Серия «Математика»
2010. Т. 3, № 1. С. 61—69

Онлайн-доступ к журналу:
http://isu.ru/izvestia

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УДК 517.946+532.517

Nonlinear diffusion and exact solutions to the Navier-Stokes equations *

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Abstract. There are considered a number of invariant or partially invariant solutions to the Navier-Stokes equations (NSE) of rank two. These solutions are determined from one-dimensional linear or quasi-linear diffusion equations. Explicit solution, which describes smoothing of initial velocity discontinuity in a liquid with initial uniform vorticity, is constructed. This problem is reduced to a linear equation with coefficients depending on time. The global existence and non-existence theorems in the problem of a longitudinal strip deformation with free boundaries are formulated. In this case, the governing quasi-linear equation is turned out to be integro-differential one. Third example demonstrates process of axially symmetric spreading of a layer on a solid plane. The corresponding free boundary problem is reduced to the Cauchy problem for the second-order degenerate quasi-linear parabolic equation. It allows us to prove the global-in-time solvability of this problem.

Keywords: linear and nonlinear diffusion, Navier-Stokes equations, free boundary problems, invariant and partially invariant solutions.

1. Group properties of NSE.

Let us consider NSE

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \Delta \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0, \tag{1}$$

describing the motion of a viscous incompressible liquid in the absence of external forces. Here t is time, ∇ and Δ are gradient and Laplacian in variables $\mathbf{x} = (x_1, x_2, x_3)$, respectively; $\mathbf{v} = (v_1, v_2, v_3)$ is velocity vector, p is pressure. Without loss of generality, the viscosity coefficient and the liquid density are taken to be equal to zero.

From the physical point of view, the first equation (1) describes the diffusive-convective process of momentum transport. The second equation

* Author is grateful to Professor S.I. Shmarev for the fruitful discussion.

(the incompressibility condition) assigns a kinematic constraint, while the pressure gradient characterizes the constraint reaction. The purpose of this communication is to determine situations, in which the above mentioned process is described in terms of solutions of a diffusive-type scalar equation. This is achieved with the help of methods of group analysis of differential equations [1, 2].

System (1) admits the infinite-dimensional Lie group G_∞. The basis of corresponding Lie algebra is formed by operators

$$X_0 = \partial/\partial t, \quad \Psi_i = \psi(t)\partial/\partial x_i + \dot{\psi}(t)\partial/\partial v_i - x_i\ddot{\psi}(t)\partial/\partial p \quad (i = 1, 2, 3),$$

$$\Phi = \varphi(t)\partial/\partial p,$$

$$X_{ij} = x_j\partial/\partial x_i - x_i\partial/\partial x_j + v_j\partial/\partial v_i - v_i\partial/\partial v_j \quad (i = 1, 2, 3;\ j = 1, 2;\ j < i),$$

$$Z = 2t\partial/\partial t + \sum_{i=1}^{3}(x_i\partial/\partial x_i - v_i\partial/\partial v_i) - 2p\partial/\partial p.$$

Here φ, ψ_i are arbitrary (of C^∞ class) functions of time; dot denotes differentiation with respect to t. Group G_∞ contains the 10-parameter Galilei group generated by operators X_0, $X_i = \partial/\partial x_i$, $Y_i = t\partial/\partial x_i + \partial/\partial v_i$, $X_{ij} = x_j\partial/\partial x_i - x_i\partial/\partial x_j + v_j\partial/\partial v_i - v_i\partial/\partial v_j$ where $i = 1, 2, 3$; $j = 1, 2;\ j < i$.

2. Free boundary problems for NSE.

Let us suppose that the flow domain Ω_t is bounded (partially or entirely) by a free surface Γ_t, which is unknown a priori. It means that the following conditions are fulfilled:

$$-p\mathbf{n} + \left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right] = 2\sigma K\mathbf{n}, \quad \mathbf{v}\cdot\mathbf{n} = V_n, \quad \mathbf{x} \in \Gamma_t, \quad 0 \le t \le T. \qquad (2)$$

Here $\mathbf{n}$ is a unit vector of external normal to Γ_t, V_n is displacement velocity of the surface Γ_t in $\mathbf{n}$ direction, σ = const ≥ 0 is the surface tension coefficient, K is the mean curvature of Γ_t.

Theorem 1. *If the free surface Γ_t is invariant under a subgroup H of G_{10} then conditions (2) on this surface will be also invariant under H.*

This theorem allows us to obtain invariant and partially invariant solutions of NSE, which are compatible in advance with conditions on the free boundary. A number of such solutions are given in [2]. One more example is presented in Section 5.

3. Linear diffusion in viscous flows.

A simple example of exact solution to NSE is the so called parallel flow, $v_1 = u(y,t),\ v_2 = v_3 = 0,\ p = 0,\ y = x_2$. Function u satisfies the linear diffusion equation $u_t = u_{yy}$. We remark that this solution is invariant under the group $G\langle X_1, X_3\rangle \subset G_{10}$. Let us consider another two parameter subgroup of $G_{10}, H = H\langle X_1 + \omega Y_2, X_3\rangle$ where ω is constant. There is the following solution among the invariant solutions of system (1) with respect to group H:

$$v_1 = q(\zeta,t),\ v_2 = \omega x_1 + \omega t q(\zeta,t),\ v_3 = 0,\ p = h(\zeta,t),\ \zeta = x_2 - \omega t x_1.$$

Function q satisfies the equation

$$\frac{\partial}{\partial t}\left[\left(1+\omega^2t^2\right)q\right] = \left(1+\omega^2t^2\right)\frac{\partial^2}{\partial\zeta^2}\left[\left(1+\omega^2t^2\right)q\right]. \qquad (3)$$

Equation (3) is also a linear diffusion equation but its diffusivity coefficient strongly depends on time.

Equation (3) has the particular solution

$$q = V\left(1+\omega^2t^2\right)^{-1}\mathrm{erf}(\zeta\tau^{-1/2}),\ \tau = t\left(1+\omega^2t^2/3\right)$$

where $V = \mathrm{const}$. The corresponding velocity field is given by formulas

$$v_1 = V\left(1+\omega^2t^2\right)^{-1}\mathrm{erf}(\zeta\tau^{-1/2}),\ v_2 = \omega x_2 + V\omega t\left(1+\omega^2t^2\right)^{-1}\mathrm{erf}(\zeta\tau^{-1/2}).$$

In the limit $t \to +0$, we have $v_1 = V$ as $x_2 > 0, v_1 = -V$ as $x_2 < 0, v_2 = \omega x_1$. Thus, we obtain solution, which describes smoothing of the initial velocity discontinuity on the plane $x_2 = 0$, while the initial vorticity is constant outside of this plane. If $\omega = 0$, we arrive to the well-known solution describing the vortex sheet diffusion in a parallel flow.

4. Longitudinal deformation of a strip with free boundaries.

In 2D case, the analog of Th. 1 takes place with replacement of group G_{10} by group G_6 formed by operators X_0, X_k, Y_k, X_{12} $(k = 1,2)$. Let us take the subgroup $G\langle X_2, Y_2\rangle \subset G_6$. There is no invariant solution of system (1) with respect to this subgroup. However, we can look for its partially invariant solution in the form

$$u = -\int_0^x f(z,t)dz,\ v = yf(x,t),\ p = p(x,t) \qquad (4)$$

where $x = x_1,\ y = x_2,\ u = v_1,\ v = v_2$.

We note that any curve $x = s(t)$ is an invariant manifold of group G. Therefore, we can utilize the form (4) for constructing NSE solutions, which are conjugated with free boundary conditions on the surface $x = s(t),\ y \in \Re$. Substitution of (4) into the system (1) leads to equations

$$p_x = \int_0^x f_t dz - f \int_0^x f_x dz - f_x, \tag{5}$$

$$f_t + f^2 - f_x \int_0^x f(z,t)dz = f_{xx}. \tag{6}$$

The problem is to find the positive function $s(t)$ and the solution $f(x,t)$ of the equation (6) in the domain $S_T = \{x, t : 0 < x < s(t), 0 < t < T\}$ satisfying initial and boundary conditions

$$s(0) = 1,\ \ f(x,0) = f_0(x),\ \ 0 \le x \le 1, \tag{7}$$

$$f_x(0,t) = 0,\ \ f_x(s(t),t) = 0,\ \ \frac{ds}{dt} = -\int_0^{s(t)} f(x,t)dx,\ \ 0 < t < T. \tag{8}$$

The solution of problem (5)–(8) describes the symmetric deformation of a viscous strip, both boundaries $|x| = s(t)$ of which are free. First condition (8) together with (4) provides the motion symmetry. Other conditions (8) guarantee the fulfillment of two out of three scalar conditions (2). To satisfy the third condition (2), we use the arbitrariness in the determination of function p from equation (5). We note also that the assumption $s(0) = 1$ does not restrict generality because of the scaling transform admitted by relations (6), (8). Below $\bar{f}_0$ denotes the mean value of function $f_0(x)$ over interval [0, 1], prime denotes differentiation of f_0 with respect to x.

Proposition 1. *[3] Let us assume that* $f_0 \in C^{2+\alpha}[0,1],\ 0 < \alpha < 1$; $f_0'(0) = f'(1) = 0$. *If* $f_0(x) \ge \delta > 0$ *for* $x \in [0,1]$ *or* $\bar{f}_0 > 0$ *and*

$$\int_0^1 \left[f_0(x) - \bar{f}_0\right]^2 dx < min\left(\frac{4}{9\pi^2}, \frac{\bar{f}_0^2}{2}\right),$$

then problem (6) – (8) has a unique solution $f \in C^{2+\alpha,1+\alpha/2}(\bar{S_T}), s \in$ $\in C^{2+\alpha/2}[0,T]$ *for an arbitrary* $T > 0$. *If* $\bar{f}_0 < 0$, *then the lifespan of problem (6)–(8) solution is less than or equal to* $-1/\bar{f}_0$.

The structure of blowing up solution to problem (6)–(8) is investigated in [4]. If the initial function $f_0(x)$ monotonously decreasing and satisfies certain additional conditions (including a "steepness condition"), then function f has asymptotics [4]

$$f(x,t) \sim -(t^* - t)^{-1} \cos^2 \left[x\sqrt{\gamma(t^* - t)} \right] \quad \text{as } t \sim t^*$$

for $0 \leq x \leq s(t)$ and $s(t) \sim \pi/2\sqrt{\gamma(t_* - t)}$ where $\gamma = \gamma(f_0) = \text{const} > 0$. In a "favourable"case $f_0(x) \geq \delta > 0$, a universal asymptotics is valid [3]

$$f(x,t) = t^{-1} \left[1 + O\left(t^{-1}\right) \right] \quad \text{as} \quad t \to \infty$$

uniformly in $x \in [0, s(t)]$, moreover, $s(t) = Ct^{-1}[1 + O(t^{-1})]$ with some positive constant $C = C(f_0)$. Both possibilities of an evolution of the solution to general problem (6)-(8) are well demonstrated on the example of exact solutions to the equation (6)

$$f(x,t) = l(t) + m(t) \cos[\pi n x / s(t)]$$

where n is natural and l, m, s satisfy a certain dynamical system [3].

Effective investigation of the free boundary problem (6)-(8) is reached by transition to the Lagrangian coordinate ξ instead of x, which is introduced by relations $x_t(\xi,t) = -\int_0^x f(z,t)dz$ for $t > 0$, $x(\xi, 0) = \xi$. Function $f[x(\xi,t),t] = \lambda(\xi,t)$ is a solution of the second initial boundary value problem for the equation

$$\frac{\partial \lambda}{\partial t} = \exp\left[\int_0^t \lambda(\xi,\tau)d\tau\right] \frac{\partial}{\partial \xi} \left\{ \exp\left[\int_0^t \lambda(\xi,\tau)d\tau\right] \frac{\partial \lambda}{\partial \xi} \right\} - \lambda^2 \tag{9}$$

in a fixed domain $\Pi_T = \{\xi, t : 0 < \xi < 1,\ 0 < t < T\}$. Equation (9) is similar to a nonlinear diffusion equation with the source term, but the diffusivity coefficient is not constant now; moreover, it depends on prehistory of the process.

5. Axially symmetric spreading of a layer on a solid plane.

In this section we consider a class of solutions of system (1), in which

$$v_r = r g(z,t), \quad v_\theta = 0, \quad v_z = -2\int_0^z g(x,t)dx, \quad p = p(z,t) \tag{10}$$

where function g satisfies the equation

$$g_t + g^2 - 2g_z \int\limits_0^z g(x,t)dx = g_{zz}. \tag{11}$$

Here v_r, v_θ, v_z are projections of the velocity vector on axes r, θ, z of a cylindrical system of coordinates, respectively.

Solution (10) belongs to a wide class of partially invariant solutions of system (1) with respect to subgroup $G_5 \subset G_{10}$ with generators $X_1,\ X_2,\ Y_1,\ Y_2,\ X_{12}$. Any plane $z = s(t)$ is an invariant manifold of G_5. This allows us to rewrite free boundary conditions (2) in terms of invariant functions g, s as follows

$$g_z(s(t),t) = 0,\ \ \frac{dg}{dt} = -2 \int\limits_0^{s(t)} g(x,t)dx,\ \ 0 < t < T. \tag{12}$$

Besides, we demand that

$$g(0,t) = 0,\ \ 0 \leq t \leq T. \tag{13}$$

Posing the initial condition

$$g(z,0) = g_0(z),\ 0 \leq x \leq 1 \tag{14}$$

we complete formulation of the free boundary problem: to find the positive function $s(t)$ and the solution of equation (11) in domain $S_T = \{z,t : 0 < z < s(t), 0 < t < T\}$ satisfying conditions (12)–(14). In view of (10), (13), the plane $z = 0$ can be taken as a solid plane since the no slip condition is fulfilled on this plane.

Further we suppose that function g_0 satisfies the smoothness and compatibility conditions,

$$g_0(z) \in C^{3+\alpha}[0,1],\ \ g_0(0) = g_0''(0) = 0,\ \ g_0'(1) = g_0'''(1) = 0. \tag{15}$$

The global solvability of problem (11)–(14) takes place if g_0 is monotonously increasing,

$$g_0'(z) > 0 \ \text{ as } \ 0 \leq z < 1. \tag{16}$$

Under conditions (15), (16), functions g and g_z are strictly positive according to the strong maximum principle [5]. This gives a possibility to reformulate problem (11)–(14) introducing the new space variable β and unknown function χ so that

$$\beta = g(z,t),\ \ \chi(\beta,t) = g_z^2. \tag{17}$$

In consequence of (11), (17), function χ is the solution of the following equation:

$$\chi_t - \chi\chi_{\beta\beta} + \frac{1}{2}\chi_\beta^2 - \beta^2\chi_\beta = 0, \tag{18}$$

which is similar to nonlinear diffusion equation.

The reduction of integro-differential equation (11) to a second-order differential equation (18) has a group-theoretic origin. In fact, equation (11) is equivalent to the system

$$g_t + wg_z + g^2 = g_{zz}, \;\; w_z = -2g,$$

which admits the infinite Lie group with operator $\Psi = \psi(t)\partial/\partial z + \dot{\psi}(t)\partial/\partial w$ (ψ is an arbitrary smooth function of t). It turns out that transform (17) realizes the so called group stratification [1] of this system on the basis of the above mentioned infinite group.

Let us denote $\beta_0 = g_0(1)$ and define $\chi_0(\beta)$ for $\beta \in [0, \beta_0]$ by relations $\beta = g_0(z),\; \chi_0(\beta) = [g_0'(z)]^2$; then let $\chi_0 = 0$ for $\beta \geq \beta_0$. Joining to (18) conditions

$$\chi(\beta, 0) = \chi_0(\beta) \;\text{ as }\; \beta \geq 0, \;\; \chi_\beta(0, t) = 0 \;\text{ as }\; 0 \leq t \leq T \tag{19}$$

(second of them is the corollary of (11), (13)), we arrive to the initial boundary value problem for a degenerate parabolic equation in a semistrip $\sum_T^+ = \{\beta, t : \beta > 0,\; 0 < t < T\}$. According to (12), (17), the free boundary in plane z, t corresponds to the line of degeneration of equation (18) (or the interface) in plane β, t. Dual setting (18), (19) of free boundary problem (11)–(14) will be used for obtaining an a priori estimate of its solution.

Proposition 2. *Under conditions (15), (16), problem (11)-(14) has a unique solution* $g \in C^{3+\alpha, 3/2+\alpha/2}(S_T),\; s \in C^{5/2+\alpha/2}[0, T]$ *for any* $T > 0$.

Proof. To prove the solvability of problem (11)-(14) for small $T > 0$, we pass to Lagrangian coordinates and apply methods developed in [5] to initial boundary value problem in a fixed domain. A local behavior of $\chi(\beta, t)$ near the interface $\beta = \eta(t)$ is governed by the corresponding properties of the function $g(z, t)$. If g is the solution of problem (11)-(14) and $|g(z, t)| \leq M < \infty$ in $\overline{S_T}$ then , $g_{zt},\; g_{zzz}$ are Holder-continuous in $\overline{S_T}$ and $g_{zzz}(s(t), t) = 0$. Exploiting these properties of function g and relations (17), we derive that functions $\chi,\; \chi_\beta$ are continuous in the image $\overline{Q_T}$ of the domain $\overline{S_T}$ under the map (17); besides, $\chi^{1/2}\chi_{\beta\beta} \to 0$ as $\beta \to \eta(t) - 0,\; t \in [0, T]$. That makes possible to get the identity

$$\frac{d}{dt}\int\limits_0^{\eta(t)} \chi d\beta + \int\limits_0^{\eta(t)} \left(\frac{3}{2}\chi_\beta^2 + 2\beta\chi\right) d\beta = 0$$

for solution of (18), (19). In turn, this identity together with inequalities $g \geq 0$, $g_z \geq 0$ in $\overline{S_T}$, $s \leq 1$ if $t \in [0,T]$ leads to an estimate $g(z,t) \leq \|g_0'\|_{L^3(0,1)}$ in $\overline{S_T}$ for an arbitrary $T > 0$. This allows us to prove the statement of Proposition 2 for any $T > 0$.

Let us assume now that function g_0 satisfies conditions (15) and $g_0'(z) < 0$ for $0 \leq x < 1$. In this case, passing to new variables $\beta = g(z,t)$, $g_z = -[\chi(\beta,t)]^{1/2}$ results to the problem (19) for equation (18) in the domain $\sum_T^- = \{\beta, t : \beta < 0,\ 0 < t < T\}$ where the first condition is replaced by $\chi(\beta,0) = \chi_0(\beta)$ as $\beta \leq 0$. Arising problem has a unique solution at least for small $T > 0$. Its important property is presence of the full set of self-similar solutions, $\chi = (t+c)^{-3}\omega(\mu)$, $\mu = \beta(t+c)$ where $c = \text{const} > 0$ and $\omega(\mu)$ is compactly supported, $\omega(0) = k > 0$. (We note that problem (18), (19) has no nontrivial self-similar solutions). In view of results of [6], we can suppose that the new problem for equation (18) has also a global in time solution.

□

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В. В. Пухначев

Нелинейная диффузия и точные решения уравнений Навье-Стокса

Аннотация. Рассматриваются примеры инвариантных или частично инвариантных решений уравнений Навье-Стокса ранга два. Эти решения определяются из одномерных линейных или квазилинейных уравнений диффузии. Построено точное решение, описывающее сглаживание начального разрыва поля скоростей в жидкости, которая в начальный момент имеет равномерную завихренность. Эта задача сводится к линейному уравнению диффузии с коэффициентами, зависящими от времени. Сформулированы теоремы существования и несуществования в целом по времени решения задачи о продольной деформации полосы со свободными границами. В этом случае основное квазилинейное уравнение диффузии оказывается интегро-дифференциальным. Третье решение описывает осесимметричный процесс растекания жидкого слоя на твердой плоскости. Здесь соответствующая задача со свободной границей редуцируется к задаче Коши для квазилинейного вырождающегося параболического уравнения второго порядка. Это позволяет доказать ее глобальную разрешимость.

Ключевые слова: линейная и нелинейная диффузия, уравнения Навье-Стокса, задачи со свободной границей, инвариантные и частично инвариантные решения.

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