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On certain classes of fractional p-valent analytic functions *

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Abstract. The theory of analytic functions and more specific p-valent functions, is one of the most fascinating topics in one complex variable. There are many remarkable theorems dealing with extremal problems for a class of p-valent functions on the unit disk $\mathbb{U}$. Recently, many researchers have shown great interests in the study of differential operator. The objective of this paper is to define a new generalized derivative operator of p-valent analytic functions of fractional power in the open unit disk $\mathbb{U}$ denoted by $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)$. This operator generalized some well-known operators studied earlier, we mention some of them in the present paper. Motivated by the generalized derivative operator $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)$, we introduce and investigate two new subclasses $S^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$ and $TS^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$, which are subclasses of starlike p-valent analytic functions of fractional power with positive coefficients and starlike p-valent analytic functions of fractional power with negative coefficients, respectively. In addition, a sufficient condition for functions $f \in \Sigma_{p,\alpha}$ to be in the class $S^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$ and a necessary and sufficient condition for functions $f \in T_{p,\alpha}$ will be obtained. Some corollaries are also pointed out. Moreover, we determine the extreme points of functions belong to the class $TS^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$.

Keywords: analytic functions, p-valent functions, starlike functions, derivative operator.

1. Introduction

Let $\Sigma_{p,\alpha}$ denote the class of functions of the form

$$f(z) = z^{p+\alpha} + \sum_{n=p+1}^{\infty} a_n z^{n+\alpha},\ (z \in \mathbb{U}), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$, where $p \in \mathbb{N}$, $\alpha \geq 0$, $p > \alpha$, $z \in \mathbb{U}$.

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For the Hadamard product or convolution of two power series f defined in 1.1 and a function g where

$$g(z)=z^{p+\alpha}+\sum_{n=p+1}^{\infty}b_nz^{n+\alpha},$$

is

$$f(z)*g(z)=z^{p+\alpha}+\sum_{n=p+1}^{\infty}a_nb_nz^{n+\alpha},\ \ (z\in\mathbb{U}).$$

We also denote by $T_{p,\alpha}$ the subclass of $\Sigma_{p,\alpha}$ consisting of functions of the form

$$f(z)=z^{p+\alpha}-\sum_{n=p+1}^{\infty}|a_n|z^{n+\alpha}, \tag{1.2}$$

which are analytic and univalent in the open unit disk $\mathbb{U}$. For the Hadamard product or convolution of two power series f defined in (1.2) and a function g where

$$g(z)=z^{p+\alpha}-\sum_{n=p+1}^{\infty}|b_n|z^{n+\alpha},\ \ (z\in\mathbb{U})$$

is

$$f(z)*g(z)=z^{p+\alpha}-\sum_{n=p+1}^{\infty}|a_n||b_n|z^{n+\alpha},\ \ (z\in\mathbb{U}).$$

For a function $f\in\Sigma_{p,\alpha}$ given by 1.1, we define the derivative operator $D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}$ by

$$D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)=z^{p+\alpha}+\sum_{n=p+1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m a_nz^{n+\alpha},\ (z\in\mathbb{U}), \tag{1.3}$$

where $m,b\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\lambda_2\geq\lambda_1\geq 0$, $p\in\mathbb{N}$, $\alpha\geq 0, p>\alpha$.

Remark 1. It should be remarked that the differential operator $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)$ is a generalization of many operators considered earlier. Let us see some of the examples: for $\lambda_1=1,\lambda_2=b=0,p=1$ and $\alpha=0$, we get the operator introduced by Sălăgean [11]. For $\lambda_2=b=0,p=1$, and $\alpha=0$, we get the generalized Sălăgean derivative operator introduced by Al-Oboudi [1]. For $\lambda_1=1,\lambda_2=0,p=1$ and $\alpha=0$, we obtain the operator introduced by Flett [7]. For $\lambda_1=1,\lambda_2=0,b=1,p=1$ and $\alpha=0$, we obtain the operator introduced by Uralegaddi and Somanatha [13]. For $\lambda_2=0$ and $\alpha=0$, we get the operator introduced by Cátás [2]. For $\lambda_2=0,\lambda_1=1$ and $\alpha=0$, we get the operator introduced by Kumar et al. [9].

Clearly, by applying the operator $D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}$ successively, we can obtain the following:

$$D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z) = \begin{cases} D^{m-1,b}_{\lambda_1,\lambda_2,p,\alpha}(D^{b}_{\lambda_1,\lambda_2,p,\alpha})f(z), m \in \mathbb{N}, \\ f(z), m = 0. \end{cases}$$

$$f \in \Sigma_{p,\alpha} \Rightarrow D^{m,b}_{\lambda_1,\lambda_2,p,\alpha} \in \Sigma_{p,\alpha}.$$

A function $f \in \Sigma_{p,\alpha}$ is said to be in the class $P_\alpha(p,\mu), (0 < \mu < p+\alpha)$ if and only if it satisfies the inequality

$$\Re\{\frac{f'(z)}{z^{p+\alpha-1}}\} > \mu, \quad (z \in \mathbb{U}). \tag{1.4}$$

The classes $P_0(1,0)$ and $P_0(p,0)$ were investigated in [10] and [12], respectively.

Now we define the subclass $S^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$ of $\Sigma_{p,\alpha}$ consisting of functions of the form 1.1 and satisfying the analytic criterion

$$\Re\Big\{\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)} - \mu\Big\} > \nu\Big|\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)} - (p+\alpha)\Big|, \quad (z \in \mathbb{U}), \tag{1.5}$$

where $m, b \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $p \in \mathbb{N}$, $\alpha \geq 0, p > \alpha$, $0 \leq \mu < p+\alpha$, $\nu \geq 0$.

2. Main results

We obtain a necessary and sufficient condition for functions $f(z) \in \Sigma_{p,\alpha}$.

Theorem 1. *Let $f \in \Sigma_{p,\alpha}$. A sufficient condition for a function of the form 1.1 to be in $S^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$ is that*

$$\sum_{n=p+1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m\big[\frac{(n-p)(1+\nu)-(p+\alpha-\mu)}{p+\alpha-\mu}\big]|a_n| \leq 1, \tag{2.1}$$

where $z \in \mathbb{U}$, $m, b \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $p \in \mathbb{N}$, $\alpha \geq 0, p > \alpha$, $0 \leq \mu < p+\alpha$, $\nu \geq 0$.

Proof. Let f be of the form 1.1. Our aim is to show that

$$\Re\Big\{\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-\mu\Big\}>\nu\Big|\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big|$$

$$\Rightarrow \nu\Big|\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big|-\Re\Big\{\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-\mu\Big\}<0$$

$$\Rightarrow \nu\Big|\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big|-\Re\Big\{\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-\mu\Big\}+p+\alpha-\mu$$
$$<p+\alpha-\mu$$

$$\Rightarrow \nu\Big|\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big|-\Re\Big\{\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big\}$$
$$<p+\alpha-\mu.$$

Hence it suffices to prove that

$$\nu\Big|\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big|-\Re\Big\{\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big\} \le p+\alpha-\mu,\quad (z\in\mathbb{U}). \tag{2.2}$$

Yields

$$\nu\Big|\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big|-\Re\Big\{\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big\}\le$$
$$\le \nu\Big|\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big|+\Big|\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big|=$$
$$=(1+\nu)\Big|\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'-(p+\alpha)D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}\Big|\le$$
$$\le \frac{(1+\nu)\sum_{n=p+1}^{\infty}(n-p)\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m|a_n||z|^{n+\alpha}}{|z|^{p+\alpha}+\sum_{n=p+1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m|a_n||z|^{n+\alpha}}\le$$
$$\le \frac{(1+\nu)\sum_{n=p+1}^{\infty}(n-p)\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m|a_n|}{1+\sum_{n=p+1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m|a_n|},$$

where $z \to 1$ along the real axis. This last expression is bounded by $p+\alpha-\mu$ if

$$\sum_{n=p+1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m\big[(n-p)(1+\nu)-(p+\alpha-\mu)\big]|a_n| \leq$$

$$\leq p+\alpha-\mu, \ \ (z \in \mathbb{U}).$$

The proof of theorem 1 is complete. □

Next result describes the starlikeness for functions in the class $(S_{p,\alpha}(\mu,\nu))$, which is an extension to the class of starlike functions defined by [3].

Corollary 1. *Let the assumptions of theorem 1 hold. Then*

$$\Re\left\{\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}\right\} > \mu.$$

Proof. By letting $\nu = 0$ in theorem 1, we obtain the desired result. □

We also introduce the class of starlike functions of fractional power of order μ $(S^*_\alpha(\mu))$.

Corollary 2. *Let the assumptions of theorem 1 hold. Then*

$$\Re\left\{\frac{z\big(f(z)\big)'}{f(z)}\right\} > \mu.$$

Proof. By setting $m = 0, \nu = 0$ in theorem 1, we have the required result. □

Finally, we have the next result which is an extension to the class of starlike function S^*.

Corollary 3. *Let the assumptions of theorem 1 hold. Then*

$$\Re\left\{\frac{z\big(f(z)\big)'}{f(z)}\right\} > 0.$$

Proof. By setting $m = 0, \mu = \nu = 0$ in theorem 1, we have the required result. □

Now we prove a sufficient condition for $f \in T_{p,\alpha}$. Consider the subclass $TS^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$ of functions in $T_{p,\alpha}$.

Theorem 2. *Let f be defined by 1.2. Then $f \in TS^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$ if and only if the condition the following condition is satisfied.*

$$\sum_{n=p+1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m\left[\frac{(p-n)\nu+(n+\alpha-\mu)}{p+\alpha-\mu}\right]|a_n|\leq 1, \tag{2.3}$$

where $z\in\mathbb{U}$, $m,b\in\mathbb{N}_0$, $\lambda_2\geq\lambda_1\geq 0$, $p\in\mathbb{N}$, $\alpha\geq 0, p>\alpha$, $0\leq\mu< p+\alpha$, $\nu\geq 0$.

Proof. The sufficiency as in theorem 1, we need only to prove the necessity. Let $f\in TS^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$ then we obtain

$$\Re\Big\{\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-\mu\Big\}>\nu\Big|\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big|$$
$$\geq\nu\Re\Big\{\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)\Big\}$$
$$\Rightarrow\Re\Big\{\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-\mu-\nu[\frac{z\big(D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)\big)'}{D^{m,b}_{\lambda_1,\lambda_2,p,\alpha}f(z)}-(p+\alpha)]\Big\}$$
$$\geq 0,\quad(\Re(z)\leq|z|).$$

Thus when $z\to 1$ along the real axis, we pose

$$\frac{p+\alpha-\sum_{n=p+1}^{\infty}(n+\alpha)\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m a_n}{1-\sum_{n=p+1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m a_n}-\mu$$
$$--\frac{\nu\sum_{n=p+1}^{\infty}(p-n)\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m a_n}{1-\sum_{n=p+1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m a_n}\geq 0\Rightarrow$$
$$p+\alpha-\mu-\sum_{n=p+1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m(n+\alpha-\mu+\nu(p-n))a_n\geq 0,$$

and obtain the desired inequality:

$$\sum_{n=p+1}^{\infty}\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m\big[(p-n)\nu+(n+\alpha-\mu)\big]a_n\leq p+\alpha-\mu,\quad(z\in\mathbb{U}).$$

Hence The proof of theorem 2 is complete. □

Corollary 4. *Let f be defined by 1.2 be in the class $TS^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$. Then we have*

$$|a_n| \le \frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \big[(p-n)\nu+(n+\alpha-\mu)\big]}, \tag{2.4}$$

where $m,b \in \mathbb{N}_0,\ \lambda_2 \ge \lambda_1 \ge 0,\ p \in \mathbb{N},\ \alpha \ge 0, p > \alpha,\ 0 \le \mu < p+\alpha,\ \nu \ge 0.$

We shall now determine the extreme points of the class $TS^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$.

Theorem 3. *Let $f(z) = z^{p+\alpha}$ and*

$$f_n(z) = z^{p+\alpha} - \frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \big[(p-n)\nu+(n+\alpha-\mu)\big]},$$

where $m,b \in \mathbb{N}_0,\ \lambda_2 \ge \lambda_1 \ge 0,\ p \in \mathbb{N},\ \alpha \ge 0, p > \alpha,\ 0 \le \mu < p+\alpha,\ \nu \ge 0.$ *Then $f \in TS^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$ if and only if it can be expressed in the form*

$$f(z) = \sum_{n=p}^{\infty} \omega_n f_n(z), \tag{2.5}$$

where $\omega_n \ge 0$ and $\sum_{n=p}^{\infty} \omega_n = 1$.

Proof. Suppose that

$$f(z) = \sum_{n=p}^{\infty} \omega_n f_n(z) = \omega_p f_p(z) + \sum_{n=p+1}^{\infty} \omega_n f_n(z) = \omega_p f_p(z) +$$

$$+ \sum_{n=p+1}^{\infty} \omega_n \left[z^{p+\alpha} - \frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \big[(p-n)\nu+(n+\alpha-\mu)\big]} \right] =$$

$$= \sum_{n=p}^{\infty} \omega_n z^{p+\alpha} - \sum_{n=p+1}^{\infty} \omega_n \left[\frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \big[(p-n)\nu+(n+\alpha-\mu)\big]} \right] =$$

$$= z^{p+\alpha} - \sum_{n=p+1}^{\infty} \omega_n \left[\frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \big[(p-n)\nu+(n+\alpha-\mu)\big]} \right].$$

Now

$$f(z) = z^{p+\alpha} - \sum_{n=p+1}^{\infty} |a_n| =$$

$$= z^{p+\alpha} - \sum_{n=p+1}^{\infty} \omega_n \left[\frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \big[(p-n)\nu + (n+\alpha-\mu)\big]} \right],$$

therefore,

$$|a_n| = \omega_n \left[\frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \big[(p-n)\nu + (n+\alpha-\mu)\big]} \right].$$

Now, we have that

$$\sum_{n=p+1}^{\infty} \omega_n = 1 - \omega_p \le 1.$$

Setting

$$\sum_{n=p+1}^{\infty} \omega_n \left[\frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \big[(p-n)\nu + (n+\alpha-\mu)\big]} \right] \left[\frac{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \big[(p-n)\nu + (n+\alpha-\mu)\big]}{p+\alpha-\mu} \right] \le 1,$$

we get

$$\sum_{n=p+1}^{\infty} \frac{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \big[(p-n)\nu + (n+\alpha-\mu)\big]}{p+\alpha-\mu} |a_n| \le 1.$$

Therefore,

$$\sum_{n=p+1}^{\infty} \left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \big[(p-n)\nu + (n+\alpha-\mu)\big] |a_n| \le p+\alpha-\mu.$$

It follows from theorem 2 that $f \in TS^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$.

Conversely, we suppose that $f \in TS^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$, it is easily seen that

$$f(z) = z^{p+\alpha} - \sum_{n=p+1}^{\infty} |a_n|$$

$$= z^{p+\alpha} - \sum_{n=p+1}^{\infty} \omega_n \left[\frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \left[(p-n)\nu + (n+\alpha-\mu)\right]} \right],$$

which suffices to show that

$$|a_n| = \omega_n \left[\frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \left[(p-n)\nu + (n+\alpha-\mu)\right]} \right].$$

Now, we have that $f \in TS^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$, then by previous corollary 4,

$$|a_n| \leq \frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \left[(p-n)\nu + (n+\alpha-\mu)\right]},$$

which is

$$\frac{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \left[(p-n)\nu + (n+\alpha-\mu)\right]}{p+\alpha-\mu} |a_n| \leq 1.$$

Since $\sum_{n=p}^{\infty} \omega_n = 1$, we see $\omega_n \leq 1$, for each $n \geq p$. We can set that

$$\omega_n = \frac{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \left[(p-n)\nu + (n+\alpha-\mu)\right]}{p+\alpha-\mu} |a_n|.$$

Thus,

$$|a_n| = \omega_n \left[\frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \left[(p-n)\nu + (n+\alpha-\mu)\right]} \right].$$

The proof of theorem 3 is complete. □

Corollary 5. *The extreme points of* $TS^{m,b}_{\lambda_1,\lambda_2,p,\alpha}(\mu,\nu)$ *are the functions given by*

$$f_n(z) = z^{p+\alpha} - \frac{p+\alpha-\mu}{\left[\frac{p+(\lambda_1+\lambda_2)(n-p)+b}{p+\lambda_2(n-p)+b}\right]^m \left[(p-n)\nu + (n+\alpha-\mu)\right]}, \quad (z \in \mathbb{U}),$$

where $n = 2, 3, ...,\ m, b \in \mathbb{N}_0,\ \lambda_2 \geq \lambda_1 \geq 0,\ p \in \mathbb{N},\ \alpha \geq 0, p > \alpha,\ 0 \leq \mu < p + \alpha,\ \nu \geq 0.$

Some other works related to subclasses of p-valent functions defined by other differential operators for different types of problems can be seen in [4; 5; 6; 8].

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О некоторых классах дробных p-валентных аналитических функций

Аннотация. Введены новые классы функций, обобщающие хорошо известные классы p-валентных функций, введенных ранее Т. Умезавой. Введены новые классы p-валентных функций, возникающих при применении формальных дифференциальных операторов и установлены достаточные условия принадлежности к ним. Также найдены определенные соотношения между этими классами.

Ключевые слова: аналитические функции; p-валентные функции; дифференциальный оператор; звездообразные функции

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