6-3u^4+9u^2+27)\mathrm{e}^{6\gamma}+4((u^2+5)\mathrm{e}^{8\gamma}-u^2+3)u^2\Big],
\end{aligned}$$

$$\begin{aligned}
L^1_{101} = {}& -\frac{k\kappa\mathrm{e}^{-6\gamma}}{2\sin^2(\gamma u)}\Big[\{(9-100u^2+u^8+12u^6+30u^4)\mathrm{e}^{2\gamma}+(9-100u^2+u^8\\
&+12u^6+30u^4)\mathrm{e}^{6\gamma}-2\mathrm{e}^{4\gamma}(u^2+9)(u^2+1)^3\}\cos^2(\gamma u)+\{16(-3+u^4\\
&+6u^2)(-\mathrm{e}^{2\gamma}+\mathrm{e}^{6\gamma})\sin(\gamma u)+128u(\mathrm{e}^{5\gamma}+\mathrm{e}^{\gamma}+\mathrm{e}^{3\gamma}+\mathrm{e}^{7\gamma})\}u\cos(\gamma u)\\
&-16u(-3+u^4+6u^2)(-\mathrm{e}^{\gamma}-\mathrm{e}^{5\gamma}+\mathrm{e}^{3\gamma}+\mathrm{e}^{7\gamma})\sin(\gamma u)-(-3+u^4\\
&+6u^2)^2\mathrm{e}^{2\gamma}-(-3+u^4+6u^2)^2\mathrm{e}^{6\gamma}+(18+24u^6-200u^2+60u^4+2u^8)\\
&\times\mathrm{e}^{4\gamma}-64u^2(1+\mathrm{e}^{8\gamma})\Big],
\end{aligned}$$

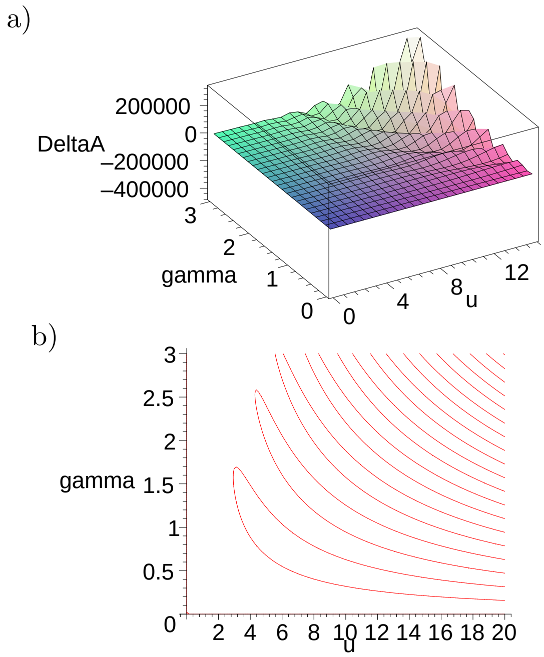


Figure 17. Boundary conditions A, $4T^3 - 27\sigma^2\chi^2 < 0$. (**a**) The 3D plot of Δ_A. (**b**) The plot of $\Delta_A = 0$ solution

L^1_{200} is omitted

for one-sided flow around, and

$$L^2_{110} = L^1_{110}, \quad L^2_{101} = 2L^1_{101}, \quad L^2_{300} \text{ is omitted}$$

for two-sided flow around,

C.

$$L^1_{110} = \frac{\gamma(u^2+1)\mathrm{e}^{-6\gamma}}{\sin^2(\gamma u)}\Big[(u-3)(u+3)(-\mathrm{e}^{2\gamma}+\mathrm{e}^{6\gamma})\cos^2(\gamma u) - 2u\{(3\mathrm{e}^{6\gamma}+3\mathrm{e}^{2\gamma}) \times \sin(\gamma u) + u(-\mathrm{e}^{5\gamma}+\mathrm{e}^{3\gamma})\}\cos(\gamma u) + 6u(\mathrm{e}^{3\gamma}+\mathrm{e}^{5\gamma})\sin(\gamma u) + u^2 - \mathrm{e}^{8\gamma}u^2 + 9\mathrm{e}^{6\gamma} - 9\mathrm{e}^{2\gamma}\Big],$$

$$L^1_{101} = -\frac{k\kappa\mathrm{e}^{-6\gamma}}{2\sin^2(\gamma u)}\Big[\{(9+2u^2+u^4)(\mathrm{e}^{2\gamma}+\mathrm{e}^{6\gamma}) + (-2u^4-20u^2-18)\mathrm{e}^{4\gamma}\} \times \cos^2(\gamma u) - 4u\{(u^2+3)(-\mathrm{e}^{2\gamma}+\mathrm{e}^{6\gamma})\sin(\gamma u) - 2u(\mathrm{e}^{5\gamma}+\mathrm{e}^{\gamma}+\mathrm{e}^{3\gamma} + \mathrm{e}^{7\gamma})\}\cos(\gamma u) - 4u(u^2+3)(-\mathrm{e}^{3\gamma}-\mathrm{e}^{7\gamma}+\mathrm{e}^{5\gamma}+\mathrm{e}^{\gamma})\sin(\gamma u) - (u^2+3)^2 \times (\mathrm{e}^{2\gamma}+\mathrm{e}^{6\gamma}) + (18+2u^4+4u^2)\mathrm{e}^{4\gamma} - 4u^2(1+\mathrm{e}^{8\gamma})\Big],$$

L^1_{200} is omitted

a)

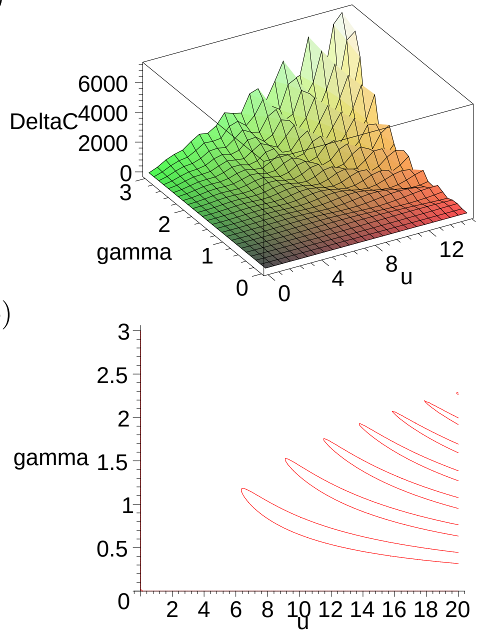


Figure 18. Boundary conditions C, $4T^3 - 27\sigma^2\chi^2 < 0$. (**a**) The 3D plot of Δ_C. (**b**) The plot of $\Delta_C = 0$ solution

for one-sided flow around, and

$$L^2_{110} = L^1_{110}, \quad L^2_{101} = 2L^1_{101}, \quad L^2_{300} \text{ is omitted}$$

for two-sided flow around.

3. Conclusions and Future Work

In this article the main terms of solution asymptotics for the nonlinear problem (1.1) with two bifurcation parameters at the boundary conditions A–D′ are obtained. The subsequent terms of solution expansions can be calculated by indeterminate coefficients method.

We have also considered the cases when the left edge is elastically supported (or elastically turned) and the right edge is rigidly fixed and visa versa. The other possible cases of fixing the edges $x = 0$ and $x = 1$ with the investigation on the presence of the divergence will be also considered.

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О дивергентной потере устойчивости удлиненной пластины в сверхзвуковом потоке газа, сжимаемой или растягиваемой внешними краевыми усилиями

Аннотация. Изучается прогиб тонкой гибкой удлиненной пластины, обтекаемой сверхзвуковым потоком газа вдоль оси Ox и сжимаемой или растягиваемой внешними краевыми усилиями по краям $x = 0$ и $x = 1$. Задача описывается нелинейным ОДУ в безразмерных переменных с двумя бифуркационными параметрами, один из которых характеризует сжатие (растяжение) пластины перпендикулярно оси Oy, а другой - число Маха. Рассматриваются шесть краевых условий согласно различным способам закрепления краев $x = 0$ и $x = 1$. Для несимметричных краевых условий рассматриваются четыре возможных случая. Применяется метод Ляпунова-Шмидта теории ветвления. В окрестности каждой точки бифуркационных кривых вычисляется асимптотика малых решений в виде сходящихся рядов по двум малым параметрам. В отличие от наших предыдущих результатов, в нелинейное уравнение введено интегральное слагаемое, учитывающее усилия в срединной плоскости изогнутой пластины. Основные трудности возникли в исследовании соответствующих двупараметрических спектральных задач и были преодолены с помощью представления бифуркационных кривых через корни соответствующего характеристического уравнения.