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A Computational Method for Solving N-Person Game

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Abstract. The nonzero sum n-person game has been considered. It is well known that the game can be reduced to a global optimization problem [5; 7; 14]. By extending Mills' result [5], we derive global optimality conditions for a Nash equilibrium. In order to solve the problem numerically, we apply the Curvilinear Multistart Algorithm [2; 3] developed for finding global solutions in nonconvex optimization problems. The proposed algorithm was tested on three and four person games. Also, for the test purpose, we have considered competitions of 3 companies at the bread market of Ulaanbaatar as the three person game and solved numerically.

Keywords: Nash equilibrium, nonzero sum game, mixed strategies, curvilinear multistart algorithm.

1. Introduction

Game theory plays an important role in applied mathematics, mathematical modeling, economics and decision theory. There are many works devoted to game theory [6; 8; 9; 10; 11; 12; 4]. Most of them deals with zero sum two person games or nonzero sum two person games. Also, two person non zero sum game was studied in [10; 15; 16] by reducing it to D.C programming[1]. The three person game was examined in [2] by global optimization techniques. So far, less attention has been paid to computational aspects of game theory, specially N-person game. Aim of this paper to fulfill this gap. This paper considers nonzero sum n person game. The paper is organized as follows. In Section 2, we formulate non zero sum n person game and show that it can be formulated as a global optimization problem with polynomial constraints. We formulate the problem of finding a Nash equilibrium for non zero sum n-person games as a nonlinear programming problem. A Global search algorithm has been propose in Section 3. Section 4 is devoted to computational experiments.

2. Nonzero Sum n-person Game

Consider the n-person game in mixed strategies with matrices $(A_q, \ \ q = 1, 2, \dots, n)$ for players $1, 2, \dots, n$.

$$A_q = \left(a^q_{i_1 i_2 \dots i_n}\right), q = 1, 2, \dots, n$$

$$i_1 = 1, 2, \dots, k_1, \ \dots, i_n = 1, 2, \dots, k_n,$$

Denote by D_p the set

$$D_p = \{u \in R^q \mid \sum_{i=1}^{p} u_i = 1, \ u_i \geq 0, \ i = 1, \dots, p\}, \ p = k_1, k_2, \dots, k_n$$

A mixed strategy for player 1 is a vector $x^1 = (x^1_1, x^1_2, \dots, x^1_{k_1}) \in D_{k_1}$, where x^1_i represents the probability that player 1 uses a strategy i. Similarly, the mixed strategies for q-th player is $x^q = (x^q_1, x^q_2, \dots, x^q_{k_q}) \in D_{k_q}, \ q = 1, 2, \dots, n$. Their expected payoffs are given by for 1-th person :

$$f_1(x^1, x^2, \dots, x^n) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \dots \sum_{i_n=1}^{k_n} a^1_{i_1 i_2 \dots i_n} x^1_{i_1} x^2_{i_2} \dots x^n_{i_n}.$$

and for q-th person

$$f_q(x^1, x^2, \dots, x^n) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \dots \sum_{i_n=1}^{k_n} a^q_{i_1 i_2 \dots i_n} x^1_{i_1} x^2_{i_2} \dots x^n_{i_n},$$

$$q = 1, 2, \dots, n.$$

Definition 1. *A vector of mixed strategies $\tilde{x}^q \in D_{k_q}, \ q = 1, 2, \dots, n$ is a Nash equilibrium if*

$$\begin{cases} f_1(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n) \geq f_1(x^1, \tilde{x}^2, \dots, \tilde{x}^n), \ \forall x^1 \in D_{k_1} \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ f_q(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n) \geq f_q(\tilde{x}^1, \dots, \tilde{x}^{q-1}, x^q, \tilde{x}^{q+1}, \dots, \tilde{x}^n), \ \forall x^q \in D_{k_q} \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ f_n(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n) \geq f_n(\tilde{x}^1, \tilde{x}^2, \dots, x^n), \ \forall x^n \in D_{k_n}. \end{cases}$$

It is clear that

$$f_1(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) = \max_{x^1 \in D_{k_1}} f_1(x^1, \tilde{x}^2, \ldots \quad \ldots, \tilde{x}^n),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$f_q(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) = \max_{x^q \in D_{k_q}} f_q(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^{q-1}, x^q, \tilde{x}^{q+1}, \ldots, \tilde{x}^n),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$f_n(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) = \max_{x^n \in D_{k_n}} f_n(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^{n-1}, x^n).$$

Denote by

$$\sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \cdots \sum_{i_{q-1}=1}^{k_{q-1}} \sum_{i_{q+1}=1}^{k_{q+1}} \cdots \sum_{i_n=1}^{k_n} a^q_{i_1 i_2 \ldots i_n} x^1_{i_1} x^2_{i_2} \ldots x^{q-1}_{i_{q-1}} x^{q+1}_{i_{q+1}} \ldots x^n_{i_n} \triangleq$$
$$\triangleq \varphi_{i_q}(x^1, x^2, \ldots, x^{q-1}, x^{q+1}, \ldots, x^n) = \varphi_{i_q}(x|x^q),$$
$$i_q = 1, 2, \ldots, k_q, \quad q = 1, 2, \ldots, n.$$

For further purpose, it is useful to formulate the following statement.

Theorem 1. *A vector strategy* $(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n)$ *is a Nash equilibrium if and only if*

$$f_q(\tilde{x}) \geq \varphi_{i_q}(\tilde{x}|\tilde{x}^q) \tag{2.1}$$

for

$$\tilde{x} = (\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n)$$
$$i_q = 1, 2, \ldots, k_q,$$
$$q = 1, 2, \ldots, n.$$

Proof. Necessity: Assume that $\tilde{x}$ is a Nash equilibrium. Then by the definition, we have

$$f_q(\tilde{x}) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \cdots \sum_{i_n=1}^{k_n} a^q_{i_1 i_2 \ldots i_n} \tilde{x}^1_{i_1} \ldots \tilde{x}^n_{i_n} \geq$$
$$\geq \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \cdots \sum_{i_{q-1}=1}^{k_{q-1}} \sum_{i_q=1}^{k_q} \sum_{i_{q+1}=1}^{k_{q+1}} \cdots \sum_{i_n=1}^{k_n} a^q_{i_1 i_2 \ldots i_n} \tilde{x}^1_{i_1} \ldots \tilde{x}^{q-1}_{i_{q-1}} x^q_{i_q} \tilde{x}^{q+1}_{i_{q+1}} \ldots \tilde{x}^n_{i_n}$$
$$= f_q(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^{q-1}, x^q, \tilde{x}^{q+1}, \ldots, \tilde{x}^n), \tag{2.2}$$
$$q = 1, 2, \ldots, n.$$

In the inequality 2.2, successively choose $x^q_{i_q} = 1, \quad i_q = 1, 2, \ldots, k_q$. We can easily see that $f_q(\tilde{x}) = \varphi_{i_q}(\tilde{x}|\tilde{x}^q)$, for $i_q = 1, 2, \ldots, k_q; \quad q = 1, 2, \ldots, n$.
Sufficiency: Suppose that for a vector $\tilde{x} \in D_{k_1} \times D_{k_2} \times \ldots \times D_{k_n}$,

conditions 2.1 are satisfied. We choose $x^q \in D_{k_q}$, $q = 1, 2, \dots, n$ and multiply 2.1 by $x^q_{i_q}$ respectively. We obtain

$$\sum_{i_q=1}^{k_q} x^q_{i_q} f_q(\tilde{x}) \geq \sum_{i_1=1}^{k_1} \dots \sum_{i_q=1}^{k_q} \dots \sum_{i_n=1}^{k_n} a^q_{i_1 i_2 \dots i_n} \tilde{x}^1_{i_1} \dots \tilde{x}^{q-1}_{i_{q-1}} x^q_{i_q} \tilde{x}^{q+1}_{i_{q+1}} \dots \tilde{x}^n_{i_n},$$

$$q = 1, 2, \dots, n.$$

Taking into account that $\sum_{i_q=1}^{k_q} x^q_{i_q} = 1,\ q = 1, 2, \dots, n,$ we have

$$f_q(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n) \geq f_q(\tilde{x}^1, \dots, \tilde{x}^n),\ \forall x^q \in D_{k_q},\ q = 1, 2, \dots, n$$

which shows that $\tilde{x}$ is a Nash equilibrium. The proof is complete.□

Theorem 2. *A mixed strategy $\tilde{x}$ is a Nash equilibrium for the nonzero sum n-person game if and only if there exists vector $\tilde{p} \in R^n$ such that vector $(\tilde{x}, \tilde{p})$ is a solution to the following bilinear programming problem :*

$$\max_{(x,p)} F(x,p) = \sum_{q=1}^{n} f_q(x^1, x^2, \dots, x^n) - \sum_{q=1}^{n} p_q \tag{2.3}$$

subject to :

$$\varphi_{i_q}(x|x^q) \leq p_q,\ i_q = 1, 2, \dots, k_q, \tag{2.4}$$

Proof. Necessity: Now suppose that $\tilde{x}$ is a Nash point. Choose vector $\tilde{p}$ as : $\tilde{p}_q = f_q(\tilde{x}),\ q = 1, 2, \dots, n$. We show that $(\tilde{x}, \tilde{p})$ is a solution to problem 2.3-2.4. First, we show that $(\tilde{x}, \tilde{p})$ is a feasible point for the problem. By theorem 1, the equivalent characterization of a Nash point, we have

$$\varphi_{i_q}(\tilde{x}|\tilde{x}^q) \geq f_q(\tilde{x}^1, \dots, \tilde{x}^n),\ q = 1, 2, \dots, n.$$

The rest of the constraints are satisfied because $\tilde{x}^q \in D_{kq},\ q = 1, 2, \dots, n$. It meant that $(\tilde{x}, \tilde{p})$ is a feasible point. Choose any $x^q \in D_{kq},\ q = 1, 2, \dots, n$. Multiply 2.4 by $x^q_{i_q},\ q = 1, 2, \dots, n$. respectively. If we have sum up these inequalities, we obtain

$$f_q(x) \leq p_q, \quad q = 1, 2, \dots, n.$$

Hence, we get

$$F(x,p) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \dots \sum_{i_n=1}^{k_n} \left(\sum_{q=1}^{n} a^q_{i_1 i_2 \dots i_n} \right) x^1_{i_1} x^2_{i_2} \dots x^n_{i_n} - \sum_{q=1}^{n} p_q \leq 0$$

for all $x^q \in D_q,\ q = 1, 2, \dots, n.$
But with $\tilde{p_q} = f_q(\tilde{x})$, we have $F(\tilde{x}, \tilde{p}) = 0$ Hence, the point $(\tilde{x}, \tilde{p})$ is a

solution to the problem 2.3-2.4.
Sufficiency: Now we have to show reverse, namely, that any solution of problem 2.3-2.4 must be a Nash point. Let $(\bar{x}, \bar{p})$ be any solution of problem 2.3-2.4. Let $\tilde{x}$ be a Nash point for the game, and set $\bar{p}_q = f_q(\bar{x})$.
We will show that $\bar{x}$ must be a Nash equilibrium of the game. Since $(\bar{x}, \bar{p})$ is a feasible point, we have

$$\varphi_{i_q}(\bar{x}|\bar{x}^q) \leq \bar{p}_q, \quad i_q = 1, 2, \ldots, k_q, \; q = 1, 2, \ldots, n. \tag{2.5}$$

Hence, we have

$$f_q(\bar{x}) \leq \bar{p}_q, \; q = 1, 2, \ldots, n.$$

Adding these inequalities, we obtain

$$F(\bar{x}, \bar{p}) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \cdots \sum_{i_n=1}^{k_n} \left(\sum_{q=1}^{n} a^q_{i_1 i_2 \ldots i_n} \right) \bar{x}^1 \bar{x}^2 \ldots \bar{x}^n - \sum_{q=1}^{n} \bar{p}_q \leq 0 \tag{2.6}$$

We know that at a Nash equilibrium $F(\tilde{x}, \tilde{p}) = 0$. Since $(\bar{x}, \bar{p})$ is also a solution, $F(\bar{x}, \bar{p})$ be equal to zero:

$$F(\bar{x}, \bar{p}) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \cdots \sum_{i_n=1}^{k_n} \left(\sum_{q=1}^{n} a^q_{i_1 i_2 \ldots i_n} \right) \bar{x}^1 \bar{x}^2 \ldots \bar{x}^n - \sum_{q=1}^{n} \bar{p}_q = 0 \tag{2.7}$$

Consequently,

$$f_q(\bar{x}) = \bar{p}_q, \quad q = 1, 2, \ldots, n.$$

Since a point $(\bar{x}, \bar{p})$ feasible, we can write the constraints (2.5) as follows:

$$\sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \cdots \sum_{i_n=1}^{k_n} a^q_{i_1 i_2 \ldots i_n} \bar{x}^1_{i_1} \bar{x}^2_{i_2} \ldots \bar{x}^n_{i_n} \geqslant$$
$$\geqslant \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \cdots \sum_{i_{q-1}=1}^{k_{q-1}} \sum_{i_{q+1}=1}^{k_{q+1}} \cdots \sum_{i_n=1}^{k_n} a^q_{i_1 i_2 \ldots i_n} \bar{x}^1_{i_1} \bar{x}^2_{i_2} \ldots \bar{x}^{q-1}_{i_{q-1}} \bar{x}^{q+1}_{i_{q+1}} \ldots \bar{x}^n_{i_n},$$

for

$$i_q = 1, 2, \ldots, k_q, \; q = 1, 2, \ldots, n.$$

Now taking into account the above results, by theorem 1 we conclude that $\bar{x}$ is a Nash point which a completes the proof. □

3. The Curvilinear Multistart Algorithm

In order to solve considered problems, we use curvilinear multistart algorithm. The algorithm was originally developed for solving box-constrained optimization problems, therefore, we convert our problem from the constrained to unconstrained form using penalty function techniques. For each equality constraint $g(x) = 0$, we construct a simple penalty function $\hat{g}(x) = g^2(x)$. For each inequality constraint $q(x) \leq 0$, we also construct the corresponding penalty function as follows:

$$\hat{q}(x) = \begin{cases} 0, & \text{if } q(x) \leq 0, \\ q^2(x), & \text{if } q(x) > 0. \end{cases}$$

Thus, we have the following box-constrained optimization problem:

$$\hat{f}(x) = f(x) + \frac{\gamma}{2} \sum_i \hat{g}_i(x) + \frac{\gamma}{2} \sum_j \hat{q}_j(x) \to \min_X,$$

$$X = \left\{ x \in \mathbb{R}^n | \underline{x_i} \leq x_i \leq \overline{x_i},\ i = 1, ..., n \right\}.$$

where γ is a penalty parameter, $\underline{x}$ and $\overline{x}$ - are lower and upper bounds. For original x-variables the constraint is the box $[0, 1]$; for p-variables box constraints are $[0, \overline{p_q}]$. Values of $\overline{p_q}$ are chosen from some intervals. An initial value of a penalty parameter γ is chosen not too large (something about 1000) and after finding some local minimums we increase it for searching another local minimum.

The proposed algorithm starts from some initial point $x^1 \in X$. At each $k-$th iteration the algorithm performs randomly "drop" of two auxiliary points $\tilde{x}^1$ and $\tilde{x}^2$ and generating a curve (parabola) which passes through all three points x^k, $\tilde{x}^1$ and $\tilde{x}^2$. Then we generate some random grid along this curve and try to found all convex triples inside the grid. For each founded triple we perform refining the triple minima value with using golden section method. The best triple became a start point for local optimization algorithm, the final point of which will be a start point for the next iteration of global method. Details are presented in Algorithms 1 and 2.

4. Computational Experiments

The proposed method was implemented in C language and tested on compatibility with using the GNU Compiler Collection (GCC, versions: 4.8.5, 4.9.3, 5.4.0), clang (versions: 3.5.2, 3.6.2, 3.7.1, 3.8) and Intel C Compiler (ICC, version 15.0.6) on both GNU/Linux, Microsoft Windows and Mac OS X operating systems.

The results of numerical experiments presented below were obtained on a personal computer with the following characteristics:

- Ubuntu server 16.04, x86_64
- Intel Core i5-2500K, 16 Gb RAM
- used compiler — gcc-5.4.0,
  build flags: `-O2 -DNDEBUG`

Algorithm 1 The Curvilinear Multistart Algorithm

Input: $x^1 \in X$ – initial (start) point; $K > 0$ – iterations count; $\delta > 0$; $N > 0$; $\varepsilon_\alpha > 0$ — algorithm parameters.

Output: Global minimum point x^* and $f^* = f(x^*)$

for $k \leftarrow 1, K$ **do** $f^k \leftarrow f(x^k)$

  generate stochastic point $\tilde{x}^1 \in X$

  generate stochastic point $\tilde{x}^2 \in X$

  generate stochastic α-grid:

$$-1 = \alpha_1 \leq .... \leq \alpha_i \leq -\delta \leq 0 \leq \delta \leq \alpha_{i+1} \leq ... \leq \alpha_N = 1$$

  Let $\hat{x}(\alpha) = \mathrm{Proj}_X \left(\alpha^2 \left((\tilde{x}^1 + \tilde{x}^2)/2 - x^k\right) + \alpha/2 \left(\tilde{x}^2 - \tilde{x}^1\right) + x^k\right)$ where $\mathrm{Proj}_X(z)$ - projection of point z onto set X.

  //note that $\hat{x}(-1) = \hat{x}^1$, $\hat{x}(1) = \hat{x}^2$, $\hat{x}(0) = x^k$.

  $f_*^k \leftarrow f^k$

  $\alpha_*^k \leftarrow 0$

  for $i \leftarrow 1, (N-2)$ **do**

//Convex triplet

    if $f(\hat{x}(\alpha_i)) > f(\hat{x}(\alpha_{i+1}))$ **and** $f(\hat{x}(\alpha_{i+1})) < f(\hat{x}(\alpha_{i+2}))$ **then**

//Refining the value of minima using

//Golden-Section search method with accuracy ε_α

      $\alpha_*^k \leftarrow \mathrm{GoldenSectionSearch}(f, \alpha_i, \alpha_{i+1}, \alpha_{i+2}, \varepsilon_\alpha)$

      if $f(\hat{x}(\alpha_*^k)) < f_*^k$ **then**

        $f_*^k \leftarrow f(\hat{x}(\alpha_*^k))$

        $\alpha_*^k \leftarrow \alpha_*^k$

      end if

    end if

  end for

//Start local optimization algorithm

  $x^{k+1} \leftarrow \mathrm{LOptim}(\hat{x}(\alpha_*^k))$

end for

$x^* \leftarrow x^k$

$f^* \leftarrow f(x^k)$

The proposed algorithm was applied for numerically solving number of problems with 3 and 4 players. In all cases, Nash equilibrium points were found successfully. Problems 3.1-3.3 are of type (2.3)−(**??**) have been solved numerically for dimensions $2 \times 2 \times 2$.

Algorithm 2 The Local Optimization Algorithm

Input: $x^1 \in X$ – initial (start) point; $\varepsilon_x > 0$ — accuracy parameter.
Output: Local minimum point x^* and $f^* = f(x^*)$

1: **repeat**
2: $d^k = x^k - \text{Proj}_X(x^k - \nabla f(x^k))$
//Perform local relaxation step, for example, with using standard convex interval capture technique.
3: $x^{k+1} = \underset{\alpha \geq 0}{\text{argmin}}\, f(x^k + \alpha d^k)$
4: **until** $\|x^{k+1} - x\|_2 \leq \varepsilon_x$

Problem 3.1

Let A_1 is $a^1_{111} = 2$, $a^1_{112} = 3$, $a^1_{121} = -1$, $a^1_{122} = 0$, $a^1_{211} = 1$, $a^1_{212} = -2$, $a^1_{221} = 4$, $a^1_{222} = 3$, A_2 is $a^2_{111} = 1$, $a^2_{112} = 2$, $a^2_{121} = 0$, $a^2_{122} = -1$, $a^2_{211} = -1$, $a^2_{212} = 0$, $a^2_{221} = 2$, $a^2_{222} = 1$, and A_3 is $a^3_{111} = 3$, $a^3_{112} = 2$, $a^3_{121} = 1$, $a^3_{122} = -3$, $a^3_{211} = 0$, $a^3_{212} = 2$, $a^3_{221} = -1$, $a^3_{222} = 2$.

The optimization problem is :

$$F(x^1, x^2, x^3, p_1, p_2, p_3) = 6x^1_1x^2_1x^3_1 + 7x^1_1x^2_1x^3_2 - 3x^1_1x^2_2x^3_2 + 5x^1_2x^2_1x^3_2 + \\ + 6x^1_2x^2_2x^3_2 - p_1 - p_2 - p_3 \to max$$

$$\begin{cases} 2x^2_1x^3_1 + 3x^2_1x^3_2 - x^2_2x^3_1 - p_1 & \leq 0 \\ x^2_1x^3_1 - 2x^2_1x^3_2 + 4x^2_2x^3_1 + 3x^2_2x^3_2 - p_1 & \leq 0 \\ x^1_1x^3_1 + 2x^1_1x^3_2 - x^1_2x^3_1 - p_2 & \leq 0 \\ -x^1_1x^3_2 + 2x^1_2x^3_1 + x^1_2x^3_2 - p_2 & \leq 0 \\ 3x^1_1x^2_1 + x^1_1x^2_2 - x^1_2x^2_2 - p_3 & \leq 0 \\ 2x^1_1x^2_1 - 3x^1_1x^2_2 + 2x^1_2x^2_1 + 2x^1_2y_2 - p_3 & \leq 0 \\ x^1_1 + x^1_2 & = 1 \\ x^2_1 + x^2_2 & = 1 \\ x^3_1 + x^3_2 & = 1 \\ x^1_1 \geq 0\,, x^1_2 \geq 0\,, x^2_1 \geq 0\,, x^2_2 \geq 0 & \\ x^3_1 \geq 0\,, x^3_2 \geq 0\,, p_1 \geq 0\,, p_2 \geq 0\,, p_3 \geq 0 & \end{cases}$$

4 Nash equilibrium points have been found:

points	Player	${x^{i}}^*_1$	${x^{i}}^*_2$	p_i^*	F^*
1.	1	0	1	3	0.0
	2	0	1	1	
	3	0	1	2	
2.	1	1	0	2	0.0
	2	1	0	1	
	3	1	0	3	
3.	1	0.5191	0.4809	1.2281	$2.08 \cdot 10^{-8}$
	2	0.5888	0.4112	0.5	
	3	0.5382	0.4618	0.9327	
4.	1	0.75	0.25	1.5	$3.37 \cdot 10^{-8}$
	2	0.8333	0.1667	0.5	
	3	1	0	1.9583	

Problem 3.2

Let A_1 is $a^1_{111} = 5$, $a^1_{112} = 3$, $a^1_{121} = 6$, $a^1_{122} = 7$, $a^1_{211} = 0$, $a^1_{212} = 8$, $a^1_{221} = 2$, $a^1_{222} = 1$, A_2 is $a^2_{111} = 2$, $a^2_{112} = 4$, $a^2_{121} = -1$, $a^2_{122} = 0$, $a^2_{211} = 3$, $a^2_{212} = 5$, $a^2_{221} = 4$, $a^2_{222} = 9$, and A_3 is $a^3_{111} = 2$, $a^3_{112} = 0$, $a^3_{121} = -4$, $a^3_{122} = -1$, $a^3_{211} = -2$, $a^3_{212} = 6$, $a^3_{221} = 8$, $a^3_{222} = 9$.

For this problem Nash equilibrium points are:

points	Player	${x^{i}}^*_1$	${x^{i}}^*_2$	p_i^*	F^*
1.	1	1	0	5	0.0
	2	1	0	2	
	3	1	0	2	
2.	1	0.5	0.5	4.8181	$2.6 \cdot 10^{-8}$
	2	0.5454	0.4545	4.5	
	3	0	1	3.4545	
3.	1	0.8	0.2	4.0	$9.9 \cdot 10^{-8}$
	2	1	0	3.2	
	3	0.5	0.5	1.2	

Problem 3.3 Let A_1 is $a^1_{111} = 3$, $a^1_{112} = 2$, $a^1_{121} = 1$, $a^1_{122} = 5$, $a^1_{211} = 8$, $a_{212} = 4$, $a_{221} = 1$, $a^1_{222} = 3$, A_2 is $a^2_{111} = 3$, $a^2_{112} = 2$, $a^2_{121} = 4$, $a^2_{122} = 0$, $a^2_{211} = 1$, $a^2_{212} = 8$, $a^2_{221} = 6$, $a^2_{222} = 6$, and A_3 is $a^3_{111} = 3$, $a^3_{112} = 1$, $a^3_{121} = 9$, $a^3_{122} = 2$, $a^3_{211} = 4$, $a^3_{212} = 7$, $a^3_{221} = 2$, $a^3_{222} = 3$.

Also, we found 4 Nash equilibrium points are:

points	Player	x^{i*}_1	x^{i*}_2	p_i^*	F^*
	1	1	0	1	
1.	2	0	1	4	0.0
	3	1	0	9	
	1	0	1	4	
2.	2	1	0	8	0.0
	3	0	1	7	
	1	0.5	0.5	1	
3.	2	0.0	1.0	5	0.0
	3	1.0	0.0	5.5	
	1	0.7	0.3	1	
4.	2	0	1	4.6	$-1.33 \cdot 10^{-15}$
	3	1.0	0.0	6.9	

Problem 3.4 We have considered competitions of 3 companies sharing the bread market of city Ulaanbataar where each company maximizes own profit depending on its manufacturing strategies. The problem was formulated as the three-person game with profit matrices $A = \{a_{ijk}\}$, $B = \{b_{ijk}\}$, $C = \{c_{ijk}\}$, $i = \overline{1,5}$, $j = \overline{1,6}$, $k = \overline{1,4}$. The matrix data can be downloaded from [17]. In this case the problem had 18 variables with 18 constraints. The solution of the problem found by the curvilinear multistart algorithm was:

Player	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	p^*	F^*
1	0	0	0	0	1		65	
2	0	0	0	0	0	1	160	0.0
3	1	0	0	0			53	

It means that first and second companies must follow their 5-th and 6-th production strategies while third company applies for its 1-st production strategy. Companies's maximum profits were 65, 160 and 53 respectively.

Problem 4.1 Let A_1 is $a^1_{1111} = 1, a^1_{1211} = 0, a^1_{1121} = 0, a^1_{1112} = 0, a^1_{2111} = 1, a^1_{2112} = 0, a^1_{2121} = 0, a^1_{2211} = 0, a^1_{1122} = 1, a^1_{1212} = 0, a^1_{1221} = 0, a^1_{2221} = 0, a^1_{2212} = 0, a^1_{2122} = 0, a^1_{1222} = 0, a^1_{2222} = 1$,

A_2 is $a^2_{1111} = 0, a^2_{1211} = 1, a^2_{1121} = 0, a^2_{1112} = 0, a^2_{2111} = 0, a^2_{2112} = 1, a^2_{2121} = 0, a^2_{2211} = 0, a^2_{1122} = 0, a^2_{1212} = 0, a^2_{1221} = 1, a^2_{2221} = 0, a^2_{2212} = 1, a^2_{2122} = 1, a^2_{1222} = 0, a^2_{2222} = 1$.

and A_3 is $a^3_{1111} = 0, a^3_{1211} = 1, a^3_{1121} = 0, a^3_{1112} = 0, a^3_{2111} = 0, a^3_{2112} = 1, a^3_{2121} = 1, a^3_{2211} = 1, a^3_{1122} = 0, a^3_{1212} = 1, a^3_{1221} = 0, a^3_{2221} = 1, a^3_{2212} = 0, a^3_{2122} = 1, a^3_{1222} = 1, a^3_{2222} = 0$,

A_4 is $a^4_{1111} = 0, a^4_{1211} = 0, a^4_{1121} = 0, a^4_{1112} = 1, a^4 2111 = 0, a^4 2112 = 0, a^4_{2121} = 1, a^4_{2211} = 0, a^4_{1122} = 1, a^4_{1212} = 2, a^4_{1221} = 0, a^4_{2221} = 1, a^4_{2212} =$

$1, a^4_{2122} = 0, a^4_{1222} = -1, a^4_{2222} = 0.$

Solution of this problem is also not unique and consist of several sets, such as:

1) $F^* = 0,\ x^{1^*} = (0,1)^T,\ x^{2^*} = (0,1)^T, x^{3^*} = (t, 1-t)^T, x^{4^*} = (1,0)^T,$ $p_1^* = 0,\ p_2^* = 0,\ p_3^* = 1$, and $p_4^* = 1 - t$, where $t \in [0, 0.5]$.
2) $F^* = 0,\ x^{1^*} = (0,1)^T,\ x^{2^*} = (u, 1-u)^T, x^{3^*} = (0,1)^T, x^{4^*} = (1,0)^T,$ $p_1^* = 0,\ p_2^* = 0,\ p_3^* = 1$, and $p_4^* = 1$, where $u \in [0, 1]$.
3) $F^* = 0,\ x^{1^*} = (0,1)^T,\ x^{2^*} = (0,1)^T, x^{3^*} = (v, 1-v)^T, x^{4^*} = (0,1)^T,$ $p_1^* = 1 - z_1^*,\ p_2^* = 1,\ p_3^* = 0$, and $p_4^* = v$, where $v \in [0.5, 1]$.

Solution (1) and (2) meets in point $x^{1^*} = (0,1)^T, x^{2^*} = (0,1)^T, x^{3^*} = (0,1)^T, x^{4^*} = (1,0)^T$.
Some single points of Nash equilibrium also are:

F^*	x^{1^*}	x^{2^*}	x^{3^*}	x^{4^*}	p_1^*	p_2^*	p_3^*	p_4^*
0	(1, 0)	(0, 1)	(1, 0)	(0, 1)	0	0	1	2
0	(1, 0)	(1, 0)	(1, 0)	(0, 1)	0	0	0	1

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Вычислительный метод для игр с ненулевой суммой для N-лиц

Аннотация. Рассматривается игра с ненулевой суммой для N-игроков. Хорошо известно, что игра может быть сведена к глобальной задаче оптимизации [5; 7; 14]. Обобщая результаты, полученные Миллсом [5], мы имеем условия гло-

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бальной оптимальности для равновесия по Нэшу. Для отыскания равновесий по Нэшу в построенной игре используется подход, базирующийся на ее редукции к невыпуклой задаче оптимизации; для решения последней применяется алгоритм глобального поиска, мы применяем Curvilinear Multistart Algorithm [2; 3], специально модифицированный для нашей редуцированной задачи невыпуклой оптимизации. Предложенный алгоритм протестирован на играх с тремя и четырьмя игроками. Кроме того, мы рассматривали маркетинговую задачу соревнования по ценам трех компаний на хлебном рынке Улан-Батора. Приводятся и анализируются результаты вычислительного эксперимента.

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