Серия «Математика»
Том 1 (2007), № 1, С. 254—266

Онлайн-доступ к журналу:
http://isu.ru/izvestia

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УДК 517.968

Difference-differential equations with fredholm operator in the main part

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Abstract. We consider the reduction of the degenerate difference-differential equations with Fredholm operator in the main expression to the regular problems. It is shown how the question of the choice of boundary conditions is connected with the Jordan structure of operator coefficients of the equations. The problem of the choice of boundary conditions is solved. The theorems of existence and uniqueness of boundary value problems are proved. The abstract theorems are used for statement and investigation of boundary value problems for partial differential equations and difference equations with degeneration.

Keywords: degenerate equations, Jordan sets, Fredholm operator, boundary value problems.

Introduction

The correct statement and investigation of a boundary value problem for degenerated systems is the major part in the modern theory of partial differential equations [2]. In case of Fredholm operator at the main differential expression the following methods were used: the analytical Nekrasov-Nazarov method [1], [14], the Lyapunov-Schmidt method [23], the Jordan set technique [23], a technique of psedoinverse operators [5] and their modifications [7], [11]-[18], a group symmetry methods [20], [22], a method of differential inclusions [3], a topological method [14], methods of regularization of ill–posed problems [12]. It should be noted that, unlike the classical bifurcation theory [23], for these problems the analog of the Lyapynov-Schmidt bifurcation equation is differential or integral [12]. Finally, the theory of analytical groups and semigroups of generating operators was developed in [19] in order to solve singular problems. The theory of generalized solutions of such problems in the class of Schwartz distributions was constructed in [8], [24]. The results of these investigations have many applications (see references in [12], [19], [24]).

However, in all papers mentioned above the problems were considered, which could be interpreted as ordinary differential equations in Banach spaces. On the other hand, there are a lot of systems of partial difference differential equations with a noninvertible operator in the main part, which can not be interpreted in this way. So, in papers [6],[9], [10], significantly more complex degenerated partial differential equations in Banach spaces are considered. In this paper we develop the similar results for difference-differential equations.

Let us consider the following equation

$$\Lambda u \equiv L_0 B u + L_1 A_1 u + \cdots + L_q A_q u = f(x), \tag{0.1}$$

where B and $A_i, i = \overline{1,q}$ are closed linear operators with the dense domains from E_1 to E_2; E_1, E_2 are Banach spaces and $D(B) \subseteq D(A_i), i = \overline{1,q}$, $x \in \Omega \subset R^r$, B is Fredholm operator with $dimN(B) = dimN(B^*) = n, \overline{R(B)} = R(B)$, $f(x) : \Omega \subset R^r \to E_2$ is a sufficiently smooth function; linear operators L_i act on the sets of abstract functions, defined on $x \in \Omega \subset R^r$ with the values in Banach spaces $E_2(E_1)$ and satisfy the following conditions:

1)$D(L_0) \subset D(L_1) \subset D(L_i), i = 2, \ldots, q$;

2)$L_0 B u(x) = B L_0 u(x), L_i A_i u(x) = A_i L_i u(x)$ on $u(x) \in D(B) \cap D(L_0)$.

Conditions 1) and 2) are carried out, for example, for difference- differential operators of the following form

$$L_i(\frac{\partial}{\partial x}, \Delta) = \sum_{|k| \leq q_i} a_k^i(x) D^k + \sum_{|k| \leq q_i} b_k^i(x) \Delta^k,$$

where

$$D^k = \frac{\partial^k}{\partial x_1^{k_1} \ldots \partial x_r^{k_r}},$$

$$\Delta^k u = \sum_{i_1=0}^{k_1} \ldots \sum_{i_r=0}^{k_r} (-1)^{|k|-|i|} C_{k_1}^{i_1} \ldots C_{k_r}^{i_r} u(x_1 + i_1 h_1, \ldots, x_r + i_r h_r),$$

$q_0 > q_1 > q_2 \geq \ldots \geq q_q$, $a_k^i(x), b_k^i(x) : \Omega \subset R^r \longrightarrow R^1$. In what follows L_i will denote this concrete form of operators. It will be more convenient not to specify concretely $D(L_0)$.

The investigation of singular equation (0.1) is reduced to regular problems, i.e. to equations solvable with respect to the main part L_0. A special decompositions of the Banach spaces E_1 and E_2 in accordance with the generalized Jordan structure of the operator coefficients $B, A_i, i = \overline{1,q}$ are used. This reduction makes it possible to pose boundary value problems for (0.1).

1. P, Q-commutability of Linear Operators in Case of Fredholm Operator

Let $E_1 = M_1 \oplus N_1, E_2 = M_2 \oplus N_2$, P be a projector on M_1 along N_1, Q be a projector on M_2 along N_2, A be a linear closed operator from E_1 to E_2, $\overline{D(A)} = E_1, A \in \{A_1, \dots, A_q\}$.

Definition 1. *If $u \in D(A), Pu \in D(A), APu = QAu$ then $A(P,Q)$ - commutes.*

Suppose that $\phi_i|_1^n$ is a basis in $N(B), \psi_i|_1^n$ is a basis in $N^*(B)$, $< \phi_k, \gamma_i >= \delta_{ki}, \;\; i,k = \overline{1,n}, \; < z_i, \psi_k >= \delta_{ik}, \; i,k = \overline{1,n}$. Then based on [23] there is a unique bounded operator $\Gamma = (B + \sum_{i=1}^n < ., \gamma_i > z_i)^{-1}$.

Suppose the following condition is satisfied:

1. Fredholm operator B has a complete A_1-Jordan set $\phi_i^{(j)}, i = \overline{1,n}, j = \overline{1,p_i}$, B^* has a complete A_1^*- Jordan set $\psi_i^{(j)}, i = \overline{1,n}, j = \overline{1,p_i}$, and the systems $\gamma_i^{(j)} \equiv A_1^* \psi_i^{(p_i+1-j)}, z_i^{(j)} \equiv A_1 \phi_i^{(p_i+1-j)}$, here $i = \overline{1,n}, j = \overline{1,p_i}$, corresponding to them are biorthogonal [7].

The projectors

$$P = \sum_{i=1}^{n} \sum_{j=1}^{p_i} < ., \gamma_i^{(j)} > \phi_i^{(j)} \equiv (< ., \Upsilon > \Phi), \tag{1.1}$$

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{p_i} < ., \psi_i^{(j)} > z_i^{(j)} \equiv (< ., \Psi > Z), \tag{1.2}$$

where $k = p_1 + \dots + p_n$ -root number, generate the direct decomposition

$$E_1 = E_{1k} \oplus E_{1\infty-k}, E_2 = E_{2k} \oplus E_{2\infty-k}$$

Corollary 1. *The operator $\Gamma(Q,P)$-commutes, $A_1\Gamma(Q,Q)$-commutes, $\Gamma A_1(P,P)$-commutes, $E_{1\infty-k}, E_{1k}$ - invariant subspaces of the operator ΓA_1, $E_{2\infty-k}, E_{2k}$- invariant subspaces of the operator $A_1\Gamma$.*

The proof is carried out by the substitution of the operators $\Gamma, A_1\Gamma, \Gamma A_1$ in formulas (1.1), (1.2), taking into account that $\psi_i^{(j)} = (\Gamma^* A_1^*)^{j-1} \psi_i^1, \phi_i^{(j)} = (\Gamma A_1)^{j-1} \phi_i^1, z_i^{(p_i+1-j)} = A_1 \phi_i^{(j)}, \gamma_i^{(p_i+1-j)} = A_1^* \psi_i^{(j)}, i = \overline{1,n}, j = \overline{1,p_i}$.

Suppose the operator $A(P,Q)$-commutes, where P, Q are defined by formulas (1.1), (1.2). Then there is a matrix $\mathcal{A}$, such that $A\Phi = \mathcal{A}Z$, $\mathrm{A}^*\Psi = \mathcal{A}'\Upsilon$ [9]. This matrix is called the matrix of (P,Q)-commutability.

Corollary 2. *The operators B, A_1 (P,Q)-commute and the matrices of (P,Q)-commutability are the symmetrical cell-diagonal matrices:*

$$\mathcal{A}_{\mathrm{B}} = diag(B_1, \ldots, B_n), \mathcal{A}_1 = diag(\mathcal{A}_{11}, \ldots, \mathcal{A}_{n1}),$$

where

$$B_i = \begin{bmatrix} 0...0 \\ 0...1 \\ ..... \\ 01...0 \end{bmatrix}$$

$$\mathcal{A}_{\mathrm{i}1} = \begin{bmatrix} 0...1 \\ .... \\ 1...0 \end{bmatrix}, i = \overline{1, n}.$$

In case when $k > n$ let us introduce definition.

Definition 2. *The operator G (P,Q)-commutes quasitriangularly, if $\mathcal{A}_{\mathrm{G}}$ is upper quasitriangular matrix, whose diagonal blocks $\mathcal{A}_{\mathrm{ii}}$ of dimension $p_i \times p_i, p_i \geq 2$ are lower right triangular matrices. If in addition $p_{i^*} = \ldots = p_{i^*+s} = 1$, then in matrix $\mathcal{A}_{\mathrm{G}}$ at the left of bloc $[\mathcal{A}_{ij}]_{i,j=\overline{i^*, i^*+s}}$ are nulls.*

2. The Reduction of Equation (0.1) to Regular Problems

Suppose:

2. The operators $A_2, \ldots, A_q$ (P,Q)- commute.

Then there are matrices $\mathcal{A}_{\mathrm{i}}, i = \overline{2, q}$, such that

$$A_i \Phi = \mathcal{A}_{\mathrm{i}} Z, A_i^* \Psi = \mathcal{A}_{\mathrm{i}}' \Upsilon$$

Following formulas (1.1), (1.2) we introduce the projection operators P, Q, which generate the direct decompositions

$$E_1 = E_{1k} \oplus E_{1\infty-k}, E_2 = E_{2k} \oplus E_{2\infty-k}.$$

Note that $\Gamma E_{2\infty-k} \subset E_{1\infty-k}$

We look for the solution of equation (0.1) in the following form

$$u(x) = \Gamma v(x) + (C(x), \Phi), \tag{2.1}$$

where $\Gamma = (B + \sum_{i=1}^{n} <\cdot, \gamma_i^{(1)}> z_i^{(1)})^{-1}$ is a bounded operator, $v \in E_{2\infty-k}$, $C(x) = (C_1(x), \ldots, C_n(x))', C_i(x) = (C_{i1}(x), \ldots, C_{ip_i}(x)), \Phi = (\Phi_1, \ldots, \Phi_n)', \Phi_i = (\phi_i^1, \ldots, \phi_i^{(p_i)}), i = \overline{1, n}.$

Substituting expression (2.1) into equation (0.1) and noting that $B\Gamma v = v$, since $B\Gamma = I - \sum_{i=1}^{n} < ., \psi_i^{(1)} > z_i^{(1)}, < v, \psi_i^{(1)} >= 0$, we obtain

$$L_0 v + \sum_{i=1}^{q} L_i A_i \Gamma v + L_0 B(C, \Phi) + \sum_{i=1}^{q} L_i A_i (C, \Phi) = f(x). \tag{2.2}$$

The operator Γ (Q, P)-commutes, so from condition 2 and corollary 1 it follows that $QA_i\Gamma(I-Q) = 0, (I-Q)A_i\Gamma Q = 0$. Hence, $QA_i\Gamma v = 0, \forall v \in E_{2\infty-k}$. According to corollary 2 $B\Phi = \mathcal{A}_{\mathrm{B}} Z$, where $\mathcal{A}_{\mathrm{B}} = (B_1, \ldots, B_m)$ is a symmetrical cell-diagonal matrix. Consequently,

$$(I-Q)B\Phi = 0, (I-Q)A_i\Phi = 0, i = \overline{1,q}, \tag{2.3}$$

because $(I-Q)Z = 0$. The following equalities hold:

$$(A_i(C,\Phi), \Psi) = \mathcal{A}_{\mathrm{i}}' C, (B(C,\Phi), \Psi) = \mathcal{A}_{\mathrm{B}} C. \tag{2.4}$$

Projecting equation (2.2) onto $E_{2\infty-k}$ using (2.3) we obtain the regular equation (solved according to operator L_0):

$$\tilde{\mathcal{L}} v = (I-Q) f(x), \tag{2.5}$$

where

$$\tilde{\mathcal{L}} = L_0 + \sum_{i=1}^{q} L_i A_i \Gamma. \tag{2.6}$$

In order to determine the vector-function $C(x) : R^r \to R^k$, we project equation (2.2) onto E_{2k} and using (2.4) we obtain

$$L_0 \mathcal{A}_B C + \sum_{i=1}^{q} L_i \mathcal{A}_i' C =< f(x), \Psi > . \tag{2.7}$$

So it is proved

Theorem 1. *Suppose conditions 1 and 2 are satisfied, $f : \Omega \subset R^r \to E_2$ - sufficiently smooth function. Then any solution of equation (0.1) can be represented in the form*

$$u = \Gamma v + (C, \Phi)$$

where v satisfies regular equation (2.5), and the vector $C(x)$ is defined from system (2.7).

Since $det \mathcal{A}_B = 0$, then system (2.7) is degenerated in the general case. Let us specify the sufficient conditions on which the systems (2.7) is recurrent sequence of regular linear difference- differential equations (solved according to operator L_1).

Introduce the following condition:

3 The operators $A_2, \ldots, A_q$ commute quasitriangularly (see Def.2), or $k = n$.

Lemma 1. *Suppose conditions 1, 2, 3 are satisfied. Then system (2.7) is a recurrent sequence of linear difference- differential equations of order* q_1 *with the operators of the form*

$$\tilde{\mathcal{L}_{ks}} = L_1 + \sum_{i=2}^{q} a^{ik}_{p_k - s+1,s} L_i.$$

In particular, if condition 1 is satisfied and $A_2 = \ldots = A_q = 0$, *system (2.7) takes the form*

$$L_1 C_{ip_i}(x) =< f(x), \psi_i^{(1)} >,$$

$$L_1 C_{ip_i - s}(x) =< f(x), \psi_i^{(s+1)} > -L_0 C_{ip_i - s+1}(x), s = \overline{1, p_i - 1}, i = \overline{1, n}.$$

Proof follows from lemma 2 [6] or lemma [10].

So if the conditions of theorem 1 and lemma 1 are satisfied then equation (0.1) can be reduced to regular equation (2.5) and to sequence of regular linear difference-differential equations (2.7) with order q_1. Thus under condition 3 system (2.7) be pseudosingular system.

Remark 1. *Based on lemma 1 the vector components* c_i *in the case* $p_i \geq 2$ *are determined from the recurrent sequence of linear equations of order* q_1. *If* $p_{i^*} = \ldots = p_{i^*+s} = 1$ *then the corresponding elements* $c_{i^*}(x), \ldots, c_{i^*+s}(x)$ *are determined from the regular system of* $s+1$ *equations of order* q_1. *In particular if* $p_1 = \cdots = p_n = 1$, *then* $\mathcal{A}_{\mathcal{B}}$ *become zero.*

3. The Choice of Boundary Conditions

In this section conditions 1,2,3 are satisfied. In the preceding section for the definition of projections $(I - P)u, Pu$ the solution u of equation (0.1) we constructed equation (2.5) with regular operator

$$\tilde{\mathcal{L}} = L_0 + \sum_{i=1}^{q} L_i A_i \Gamma,$$

and system (2.7), splitting into recurrent sequence of linear equations with regular difference-differential operators

$$\tilde{\mathcal{L}_{ks}} = L_1 + \sum_{i=2}^{q} a^{ik}_{p_k - s+1,s} L_i.$$

The main part of operator $\tilde{\mathcal{L}}$ has the order q_0, the main part of operator $\tilde{\mathcal{L}_{ks}}$ - the order q_1. Because $q_1 < q_0$ then the boundary conditions on projections $Pu, (I-P)u$ may be different in the general case. If $q_1 = 0$ then boundary conditions may be given only on the projection $(I-P)u$.

Let $\omega = \Omega_1 \cap \Omega_2 \subset \Omega$.

We shall find the solution $u : \omega \subset R^r \to E_1$ of equation (0.1), whose projections $Pu, (I-P)u$ are subjected to the following boundary conditions:

$$\tau_1(I-P)u = \alpha(x), x \in \partial\Omega_1, \tag{3.1}$$

$$\tau_2 Pu = \beta(x), x \in \partial\Omega_2. \tag{3.2}$$

Here τ_1, τ_2 are linear difference-differential operators in the space of abstract functions, which commute with the projection operator P and with the operators B, Γ. So the functions $\alpha(x), \beta(x)$ must satisfy conditions $P\alpha(x) = 0, (I-P)\beta(x) = 0$. Let the conditions of theorem 1 be satisfied. We shall search for the solution u of equation (0.1), which satisfies boundary conditions (3.1), (3.2). Based on theorem 1 the solution has the form

$$u(x) = \Gamma v(x) + (C(x), \Phi), \tag{3.3}$$

where v satisfies (2.5) and the condition $Qv = 0$. The components of the vector C can be find from system (2.7).

Lemma 2. *If v, C satisfy the boundary conditions*

$$\tau_1 v = B\alpha(x), x \in \partial\Omega_1, \tag{3.4}$$

$$\tau_2(C, \Phi) = \beta(x), x \in \partial\Omega_2, \tag{3.5}$$

then solution (3.3) satisfies boundary conditions (3.1), (3.2).

Proof. Since $Pu = (C, \Phi)$, then condition (3.2) holds if and only if condition (3.5) holds. Since $Qv = 0, P\Gamma = \Gamma Q$, then $(I-P)u = \Gamma v$. By applying the operator B to the both sides of this equality, we obtain $v = B(I-P)u$. From here $\tau_1 v = \tau_1 B(I-P)u = B\alpha(x)$, and so $B[\tau_1(I-P)u - \alpha(x)] = 0$, i.e., $\tau_1(I-P)u - \alpha(x) \in \sum c_i\phi_i$. So if v satisfies condition (3.4) then v satisfies (3.1). Besides, it is not difficult to understand if v satisfies (3.1) then v satisfies (3.4).

Lemma 3. *If $Ker\tilde{\mathcal{L}} = \{0\}$ then any solution v of equation (2.5) with condition (3.4) lies in the subspace $E_{2\infty-k}$.*

Proof. Since $QB = BP, P\alpha(x) = 0, QA_i\Gamma = A_i\Gamma Q$, then the projection Qv satisfies homogeneous problem $\tilde{\mathcal{L}}Qv = 0$. As $Ker(\tilde{\mathcal{L}}) = \{0\}$, then $Qv = 0$ and $v \in E_{2\infty-k}$.

Suppose

4. Operators $\tilde{\mathcal{L}}, \tilde{\mathcal{L}_{ks}}$ have the bounded inverses.

Then based on lemma 2, lemma 3 and theorems 1 we obtain the following result

Theorem 2. *Let conditions 1, 2, 3, 4 be satisfied. Then problem (0.1) with conditions (3.1), (3.2) has the unique solution*

$$u(x) = \Gamma v(x) + (C(x), \Phi),$$

where v satisfies problem (2.5), (3.4), vector C- equations (2.7) with condition (3.5).

4. Examples

Theorem 2 can be used for the statement and the investigation nonclassical boundary value problems for partial differential and difference equations with degeneration.

1) Suppose that the coefficients $a_i(x,y), b_i(x,y)$ of elliptic equation ($\Delta \overset{def}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$):

$$(\triangle + a_1(x,y)\frac{\partial}{\partial x} + a_2(x,y)\frac{\partial}{\partial y} + a_3(x,y))Bu + (\frac{\partial}{\partial x} + b_1(x,y)\frac{\partial}{\partial y} +$$

$$+b_2(x,y))Au = f(x,y) \tag{4.1}$$

are defined in bounded domain $\Omega \subset R^2$ and belong to the space $C_{l-2,\alpha}(\overline{\Omega})$, $l \geq 2, \alpha \in (0,1), f : \Omega \to E_2$ belongs to the space $C_{l-2,\alpha}(\overline{\Omega})$. Fredholm operator B has a complete A -Jordan set, A is a compact operator.

Let the domain $\omega \subseteq \Omega$, $\partial\omega = l_1 \cup \partial_1$, where l_1 is the part of the straight line $x = x_0$(fig.1).

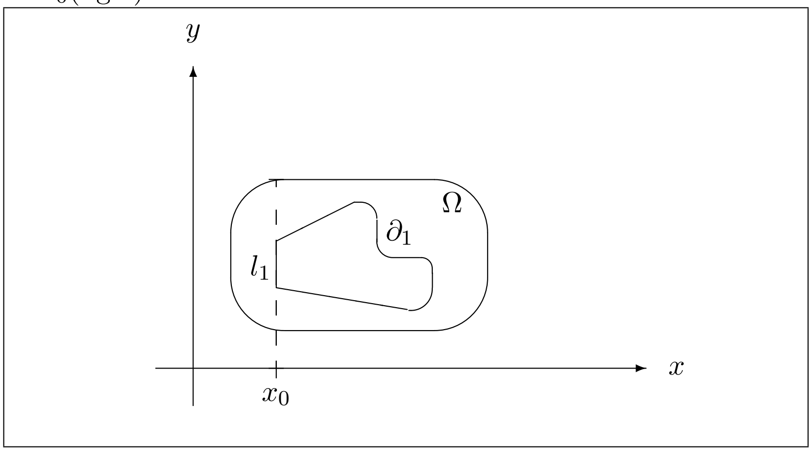

Fig. 1. Example of a domain

We shall consider the "weakened" Dirichlet problem for equation (4.1) in the domain ω, i.e. the problem of finding function u, satisfying equation (4.1) in ω, and on the boundary $\partial\omega$ the conditions

$$u|_{x=x_0} = \phi_0(y), \tag{4.2}$$

$$(I-P)u|_{\partial_1} = \phi_1(x,y), \tag{4.3}$$

where $\phi_0, \phi_1 \in C_{l,\alpha}, P\phi_1 = 0,\ (I-P)\Phi_0(y) = \Phi_1(x_0,y),\ \phi_0, \phi_1 \in D(B)$.

Note that weakening the classical statement of the Dirichlet problem is that there is no condition on projection Pu along ∂_1 of the boundary $\partial\omega$. It is connected with that Pu will be defined from the system of the first order.

Based on theorem 1 the solution has form (2.1).

If condition 4 is satisfied and $l \geq 2max_{1\leq i\leq l}p_i$ then based on theorem 2 problem (4.1)-(4.3) has the solution in the class $C_{l,\alpha}(\bar{\omega})$.

Suppose that the homogeneous problem

$$\tilde{\mathcal{L}}u \equiv (\triangle + a_1(x,y)\frac{\partial}{\partial x} + a_2(x,y)\frac{\partial}{\partial y} + a_3(x,y))u+$$

$$+(\frac{\partial}{\partial x} + b_1(x,y)\frac{\partial}{\partial y} + b_2(x,y))A\Gamma u = 0,$$

$$u|_{\partial\omega} = 0$$

has only the trivial solution. Notice that this condition will be satisfied if the diameter of the domain ω will be sufficiently small. It is not difficult to show that $\tilde{\mathcal{L}}$ is continuously invertible. In fact A is a compact operator, Γ is a bounded operator, so $A\Gamma$ is a compact operator. So the problem

$$\tilde{\mathcal{L}}v = (I-Q)f(x,y)$$

with the ordinary Dirichlet conditions

$$v|_{x=x_0} = B(I-P)\phi_0$$

$$v|_{\partial_1} = B\phi_1$$

can be reduced to the equation of the second kind with the compact operator. (In the case when $E_1 = E_2 = R^n$ it was shown in [2]). So the operator $\tilde{\mathcal{L}}$ is Fredholm operator, and also $N(\tilde{\mathcal{L}}) = \{0\}$. Therefore it is continuously invertible. Based on lemma 1 coefficients of the vector $C(x,y)$ are defined from the split recurrent sequence of linear differential equations of the first order with the boundary condition

$$(C,\Phi)|_{x=x_0} = P\phi_0(y).$$

So the conditions of theorem 2 for problem (4.1), (4.2), (4.3) are satisfied and it has unique solution in the class $C_{l,\alpha}(\bar{\omega})$.

Remark 2. *If operator B has a complete A_1- Jordan set, A_1, A_2 are compact operators, $a_{ij}\xi_i\xi_j \geq \nu \sum_1^r \xi_i^2, \nu - const > 0$,*

$$L_0(\frac{\partial}{\partial x}) = \sum_1^r a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_1^r a_i(x)\frac{\partial}{\partial x_i} + a(x),$$

$$L_1(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x_1} + \sum_2^r b_i(x)\frac{\partial}{\partial x_i},$$

$$x \in \Omega \subset R^r,$$

then similar results can be obtained for the equation

$$L_0(\frac{\partial}{\partial x})Bu + L_1(\frac{\partial}{\partial x})A_1 u + A_2 u = f(x), x \in \omega,$$

with the boundary conditions (fig.2):

$$(I-P)u|_{\partial\omega} = \phi_0(x),\ P\Phi_0 = 0,$$

$$Pu|_{x_1 = x_1^0} = \phi_1(x_2, \ldots, x_r)\ (I-P)\Phi_1 = 0.$$

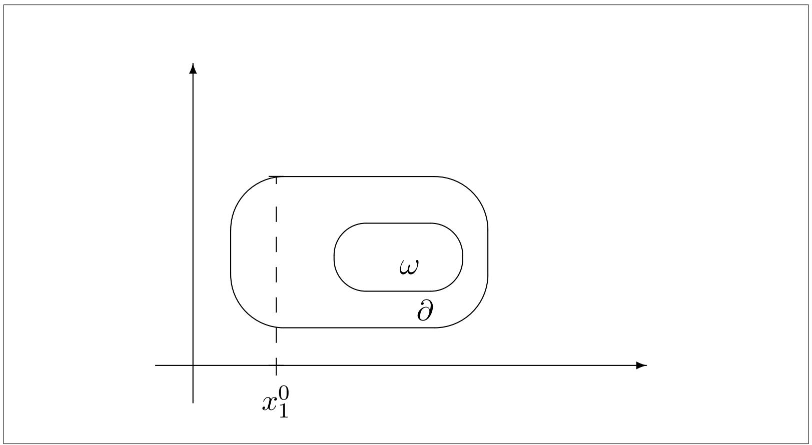

Fig. 2. The domain in the remark

2) Now we consider the initial value problem for difference equation of the first order

$$B(u(x+1) - u(x)) + Au = f(x), \tag{4.4}$$

$$(I-P)u|_{x=0} = u_0, Pu_0 = 0. \tag{4.5}$$

Here operators B, A satisfy condition 1, where $A_1 \overset{def}{=} A, x \in \Omega = \{0, 1, 2, \ldots\}$.

We look for the solution of (4.4) in the following form

$$u(x) = \Gamma v(x) + (C(x), \Phi).$$

According to results of paragraph 3 the problem (4.4)-(4.5)can be reduced to following problems:

$$v(x+1) - v(x) + A\Gamma v(x) = (I - Q)f(x), \tag{4.6}$$

$$v|_{x=0} = Bu_0 \tag{4.7}$$

$$\mathcal{A}_B(C(x+1) - C(x)) + \mathcal{A}_1' C(x) =< f(x), \Psi > . \tag{4.8}$$

and it follows from (4.5) that there is no boundary condition on $C(x)$.

By initial value (4.7) one easily arrives at the next solution

$$v(x) = (I - A\Gamma)^x Bu_0 + \sum_{\nu=0}^{x-1} (I - A\Gamma)^{x-\nu-1}(I - Q)f(\nu),$$

under $x = \{1, 2, \ldots\}$.

Based on lemma 1 coefficients of the vector $C(x)$ are defined from the split recurrent sequence

$$C_{ip_i}(x) =< f(x), \psi_i^{(1)} >,$$

$$C_{ip_i - s}(x) =< f(x), \psi_i^{(s+1)} > -(C_{ip_i - s+1}(x+1) - C_{ip_i - s+1}(x)),$$
$$s = \overline{1, p_i - 1}, i = \overline{1, n}.$$

So the conditions of theorem 2 for problem (4.4)-(4.5) are satisfied and it has unique solution.

5. Conclusion

In regular case, when the operator in the leading part of degenerate equation is continuously invertible, we can apply for investigation the well-known methods [2]. In irregular case, the problems stated for such equations in the standard way in general have no classical solutions [6]. It was shown in [6] that the solvability depends on the lowest terms. Thus, the question on influence of the lowest terms is important for statement of the boundary value problems in the theory of difference- differential equations with a noninvertible operator in the leading part.

Correct statement of boundary value problems for partial differential equations and difference equations with Fredholm operator in the split leading part and their investigations can be simplified significantly if to find a reasonable projection of the solution onto subspaces in accordance with properties of the Jordan structure of the operator coefficients of the equation [6], [9], [10].

In general, the choice of boundary conditions for equation (0.1) which supply the existence of the unique classical solution for arbitrary $f(x)$ is difficult. So, we need to extend the class of solutions, where we seek for the boundary value problem solutions of equation (0.1). For example, we can suppose that coefficients of projection Pu are the elements of the distributions space. This extended notion of the solution for equation (0.1) when $x \in R^1$ was investigated in [8].

The method proposed in this paper of reduction of equation (0.1) to regular problems can be applied to investigation in the case, when operator coefficients of equation (0.1) depend of x.

References

1. *Ahmedov K.T.* Analytical Method by Nekrasov-Nazarov in Nonlinear Analysis // Uspekhi Mat.Nauk. —1957. — v. 12, No 4. — P. 135–155.
2. *Bitsadze K.T.* Some Classes of Partial Differential Equations, "Nauka", Moscow. — 1981.
3. *Benedetto E.Di, Showalter R.E.* Implicit degenerate evolution equations and their applications // SIAM J.Math.Anal. 1981, v.12, N5 —p. 731–751.
4. *B.V.Loginov B.V., Rusak Ju.B.* Generalized Jordan Structure in the Problem of the Stability of Bifurcating Solutions // Nonlinear Analysis, Theory, Methods and Applications, vol.17, 3,1991. — p.219–232.
5. *Nashed M.Zuhair* Generalized Inverses and Applications (Proc.Adv.Sem., Madison, Wisc., 1973), Academic Press, New York, 1976.
6. *Sidorov N.A.,Blagodatskaya E.B.* Differential Equations with Fredholm Operator in the Leading Differential Expression // Soviet Math. Dokl., Vol. 44 (1992), No.1 — p.302–305.
7. *Sidorov N.A.,Romanova O.A.* Differentsial'nye Uravneniya 19 (1983) —p. 1516–1526; English transl.in Differential Equations 19 (1983).
8. *Sidorov N.A., Falaleev M.V.*// Differentsial'nye Uravneniya 23 (1987) —p. 726–728. (Russian)
9. *Sidorov N.A.,Blagodatskaya E.B.* Differential Equations with a Fredholm Operator in the Leading Differential Expression // Preprint no.1. Ac.Sci. USSR, SB ICC, 1991.
10. *Sidorov N.A.,Romanova O.A.* Differential Equations with the Operator of Finite Index in the Main Part // Differentsial'nye Uravneniya 30 (1994) —p.729–731.
11. *Sidorov N.A.* On the Branching of Solutions of the Cauchy Problem for one Class of Nonlinear Integro-differential Equations // Differentsial'nye Uravneniya 9 (1967) —p. 1592–1601.
12. *Sidorov N.A.* General Questions of Regularization in the Problems of Bifurcation Theory // Irkutsk university, 1982 — p.312.
13. *Sidorov N.A.* Solution of the Cauchy Problem for one Class of Nonlinear Integro-differential Equations with Analytical Nonlinearities // Differentsial'nye Uravneniya 7 (1968) —p. 1309–1316.
14. *Sidorov N.A.* Investigation of Continuous Solutions of the Cauchy Problem in the Neighborhood of a Bifurcation Point// Izv. Vuzov; Mathematics 9 (1976) — p.99–110.
15. *Sidorov N.A.* On the Branching of Solutions of Differential Equations with Degeneration// Differentsial'nye Uravneniya 8 (1973) — p.1464–1482.

16. *Sidorov N.A.* Differential Equations with the Volterra Operator under Derivative// Mathematics.Izvestiya Vuzov, 1 (1984).
17. *Sidorov N.A.* On some Class of Singular Differential Equations with Convergence// Mat.Zametki 35 (1984), —p.569–578; English transl. in Math. Notes 35 (1984).
18. *Sidorov N.A.* A-adjoint Sets of Linear Operators and their Applications to the Differential Equations// In: "Methods of optimization and operations research", Irkutsk, SEI, 1984 — p.169–185.
19. *Sviridyuk G.A.* On the General Theory of Semigroup of Operators// Uspekhi Mat. Nauk. 49 (1994) —p.47–74.
20. *Loginov B.V.* Branching theory of solutions of nonlinear equations at conditions of a group invariance. — Tashkent, "Fan", 1985.
21. *Trenogin V.A., Sidorov N.A., Loginov B.V.*// Differ. and Integral Equations — Ohio, USA, 1990, v.3, no.1, p.145–154.
22. *Trenogin V.A., Sidorov N.A., Loginov B.V.* Potentiality, Group Symmetry and Bifurcation in the Theory of Branching Equation // Differ. and Integral Equations, Volume 3, Number 1, 1990. —p.145–154.
23. *Vainberg M.M., Trenogin V.A.* The Theory of Branching of Solutions of Nonlinear Equations. — "Nauka", Moscow, 1969; English trans., Wolters-Noordhoff, Groningen, 1974.
24. *Zavalishin S.T., Sesekin A.N.* Impulse Processes, Models and Applications. — "Nauka", Moscow, 1992.

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Дифференциально-разностные уравнения с фредгольмовым оператором при главной части

Аннотация. Рассматривается редукция вырожденных дифференциально-разностных уравнений с фредгольмовым оператором при главной части к регулярным задачам. Показано, что выбор начальных условий связан с жордановой структурой операторных коэффицинтов уравнений. Решена задача выбора начальных условий. Доказаны теоремы существования и единственности для задач с заданными начальными условиями. Полученные результаты используются для постановки и исследования граничных задач для дифференциальных уравнений в частных производных и разностных уравнений с вырождением.