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On Partial Groupoids Associated with the Composition of Multilayer Feedforward Neural Networks

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Abstract. In this work, partial groupoids are constructed associated with compositions of multilayer neural networks of direct signal distribution (hereinafter simply neural networks). The elements of these groupoids are tuples of a special type. Specifying such a tuple determines the structure (i.e., architecture) of the neural network. Each such tuple can be associated with a mapping that will implement the operation of the neural network as a computational circuit. Thus, in this work, the neural network is identified primarily with its architecture, and its work is implemented by a mapping that is built using an artificial neuron model. The partial operation in the constructed groupoids is designed in such a way that the result of its application (if defined) to a pair of neural networks gives a neural network that, on each input signal, acts in accordance with the principle of composition of neural networks (i.e., the output signal of one network is sent to the input second network). It is established that the constructed partial groupoids are semigroupoids (i.e. partial groupoids with the condition of strong associativity). Some endomorphisms of the indicated groupoids are constructed, which make it possible to change the threshold values and activation functions of the neurons of the specified population. Transformations of the constructed partial groupoids are studied, which allow changing the weights of synoptic connections from a given set of synoptic connections. In the general case, these transformations are not endomorphisms. A partial groupoid was constructed for which this transformation is an endomorphism (the support of this partial groupoid is a subset in the support of the original partial groupoid).

Keywords: partial groupoid, semigroupoids, endomorphism of partial groupoid, multilayer feedforward neural network

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Научная статья

О частичных группоидах, ассоциированных с композицией многослойных нейронных сетей прямого распространения сигнала

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Аннотация. Строятся частичные группоиды, ассоциированные с композициями многослойных нейронных сетей прямого распределения сигнала (далее — нейронные сети). Элементами данных группоидов являются кортежи специального вида. Задание такого кортежа определяет структуру (т. е. архитектуру) нейронной сети. Каждому такому кортежу можно сопоставить отображение, которое будет реализовывать работу нейронной сети как вычислительной схемы. Таким образом, в данной работе нейронная сеть отождествляется в первую очередь со своей архитектурой, а ее работу реализует отображение, которое строится с помощью модели искусственного нейрона. Частичная операция в построенных группоидах устроена так, что результат ее применения (если он определен) к паре нейронных сетей дает нейронную сеть, которая на каждом входном сигнале действует в соответствии с принципом композиции нейронных сетей (т. е. выходной сигнал одной сети отправляется на вход второй сети). Установлено, что построенные частичные группоиды являются полугруппоидами (т. е. частичными группоидами с условием сильной ассоциативности). Строятся некоторые эндоморфизмы указанных группоидов, которые позволяют менять пороговые значения и функции активации нейронов указанной совокупности. Изучаются преобразования построенных частичных группоидов, которые позволяют менять веса синоптических связей из заданного множества синоптических связей. Данные преобразования в общем случае не являются эндоморфизмами. Был построен частичный группоид, для которого данное преобразование является эндоморфизмом (носитель этого частичного группоида является подмножеством в носителе исходного частичного группоида).

Ключевые слова: частичный группоид, полугруппоид, эндоморфизм частичного группоида, многослойная нейронная сеть прямого распространения сигнала

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1. Introduction

This work considers only multilayer neural networks with direct signal distribution (hereinafter, simply networks or neural networks). Information about neural networks and their structures can be found in [4; 5; 7; 9; 10]. The information technology industry's advances in the use of various neural networks can be found in [10]. We consider neural networks as mathematical objects that define the structure of the neural network. The operation of a neural network as a computational circuit is defined as a mapping defined using an artificial neuron (McCulloch–Pitts) model. For each neural network $\mathcal{N}$ with k inputs and m outputs corresponds to a mapping $F_{\mathcal{N}} : \mathbb{R}^k \to \mathbb{R}^m$, which models the operation of a neural network as a computing circuit. The definition 4, which models the neural network in this work, is constructed in a similar way to the definition of a neural network from the works [6]. The definition 4 has some differences from the definition of neural networks in [7]. The latter is explained by the context of the study.

If on each input signal $\overline{x} \in \mathbb{R}^k$ the neural network $\mathcal{N} := \mathcal{N}_1 \circ \mathcal{N}_2$ produces the signal $F_{\mathcal{N}_2}(F_{\mathcal{N}_1}(\overline{x}))$, then we will say that the network $\mathcal{N}$ is built in accordance with the *principle of composition*. This approach, from the point of view of the theory of abstract automata, can be interpreted as supplying the output signals of the first automaton to the input of the second automaton (see, for example, [2, definition 25, p. 54]).

In this work, we do not use abstract automata and related constructions (see, for example, [9]) to model neural networks. Because this approach does not provide a convenient way to consider the structure of neural networks. This circumstance is known and was noted by V.M. Glushkov in the review [2, p. 59, conclusion] for any automata represented as abstract automata.

Objectives of research. The work is aimed at creating algebraic systems that describe the composition of neural networks and studying the algebraic properties of these systems. The results of the work will be useful for developing methods for studying neural networks using algebraic objects.

There are various ways to construct algebraic systems that model the composition of neural networks. In this work, the concept of *partial groupoid* will be used to model the composition. Information about partial groupoids can be found in the works [1; 3; 8], which formed the basis for Section 2 of this work.

Main results. A partial groupoid $\mathcal{SN}(k, Q) = (\mathrm{SN}(k, Q), \circ)$ is constructed whose elements are neural networks with k inputs and k outputs, neurons are elements of the set Q . The partial operation $(\circ)$ is defined so that for any input signal $\overline{x} \in \mathbb{R}^k$ the equality holds

$$F_{\mathcal{N}_1 \circ \mathcal{N}_2}(\overline{x}) = F_{\mathcal{N}_2}(F_{\mathcal{N}_1}(\overline{x}))$$

(see the Definitions 5, 6 and the Proposition 1 in section 3 of this article). Theorem 1 shows that the partial groupoid $\mathcal{SN}(k,Q)$ is a semigroupoid, and the zero extension $\widetilde{\mathrm{SN}}(k,Q)$ of this a groupoid is a semigroup.

The main results of the work include Theorems 2 and 3. Theorem 2 gives a series of endomorphisms Γ of the partial groupoid $\mathcal{SN}(k,Q)$ that introduce changes to the structure mapping l. Theorem 3 gives a series of endomorphisms Υ of the partial groupoid $\mathcal{SN}(k,Q)$ that introduce changes to the structure mapping g. For each $S \subseteq Q \times Q$ a set of neural networks $T(k,Q,S) \subseteq \mathrm{SN}(k,Q)$ is introduced. This set is closed (in the sense of a partial operation) with respect to the operation $(\circ)$. A series of transformations Φ of the set $\mathrm{SN}(k,Q)$ are constructed, which introduce changes into the structural mapping f. Theorem 4 shows that the transformations Φ are endomorphisms of the partial groupoid $(T(k,Q,S),\circ)$.

The transformations discussed in Theorems 2, 3 and 4 can be used in modeling the learning processes of neural networks using objects of general algebra. This explains the consideration of these transformations. The following problem remains open

Problem. Give a description of the set of all endomorphisms of the partial groupoid $\mathcal{SN}(k,Q)$.

2. Partial groupoids

We denote the Cartesian square of the set X by $X^2 := X \times X$. *A partial binary operation on the set* G is the mapping $(*) : \mathrm{Dom}(*) \to G$, where $\mathrm{Dom}(*) \subseteq G^2$. We will call a tuple $\mathcal{G} = (G,*)$ *a partial groupoid* if $(*)$ is a partial binary operation on G. The set $\mathrm{Dom}(*)$ can be interpreted as the domain of definition of the partial operation $(*)$. Every groupoid $G = (G,*)$ can be interpreted as a partial groupoid $\mathcal{G} = (G,*)$ such that the condition $\mathrm{Dom}(*) = G^2$ is satisfied.

Definition 1. *Let $\mathcal{G} = (G,*)$ be a partial groupoid. Then by $\Diamond$ we denote a special element such that this element is not contained in the set G. We assume that $\widetilde{G} := G \cup \{\Diamond\}$ and $(\widetilde{*})$ is binary algebraic operation on a set $\widetilde{G}$ such that for all tuples $(x,y) \in \mathrm{Dom}(*)$ and $(u,v) \in \widetilde{G}^2 \setminus \mathrm{Dom}(*)$ the following relations are satisfied:*

$$x \widetilde{*} y = x * y, \quad u \widetilde{*} v = \Diamond.$$

We will call the system $\widetilde{G} := (\widetilde{G}, \widetilde{*})$ zero extension of the partial groupoid $\mathcal{G}$.

Usually, the element $\Diamond$ is denoted by the symbol of the empty set or zero (see work [1]), but in this work these symbols are used for their intended purpose. That's why we use the symbol $\Diamond$. The zero extension of a partial groupoid will be an ordinary groupoid. Note that the element $\Diamond$ has the multiplicative property of zero in the groupoid $\widetilde{G}$. Two different types of endomorphisms of a partial groupoid arise naturally: *endomorphisms of the zero extension of a partial groupoid* and *endomorphisms of a partial groupoid*. The last type of endomorphisms can be defined as follows. Let $I(G)$ be the symmetric semigroup of transformations of the set G.

Definition 2. *An endomorphism of a partial groupoid $\mathcal{G} = (G, *)$ is any transformation ϕ from $I(G)$ such that for all tuples $(x, y) \in \mathrm{Dom}(*)$ and $(u, v) \in G^2 \setminus \mathrm{Dom}(*)$ the conditions are satisfied:*

$$\phi(x*y) = \phi(x)*\phi(y), (\phi(x), \phi(y)) \in \mathrm{Dom}(*), (\phi(u), \phi(v)) \notin \mathrm{Dom}(*). \quad (2.1)$$

This definition is based on the definition of (strong) homomorphism of two partial groupoids given in the work [8]. This approach to defining endomorphism is quite natural and is in good agreement with the goals of the study. Endomorphisms obtained in this way will have a natural interpretation in the context of neural networks.

Definition 3. *A partial groupoid $\mathcal{G} = (G, *)$ is called a semigroupoid if for any $x, y, z \in G$ the following conditions are satisfied:*

$$(x * y, z) \in \mathrm{Dom}(*) \Rightarrow (x, y * z) \in \mathrm{Dom}(*), (x * y) * z = x * (y * z), \quad (2.2)$$

$$(x, y * z) \in \mathrm{Dom}(*) \Rightarrow (x * y, z) \in \mathrm{Dom}(*), (x * y) * z = x * (y * z). \quad (2.3)$$

The given conditions (2.2) and (2.3) are called *strong associativity* of a partial groupoid. Information about semigroupoids can be found in the review by [3] (see also the works by [1]). The implications (2.2) and (2.2) can be replaced by a single equivalence.

3. Neural networks

Further, we will use the notation $F(\mathbb{R}) := \mathrm{Hom}(\mathbb{R}, \mathbb{R})$.

Definition 4. *Let the following objects be defined:*
1) the tuple $(M_1, ..., M_n)$ of length $n \geq 1$ of finite non-empty sets, where for $i \neq j$ the condition $M_i \cap M_j = \varnothing$ is satisfied;
2) the set $S := (M_1 \times M_2) \cup (M_2 \times M_3) \cup ... \cup (M_{n-1} \times M_n)$;
3) the mapping $f : S \to \mathbb{R}$;
4) the set $A := M_1 \cup ... \cup M_n$;
5) the mapping $g : A \to F(\mathbb{R})$;

6) the mapping $l : A \to \mathbb{R}$*;*
7) the bijection $i : M_1 \to \{1, 2, ..., |M_1|\}$*;*
8) the bijection $o : M_n \to \{1, 2, ..., |M_n|\}$*.*
Then the tuple $\mathcal{N} = (M_1, ..., M_n, i, o, f, g, l)$ *will be called a multilayer feedforward neural network.*

The above definition is based on the formal definition of a neural network from the work [7] (and earlier works; coincides with a similar definition from the work [6]). There are differences. Points 7 and 8 are added (they were not in [7]), changes are made to the first point. The differences are dictated by context.

Standard characteristics of a neural network. We will associate the following notations with each neural network $\mathcal{N}=(M_1,...,M_n, i, o,f, g, l)$:

$$n(\mathcal{N}) := n, \quad A(\mathcal{N}) := \bigcup_{i=1}^{n} M_i, \quad Syn(\mathcal{N}) := \bigcup_{i=1}^{n-1} M_i \times M_{i+1}.$$

Thus, $n(\mathcal{N})$ is the number of all network layers $\mathcal{N}$, $A(\mathcal{N})$ is the set of all neurons of the network $\mathcal{N}$ and $Syn(\mathcal{N})$ is the set of all synoptic connections of network $\mathcal{N}$. We will call the mappings i, o, f, g, l *structural mappings* of the neural network $\mathcal{N}$.

Definition 5. *Let* k *be some natural number and* Q *be some set such that* $|Q| \geq k$*. Then by* $\mathrm{SN}(k, Q)$ *we denote the set of all neural networks with* k *inputs and* k *outputs whose neurons are elements of the set* Q*.*

Definition 6. *Let*

$$\mathcal{N}_1 = (M_1, ..., M_u, i_1, o_1, f_1, g_1, l_1), \ \mathcal{N}_2 = (P_1, ..., P_v, i_2, o_2, f_2, g_2, l_2)$$

these are two neural networks from $\mathrm{SN}(k, Q)$*. Let us define a partial binary operation* $(\circ)$ *on the set* $\mathrm{SN}(k, Q)$ *so that the result* $\mathcal{N}_1 \circ \mathcal{N}_2$ *of applying the operation* $(\circ)$ *to the pair* $(\mathcal{N}_1, \mathcal{N}_2)$ *is defined if and only if the condition* $A(\mathcal{N}_1) \cap A(\mathcal{N}_2) = \varnothing$ *is satisfied. Let* $(\mathcal{N}_1, \mathcal{N}_2)$ *be a pair of neural networks such that the result* $\mathcal{N}_1 \circ \mathcal{N}_2$ *is defined. Then the equality holds*

$$\mathcal{N}_1 \circ \mathcal{N}_2 := (M_1, ..., M_u, P_1, ..., P_v, i_1, o_2, f', g', l'), \tag{3.1}$$

where the structural mappings f'*,* g'*,* l' *are defined so that the following statements hold:*
1) for any $n_1 \in A(\mathcal{N}_1)$ *and* $n_2 \in A(\mathcal{N}_2)$ *the conditions are satisfied*

$$g'(n_1) := g_1(n_1), \ g'(n_2) := g_2(n_2), \ l'(n_1) := l_1(n_1), \ l'(n_2) := l_2(n_2); \tag{3.2}$$

2) for any tuples $s_1 \in Syn(\mathcal{N}_1)$*,* $s_2 \in Syn(\mathcal{N}_2)$ *and* $(k, m) \in M_u \times P_1$ *the conditions are satisfied*

$$f'(s_1) := f_1(s_1), \quad f'(s_2) := f_2(s_2), \tag{3.3}$$

$$f'((k,m)) := 1 \Leftrightarrow o_1(k) = i_2(m), \quad f'((k,m)) := 0 \Leftrightarrow o_1(k) \neq i_2(m).$$

Then the partial groupoid $\mathcal{SN}(k,Q) := (\mathrm{SN}(k,Q), \circ)$ *will be called a partial groupoid associated with the composition of multilayer neural networks.*

The relations (3.2) and (3.2) completely determine the structural mappings f', g' and l' of the network $\mathcal{N}_1 \circ \mathcal{N}_2$.

Remark 1. Let us discuss why the product ($\circ$) is introduced exclusively on pairs of neural networks $(\mathcal{N}_1, \mathcal{N}_2)$ such that the intersection of sets $A(\mathcal{N}_1)$ and $A(\mathcal{N}_2)$ are the empty set. If we assume that the intersection of the sets $A(\mathcal{N}_1)$ and $A(\mathcal{N}_2)$ is not empty, then the object defined by the relation (3.1) is not a neural network in the sense of the definition 4. The first point of this definition is violated. The first point gives the condition that the intersection of any two different layers of neurons is the empty set. The latter is not possible for an object from (3.1) when $A(\mathcal{N}_1) \cap A(\mathcal{N}_2) \neq \varnothing$. On the other hand, if $A(\mathcal{N}_1) \cap A(\mathcal{N}_2) = \varnothing$, then the relations (3.1), (3.2) and (3.3) correctly define the object $\mathcal{N}_1 \circ \mathcal{N}_2$. This object is a neural network in the sense of the definition 4.

Remark 2. It is possible to model the composition of neural networks using ordinary groupoids. In work [6] the authors of the work constructed a groupoid (ordinary groupoid) with support $X(k,Q)$ and operation ($\odot$) such that $\mathrm{SN}(k,Q) \subset X(k,Q)$ and equality $F_{\mathcal{N}_1 \odot \mathcal{N}_2}(\overline{x}) = F_{\mathcal{N}_2}(F_{\mathcal{N}_1}(\overline{x}))$ is performed for any networks $\mathcal{N}_1, \mathcal{N}_2 \in X(k,Q)$ (i.e., the operation ($\odot$) is built in accordance with the principle compositions). In this case, the equality holds

$$A(\mathcal{N}_1 \odot \mathcal{N}_2) \cap [A(\mathcal{N}_1) \cup A(\mathcal{N}_2)] = \varnothing.$$

It was proven that $(X(k,Q), \odot)$ is a free groupoid (as a result of which it is even devoid of associativity). It removes the question of describing the set of all endomorphisms, but leaves the question of describing the set of all automorphisms of this groupoid. The groupoid $(X(k,Q), \odot)$ is burdened with mathematical formalism much more than the partial groupoid $\mathcal{SN}(k,Q)$. This mathematical formalism may be unnatural for specialists specializing in the practical use of neural networks. The groupoid $(X(k,Q), \odot)$ will be useful for studying the theoretical properties of the composition of neural networks.

Operation of a neural network. The operation of the neural network $\mathcal{N} = (M_1, ..., M_d, i, o, f, g, l)$ from $\mathrm{SN}(k,Q)$ as a computational circuit will be implemented by the function $F_{\mathcal{N}} : \mathbb{R}^k \to \mathbb{R}^k$. We will describe the action of this function using the McCulloch-Pitts artificial neuron model (see, for example, [4]). Let n be a neuron of layer M_j for $j > 1$. Neuron n receives a signal (in the form of a number) from each neuron of the previous layer. Let us denote these neurons by symbols $n_1, ..., n_{|M_{j-1}|}$, and the signals (numbers) they send will be denoted by the symbols $a_1, ..., a_{|M_{j-1}|}$. Then the neuron n generates its signal G_n according to the rule

$$G_n = y_n \left(\sum_{s=1}^{|M_{j-1}|} f((n_s, n)) \cdot a_s + l(n) \right), \tag{3.4}$$

where $y_n := g(n)$ is the activation function of neuron n ($g(n)$ is the result of applying the mapping g to neuron n, which is equal to some numerical function from $F(\mathbb{R})$). Further the signal is transmitted through the appropriate synoptic connections to the next layer.

The action of the function $F_{\mathcal{N}}$ is that the signal $(x_1, ..., x_k) \in \mathbb{R}^k$ is transmitted to the input layer M_1 so that the neuron $n \in M_1$ receives signal x_s when $i(n) = s$. Neurons of the input layer transmit their signals $y_n(x_s + l(n))$ ($y_n = g(n)$, $i(n) = s$) via synoptic connections to the second layer Next, the signal propagates through the network in accordance with the artificial neuron model described above. The output layer M_d generates a vector $(u_1, ..., u_k)$, where the number u_s is generated by the neuron $n \in M_d$ such that $o(n) = s$. We assume by definition that the equality $F_{\mathcal{N}}(x_1, ..., x_k) := (u_1, ..., u_k)$ holds.

Proposition 1. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two networks from* $\mathrm{SN}(k, Q)$ *such that that $A(\mathcal{N}_1) \cap A(\mathcal{N}_2) = \varnothing$. Then for any signal $\overline{x} \in \mathbb{R}^k$ the equality $F_{\mathcal{N}_1 \circ \mathcal{N}_2}(\overline{x}) = F_{\mathcal{N}_2}(F_{\mathcal{N}_1}(\overline{x}))$ holds.*

Proof. Indeed, by virtue of the definition of the neural network $\mathcal{N}_1 \circ \mathcal{N}_2$, the signal $\overline{x}$ will pass through the network $\mathcal{N}_1 \circ \mathcal{N}_2$ to the layer with index $n(\mathcal{N}_1)$ just as it would pass through the network $\mathcal{N}_1$ (follows from the relations (3.4), (3.1), (3.2) and (3.3)). The layer with index $n(\mathcal{N}_1)$ will generate a signal $F_{\mathcal{N}_1}(\overline{x})$, which will then pass through the layers of the network $\mathcal{N}_1 \circ \mathcal{N}_2$ in the same way as it would pass through the layers of the $\mathcal{N}_2$ network. Therefore, the layer with index $n(\mathcal{N}_1) + n(\mathcal{N}_2)$ will generate a signal $F_{\mathcal{N}_2}(F_{\mathcal{N}_1}(\overline{x}))$. The statement has been proven. □

4. Formulation and proof of Theorem 1

Theorem 1. *The partial groupoid $\mathcal{SN}(k, Q)$ is a semigroupoid. Zero extension $\widetilde{\mathrm{SN}}(k, Q)$ of a partial groupoid $\mathcal{SN}(k, Q)$ is a semigroup.*

Proof. a) Let us show that the partial groupoid $\mathcal{SN}(k, Q)$ has the property of strong associativity (see (2.2) and (2.3)). Let

$$\mathcal{N}_1 = (M_1, ..., M_u, i_1, o_1, f_1, g_1, l_1),\ \mathcal{N}_2 = (P_1, ..., P_v, i_2, o_2, f_2, g_2, l_2),$$

$$\mathcal{N}_3 = (D_1, ..., D_t, i_3, o_3, f_3, g_3, l_3)$$

these are three arbitrary neural networks from $\mathrm{SN}(k, Q)$. Let us assume that the condition is satisfied $(\mathcal{N}_1 \circ \mathcal{N}_2, \mathcal{N}_3) \in \mathrm{Dom}(\circ)$. This means that the condition $(\mathcal{N}_1, \mathcal{N}_2) \in \mathrm{Dom}(\circ)$ is satisfied. Consequently, we have the relations

$$A(\mathcal{N}_1) \cap A(\mathcal{N}_2) = \varnothing,\ A(\mathcal{N}_1 \circ \mathcal{N}_2) \cap A(\mathcal{N}_3) = \varnothing.$$

Since the equality $A(\mathcal{N}_1 \circ \mathcal{N}_2) = A(\mathcal{N}_1) \cup A(\mathcal{N}_2)$ holds, we have the equalities

$$A(\mathcal{N}_1) \cap A(\mathcal{N}_2) = \varnothing, \quad A(\mathcal{N}_1) \cap A(\mathcal{N}_3) = \varnothing, \quad A(\mathcal{N}_2) \cap A(\mathcal{N}_3) = \varnothing. \quad (4.1)$$

Then, by the definition of the operation $(\circ)$, we have $(\mathcal{N}_2, \mathcal{N}_3) \in \mathrm{Dom}(\circ)$ and $(\mathcal{N}_1, \mathcal{N}_2 \circ \mathcal{N}_3) \in \mathrm{Dom}(\circ)$.

We have shown that if the product $(\mathcal{N}_1 \circ \mathcal{N}_2) \circ \mathcal{N}_3$ is defined, then the product $\mathcal{N}_1 \circ (\mathcal{N}_2 \circ \mathcal{N}_3)$ is defined. Equality $(\mathcal{N}_1 \circ \mathcal{N}_2) \circ \mathcal{N}_3 = \mathcal{N}_1 \circ (\mathcal{N}_2 \circ \mathcal{N}_3)$ is derived from the relations (3.1), (3.2) and (3.3). We have shown that the partial groupoid $\mathcal{SN}(k, Q)$ satisfies the condition (2.2). Using similar reasoning, we can show that for a given partial groupoid the condition (2.3) is satisfied. The strong associativity of the partial groupoid $\mathcal{SN}(k, Q)$ is proved.

b) If for three neural networks $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ the relations (4.1) are satisfied, then from the definition of the zero extension and the conditions of strong associativity (which we proved above) the identities follow

$$(\mathcal{N}_1 \circ \mathcal{N}_2) \circ \mathcal{N}_3 = (\mathcal{N}_1 \tilde{\circ} \mathcal{N}_2) \tilde{\circ} \mathcal{N}_3, \quad \mathcal{N}_1 \circ (\mathcal{N}_2 \circ \mathcal{N}_3) = \mathcal{N}_1 \tilde{\circ} (\mathcal{N}_2 \tilde{\circ} \mathcal{N}_3),$$

$$(\mathcal{N}_1 \tilde{\circ} \mathcal{N}_2) \tilde{\circ} \mathcal{N}_3 = \mathcal{N}_1 \tilde{\circ} (\mathcal{N}_2 \tilde{\circ} \mathcal{N}_3). \quad (4.2)$$

Let now $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ not satisfy the relations (4.1). In this case the equalities are true

$$(\mathcal{N}_1 \tilde{\circ} \mathcal{N}_2) \tilde{\circ} \mathcal{N}_3 = \mathcal{N}_1 \tilde{\circ} (\mathcal{N}_2 \tilde{\circ} \mathcal{N}_3) = \Diamond. \quad (4.3)$$

Indeed, if (4.1) are not satisfied, then at least one of the conditions is satisfied: 1) $A(\mathcal{N}_1) \cap A(\mathcal{N}_2) \neq \varnothing$, 2) $A(\mathcal{N}_1) \cap A(\mathcal{N}_3) \neq \varnothing$, 3) $A(\mathcal{N}_2) \cap A(\mathcal{N}_3) \neq \varnothing$. Let us assume that condition 1) is satisfied. Then we have the implications:

$$A(\mathcal{N}_1) \cap A(\mathcal{N}_2) \neq \varnothing \Rightarrow \mathcal{N}_1 \tilde{\circ} \mathcal{N}_2 = \Diamond \Rightarrow (\mathcal{N}_1 \tilde{\circ} \mathcal{N}_2) \tilde{\circ} \mathcal{N}_3 = \Diamond;$$

$$\mathcal{N}_2 \tilde{\circ} \mathcal{N}_3 \neq \Diamond \Rightarrow A(\mathcal{N}_1) \cap A(\mathcal{N}_2 \tilde{\circ} \mathcal{N}_3) = A(\mathcal{N}_1) \cap [A(\mathcal{N}_2) \cup A(\mathcal{N}_3)] \neq \varnothing \Rightarrow$$

$$\Rightarrow \mathcal{N}_1 \tilde{\circ} (\mathcal{N}_2 \tilde{\circ} \mathcal{N}_3) = \Diamond; \qquad \mathcal{N}_2 \tilde{\circ} \mathcal{N}_3 = \Diamond \Rightarrow \mathcal{N}_1 \tilde{\circ} (\mathcal{N}_2 \tilde{\circ} \mathcal{N}_3) = \Diamond.$$

In this case, the equality (4.3) is true. Similarly, it can be shown that when conditions 2) and 3) are met, equality (4.3) is satisfied. The equalities (4.2) and (4.3) show that the groupoid $\widetilde{\mathrm{SN}}(k, Q)$ is an associative groupoid. $\square$

5. Formulation and proof of Theorems 2, 3 and 4

In this section it will be convenient to use a special notation for writing mappings. Let $\phi : L_1 \to L_2$. Then we will use the following notation:

$$\phi : \quad x \to y \quad \Leftrightarrow \quad \phi(x) = y \quad (x \in L_1, \quad y \in L_2).$$

Mapping l**.** Let H be some subset of the set Q and $\alpha : H \to \mathbb{R}$ is some mapping of H to $\mathbb{R}$. Let us introduce the transformation $\Gamma[H, \alpha]$ of the set $\mathrm{SN}(k, Q)$ as follows

$$\Gamma[H, \alpha] : \mathcal{N} = (M_1, ..., M_n, i, o, f, g, l) \to (M_1, ..., M_n, i, o, f, g, \widehat{l}), \quad (5.1)$$

where $\widehat{l}$ is a mapping from the set $A(\mathcal{N})$ to the set $\mathbb{R}$, which on any neuron $n \in A(\mathcal{N})$ acts according to the rule

$$\widehat{l}(n) := \begin{cases} l(n), & if\ n \notin H, \\ \alpha(n), & if\ n \in H. \end{cases}$$

Theorem 2. *For any subset* H *of the set* Q *and any mapping* $\alpha : H \to \mathbb{R}$ *the mapping* $\Gamma[H, \alpha]$ *is an endomorphism of the partial groupoid* $\mathcal{SN}(k, Q)$.

Proof. a) For compactness we assume that $\phi := \Gamma[H, \alpha]$. Let us show that ϕ satisfies the conditions (2.1). Let $\mathcal{N}_1, \mathcal{N}_2, \mathcal{U}_1, \mathcal{U}_2$ be four arbitrary neural networks from $\mathrm{SN}(k, Q)$ such that the conditions are met

$$(\mathcal{N}_1, \mathcal{N}_2) \in \mathrm{Dom}(\circ), \quad (\mathcal{U}_1, \mathcal{U}_2) \in (\mathrm{SN}(k, Q))^2 \setminus \mathrm{Dom}(\circ),$$

$$\mathcal{N}_1 = (M_1, ..., M_u, i_1, o_1, f_1, g_1, l_1), \ \mathcal{N}_2 = (P_1, ..., P_v, i_2, o_2, f_2, g_2, l_2).$$

Note that the ϕ transformation does not affect the neurons of neural networks (see (5.1)). Therefore, for any network $\mathcal{N}$ from $\mathrm{SN}(k, Q)$ the equality $A(\phi(\mathcal{N})) = A(\mathcal{N})$ holds. From the definition of the partial operation $(\circ)$ it follows that the neural networks $\mathcal{U}_1$ and $\mathcal{U}_2$ have identical neurons, therefore, the neural networks $\phi(\mathcal{U}_1)$ and $\phi(\mathcal{U}_2)$ have identical neurons. Therefore, the condition $(\phi(\mathcal{U}_1), \phi(\mathcal{U}_2)) \notin \mathrm{Dom}(\circ)$ is satisfied. Because the $(\mathcal{N}_1, \mathcal{N}_2) \in \mathrm{Dom}(\circ)$, then the the tuple $(\phi(\mathcal{N}_1), \phi(\mathcal{N}_2))$ belongs to the set $\mathrm{Dom}(\circ)$. Thus, we have shown that ϕ satisfies the last two conditions from (2.1).

b) Let us show that $\phi(\mathcal{N}_1 \circ \mathcal{N}_2) = \phi(\mathcal{N}_1) \circ \phi(\mathcal{N}_2)$. The definition of the partial operation $(\circ)$ implies the equalities

$$\mathcal{N}_1 \circ \mathcal{N}_2 = (M_1, ..., M_u, P_1, ..., P_v, i', o', f', g', l'),$$

$$\phi(\mathcal{N}_1 \circ \mathcal{N}_2) = (M_1, ..., M_u, P_1, ..., P_v, i', o', f', g', \widehat{l'}),$$

$$\phi(\mathcal{N}_1) \circ \phi(\mathcal{N}_2) = (M_1, ..., M_u, P_1, ..., P_v, i', o', f', g', m[\widehat{l_1}, \widehat{l_2}]),$$

where the mapping $m[\widehat{l_1},\widehat{l_2}] : A(\phi(\mathcal{N}_1)) \cup A(\phi(\mathcal{N}_2)) \to \mathbb{R}$ defined by rule

$$m[\widehat{l_1},\widehat{l_2}](n_1) := \widehat{l_1}(n_1) \quad (\forall n_1 \in A(\phi(\mathcal{N}_1))), \tag{5.2}$$
$$m[\widehat{l_1},\widehat{l_2}](n_2) := \widehat{l_2}(n_2) \quad (\forall n_2 \in A(\phi(\mathcal{N}_2))).$$

Equalities (5.2) are defined in accordance with relations (3.2) for the structural mapping l'. Let us show that the mappings $\widehat{l'}$ and $m[\widehat{l_1},\widehat{l_2}]$ are pointwise equal. Let $n_1 \in A(\phi(\mathcal{N}_1)) = A(\mathcal{N}_1)$ and $n_2 \in A(\phi(\mathcal{N}_2)) = A(\mathcal{N}_2)$. Then we have equalities

$$\widehat{l'}(n_1) = \begin{cases} l_1(n_1), & if\ n_1 \notin H, \\ \alpha(n_1), & if\ n_1 \in H \end{cases} = \widehat{l_1}(n_1) = m[\widehat{l_1},\widehat{l_2}](n_1),$$

$$\widehat{l'}(n_2) = \begin{cases} l_2(n_2), & if\ n_2 \notin H, \\ \alpha(n_2), & if\ n_2 \in H \end{cases} = \widehat{l_2}(n_2) = m[\widehat{l_1},\widehat{l_2}](n_2),$$

which show that $\widehat{l'} = m[\widehat{l_1},\widehat{l_2}]$. Therefore, the tuples $\phi(\mathcal{N}_1) \circ \phi(\mathcal{N}_2)$ and $\phi(\mathcal{N}_1 \circ \mathcal{N}_2)$ are equal. We have shown that the first condition from (2.1) is satisfied. Thus ϕ satisfies the conditions (2.1). This means that ϕ is an endomorphism and the theorem is proven. □

Mapping g**.** Let H be some subset of the set Q and $\beta : H \to F(\mathbb{R})$ be some map of H to $F(\mathbb{R})$. Let us introduce the transformation $\Upsilon[H,\beta]$ of the set $\mathrm{SN}(k,Q)$ as follows

$$\Upsilon[H,\beta] : \mathcal{N} = (M_1,...,M_n,i,o,f,g,l) \to (M_1,...,M_n,i,o,f,\widehat{g},l),$$

where $\widehat{g}$ is a mapping from the set $A(\mathcal{N})$ to the set $F(\mathbb{R})$, which on every neuron $n \in A(\mathcal{N})$ acts according to the rule

$$\widehat{g}(n) := \begin{cases} g(n), & if\ n \notin H, \\ \beta(n), & if\ n \in H. \end{cases}$$

Theorem 3. *For any subset H of the set Q and any mapping $\beta : H \to F(\mathbb{R})$ the mapping $\Upsilon[H,\beta]$ is an endomorphism of the partial groupoid $\mathcal{SN}(k,Q)$.*

Proof. The proof is similar to the theorem 2. □

Mapping f**.** Let S be some subset of the set Q^2 and $\gamma : S \to \mathbb{R}$ be some map of S to $\mathbb{R}$. Let us introduce the transformation $\Phi[S,\gamma]$ of the set $\mathrm{SN}(k,Q)$ as follows

$$\Phi[S,\gamma] : \mathcal{N} = (M_1,...,M_n,i,o,f,g,l) \to (M_1,...,M_n,i,o,\widehat{f},g,l),$$

where $\widehat{f}$ is a mapping of the set $Syn(\mathcal{N})$ into the set $\mathbb{R}$, which on any synoptic connection $s \in Syn(\mathcal{N})$ follows the rule

$$\widehat{f}(s) := \begin{cases} f(s), & if\ s \notin S, \\ \gamma(s), & if\ s \in S. \end{cases}$$

Remark 3. The transformation $\Phi[S,\gamma]$ is not necessarily an endomorphism of the partial groupoid $\mathcal{SN}(k,Q)$. Indeed, let the connection $x := (a,b) \in Q^2$ satisfy the conditions $x \in S$ and $\gamma(x) = 3$ (you can take any number different from zero and one) . Further, we assume that $\mathcal{N}_1, \mathcal{N}_2$ are two networks from $\mathrm{SN}(k,Q)$ such that the output (i.e., last) layer network $\mathcal{N}_1$ contains neuron a and the input (i.e., first) layer of network $\mathcal{N}_2$ contains neuron b. We assume that $A(\mathcal{N}_1) \cap A(\mathcal{N}_2) = \varnothing$ (i.e. the action $\mathcal{N}_1 \circ \mathcal{N}_2$ defined). Then we can easily obtain the condition

$$\Phi[S,\gamma](\mathcal{N}_1 \circ \mathcal{N}_2) \neq \Phi[S,\gamma](\mathcal{N}_1) \circ \Phi[S,\gamma](\mathcal{N}_2).$$

In fact, the weight of the synoptic connection $x = (a,b)$ in the neural network $\Phi[S,\gamma](\mathcal{N}_1 \circ \mathcal{N}_2)$ will be equal to 3. Weight of synoptic connection $x = (a,b)$ in the network $\Phi[S,\gamma](\mathcal{N}_1) \circ \Phi[S,\gamma](\mathcal{N}_2)$ will be equal to either zero or one. Therefore, the considered networks cannot be equal.

Remark 4. Despite the fact that the transformation $\Phi[S,\gamma]$ is not an endomorphism of the partial groupoid $\mathcal{SN}(k,Q)$, this transformation is important in the context of issues related to training neural networks (since the assignment of new weights of synoptic connections is part of the iteration of the inverse error distribution method). Therefore, it is useful to find a subset X of the set $\mathrm{SN}(k,Q)$ such that on this set the operation $(\circ)$ will be closed (in the sense of a partial operation) and for any pair of networks from this set the transformation $\Phi[S,\gamma]$ will preserve the partial operation (i.e. satisfy the conditions (2.1)). The results of constructing the set X are reflected in Theorem 4.

As usual, 2^X is a Boolean of the set X. Let us introduce the mappings $U_1 : \mathrm{SN}(k,Q) \to 2^Q$ and $U_2 : \mathrm{SN}(k,Q) \to 2^Q$, which for each network $\mathcal{N} = (M_1, ..., M_n, i, o, f, g, l) \in \mathrm{SN}(k,Q)$ act according to the rule $U_1(\mathcal{N}) := M_1$, $U_2(\mathcal{N}) := M_n$.

Let S be some subset of the set Q^2. Let $D(S)$ denote the set of neurons from Q such that the condition holds: $x \in D(x) \Leftrightarrow \exists (a,b) \in S : a = x \vee b = x$. Let's introduce a lot of neural networks

$$T(k,Q,S) := \{\mathcal{N} \in \mathrm{SN}(k,Q) \mid U_1(\mathcal{N}) \cap D(S) = \varnothing,\ U_2(\mathcal{N}) \cap D(S) = \varnothing\}.$$

The set $T(k,Q,S)$ consists of neural networks such that the input and output layers of these networks do not contain neurons from $D(S)$. It is not difficult to notice that $(T(k,Q,S), \circ)$ is a partial groupoid, where $(\circ)$ is the operation introduced by the definition 6.

Theorem 4. *For any subset S of the set Q^2 and any mapping $\gamma : S \to \mathbb{R}$ the mapping $\Phi[S,\gamma]$ is an endomorphism of the partial groupoid $(T(k,Q,S), \circ)$.*

Proof. a) Let $\phi := \Phi[S,\gamma]$. The transformation ϕ is correctly defined on all elements of the set $T(k,Q,S)$ (this fact follows from the definition of this mapping). Note that $A(\phi(\mathcal{N})) = A(\mathcal{N})$. Therefore, repeating verbatim the reasoning from point a) of Theorem 2, we obtain that ϕ satisfies the last two conditions from (2.1).

b) Let us show that $\phi(\mathcal{N}_1 \circ \mathcal{N}_2) = \phi(\mathcal{N}_1) \circ \phi(\mathcal{N}_2)$ when $(\mathcal{N}_1, \mathcal{N}_2) \in \mathrm{Dom}(\circ)$ and the equalities hold

$$\mathcal{N}_1 = (M_1, ..., M_u, i_1, o_1, f_1, g_1, l_1),\ \mathcal{N}_2 = (P_1, ..., P_v, i_2, o_2, f_2, g_2, l_2).$$

We have equalities $\mathcal{N}_1 \circ \mathcal{N}_2 = (M_1, ..., M_u, P_1, ..., P_v, i', o', f', g', l')$ and

$$\phi(\mathcal{N}_1 \circ \mathcal{N}_2) = (M_1, ..., M_u, P_1, ..., P_v, i', o', \widehat{f'}, g', l'),$$

$$\phi(\mathcal{N}_1) \circ \phi(\mathcal{N}_2) = (M_1, ..., M_u, P_1, ..., P_v, i', o', b[\widehat{f_1}, \widehat{f_2}], g', l'),$$

where the mapping $b[\widehat{f_1}, \widehat{f_2}] : Syn(\phi(\mathcal{N}_1) \circ \phi(\mathcal{N})) \to \mathbb{R}$ defined in accordance with (3.3). Since $Syn(\phi(\mathcal{N})) = Syn(\mathcal{N})$ and for any pair $(\mathcal{N}_1, \mathcal{N}_2)$ of $\mathrm{Dom}(\circ)$ the equality is satisfied

$$Syn(\mathcal{N}_1 \circ \mathcal{N}_2) = Syn(\mathcal{N}_1) \cup Syn(\mathcal{N}_2) \cup (U_2(\mathcal{N}_1) \times U_1(\mathcal{N}_2)), \tag{5.3}$$

then the mapping $b[\widehat{f_1}, \widehat{f_2}]$ is defined on the set from the right side of the equality (5.3). Since $\mathcal{N}_1, \mathcal{N}_2 \in T(k, Q, S)$, then on the set $U_2(\mathcal{N}_1) \times U_1(\mathcal{N}_2)$ the mapping $b[\widehat{f_1}, \widehat{f_2}]$ acts in the same way as the mappings f' and $\widehat{f'}$ (the mapping f' is determined by the relations (3.3)). According to (3.3) the action of the mapping $b[\widehat{f_1}, \widehat{f_2}]$ on the set $Syn(\mathcal{N}_1) \cup Syn(\mathcal{N}_2)$ is defined by the equalities $b[\widehat{f_1}, \widehat{f_2}](s_1) := \widehat{f_1}(s_1)\ \ (\forall s_1 \in Syn(\mathcal{N}_1))$ and $b[\widehat{f_1}, \widehat{f_2}](s_2) := \widehat{f_2}(s_2)\ \ (\forall s_2 \in Syn(\mathcal{N}_2))$. Let us show that the mappings $\widehat{f'}$ and $b[\widehat{f_1}, \widehat{f_2}]$ are equal on the set $Syn(\mathcal{N}_1) \cup Syn(\mathcal{N}_2)$. In fact, the relations hold

$$\widehat{f'}(s_1) = \begin{cases} f_1(s_1), & if\ s_1 \notin S, \\ \gamma(s_1), & if\ s_1 \in S \end{cases} = \widehat{f_1}(s_1) = b[\widehat{f_1}, \widehat{f_2}](s_1)\ \ (s_1 \in Syn(\mathcal{N}_1)),$$

$$\widehat{f'}(s_2) = \begin{cases} f_2(s_2), & if\ s_2 \notin S, \\ \gamma(s_2), & if\ s_2 \in S \end{cases} = \widehat{f_2}(s_2) = b[\widehat{f_1}, \widehat{f_1}](s_2)\ \ (s_2 \in Syn(\mathcal{N}_2)),$$

which show that $\widehat{f'} = b[\widehat{f_1}, \widehat{f_1}]$. Therefore, the tuples $\phi(\mathcal{N}_1) \circ \phi(\mathcal{N}_2)$ and $\phi(\mathcal{N}_1 \circ \mathcal{N}_2)$ are equal. We have shown that the first condition from (2.1) is satisfied. Thus ϕ satisfies the conditions (2.1). Theorem is proven. □

6. Conclusion

The partial groupoid $\mathcal{SN}(k, Q)$ allows us to model the composition of neural networks. In this case, the composition $\mathcal{N}_1 \circ \mathcal{N}_2$ carries information about the architecture (i.e., the internal structure) of the networks $\mathcal{N}_1$, $\mathcal{N}_2$

and $\mathcal{N}_1 \circ \mathcal{N}_2$. The ability to store information about the architecture of neural networks distinguishes this method of modeling from modeling the composition of neural networks using abstract automata. The results of this work will be useful in those studies of neural networks that involve significant use of information about the structure (i.e. architecture) of neural networks. If there is no such need, then it is preferable to use automata theory or matrix implementations of neural networks. This will eliminate the need to take into account mathematical formalism, which may be unnatural in the context of solving specific applied problems.

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