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Fuzzy Volterra Integral Equations with Piecewise Continuous Kernels: Theory and Numerical Solution

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Abstract. This research focuses on addressing both linear and nonlinear fuzzy Volterra integral equations that feature piecewise continuous kernels. The problem is tackled using the method of successive approximations. The study discusses the existence and uniqueness of solutions for these fuzzy Volterra integral equations with piecewise kernels. Numerical results are obtained by applying the successive approximations method to examples for both linear and nonlinear scenarios. Error analysis graphs are plotted to illustrate the accuracy of the method. Furthermore, a comparative analysis is presented through graphs of approximate solutions for different fuzzy parameter values. To highlight the effectiveness and significance of the successive approximations method, a comparison is made with the traditional homotopy analysis technique. The results indicate that the successive approximation method outperforms the homotopy analysis method in terms of accuracy and effectiveness.

Keywords: fuzzy Volterra integral equation, piecewise kernel, successive approximation, error estimation

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Научная статья

Нечеткие интегральные уравнения Вольтерра с кусочно-непрерывными ядрами: теория и численное решение

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Аннотация. Исследуется теория линейных и нелинейных нечетких интегральных уравнений Вольтерра с кусочно-непрерывными ядрами. Проблема решается с использованием метода последовательных приближений. Рассмотрены вопросы существования и единственности решений для нечетких интегральных уравнений Вольтерра с кусочными ядрами. Численные результаты получены путем применения метода последовательных приближений как к линейным, так и нелинейным интегральным уравнениям Вольтерра с кусочно-непрерывными ядрами. Построены графики для анализа ошибок с целью иллюстрации точности метода. Кроме того, представлено сравнительное исследование, где используются графики приближенных решений для различных значений нечетких параметров. Чтобы подчеркнуть эффективность и значимость метода последовательных приближений, проводится сравнение с традиционной техникой гомотопического анализа. Результаты показывают, что метод последовательных приближений превосходит метод гомотопического анализа по точности и эффективности.

Ключевые слова: нечеткое интегральное уравнение Вольтерра, кусочное ядро, последовательная аппроксимация, оценка погрешности

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1. Introduction

Fuzzy integral equations (FIEs) represent significant and applicable challenges across various disciplines, including engineering, physics, biology, and chemistry. Research by Bede and Gal [5], Friedman and Ma [11], and Goetschel and Voxman [12] has contributed to the theoretical understanding of FIEs. Ziari and Abbasbandy addressed nonlinear FIEs through the application of fuzzy quadrature rules [30]. The Reproducing Kernel Hilbert space method was utilized by Javan et al. [10], while Asari et al. explored radial basis functions in their work [4]. Amirfakhrian et al. implemented fuzzy interpolation techniques to tackle FIEs [2]. Additionally, numerous other methods for solving FIEs are discussed in [1]. The well-known sinc-collocation method was employed in [17] to solve fuzzy Fredholm integral equations. In [8], a combination of the homotopy analysis method and Laplace transformations was used to investigate Abel-type FIEs. Furthermore, the CESTAC method and the CADNA library were applied in [9; 18] to optimize the results of the homotopy analysis method for solving FIEs.

Volterra integral equation with piecewise continuous kernel is known and applicable problem which can be employed in various balance problems including electric loading problem. Sidorov et al. in [7; 24] studied the generalized solution of Volterra integral equations. Solvability of this problem has been illustrated by Sidorov in [25; 27] and Muftahov and Sidorov in [14]. The successive approximation method was used to find the solution of Volterra integral equations in [26]. The numerical solution of this problem can be found in [15]. Also some numerical and semi-analytical methods can be found for solving Volterra integral equations with piecewise kernel such as the spline collocation method [29], Lagrange-collocation method [19], Adomian decomposition method [20], homotopy perturbation method [21], the collocation method with Taylor polynomials [22] and other [23]. For more details on the theory of Volterra integral equations with piecewise continuous kernels readers may refer to monograph [28]. Such equations naturally generalizes the non-classic Volterra equations studied in monograph [3].

This research focuses on a new category of fuzzy Volterra integral equations that feature a piecewise continuous kernel as

$$S(p) = H(p) \oplus (\mathcal{FR}) \sum_{t=1}^{m'} \int_{\delta_{t-1}(p)}^{\delta_t(p)} W_t(q,p) \odot G(S(q))dq, \quad b_1 \leq q, p \leq T \leq b_2, \tag{1.1}$$

where

$$b_1 =: \delta_0(p) < \delta_1(p) < ... < \delta_{m'-1}(p) < \delta_{m'}(p) := p, \quad b_1 \leq p \leq T \leq b_2$$

The kernel $W_t(q,p)$ is defined as a crisp and positive function over the square region $b_1 \leq q, p \leq T \leq b_2$. The function $S(p)$ represents a fuzzy real-valued function, and $G : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is continuous. Additionally, $W_t(q,p)$ is characterized as a piecewise kernel along continuous curves $\delta_t(p)$ for $t = 1, 2, ..., m'$. Consequently, the functions $W_1(q,p), W_2(q,p), ..., W_{m'}(q,p)$ exhibit uniform continuity with respect to t, and there exists a constant $M_t > 0$ such that $M_t = \max_{b_1 \leq q,p \leq b_2} |W_t(q,p)|$. We employed the method of successive approximations to address problem (1.1). The theorem regarding the existence of solutions is also examined. Furthermore, the main theorem is demonstrated below to provide error estimation for the problem. By solving various examples in both linear and nonlinear contexts and plotting error graphs along with graphs of fuzzy approximate solutions, we illustrate the capability and efficiency of the method.

This paper is organized as follows. Section 2 provides the preliminaries of fuzzy mathematics. Section 3 is the main idea of this study. Also in this section the main existence of solution theorem is illustrated. Section 4 shows the error estimation of the successive approximation method for solving problem (1.1). Section 5 provides the linear and nonlinear examples. Using some graphs we show the accuracy of the method. Section 5 is the conclusion.

2. Preliminaries

We have reported the main definitions and theorems of fuzzy mathematics [5; 6; 11–13; 16].

Definition 1. *Based on the following properties a fuzzy number* $p : \mathbb{R} \to [0,1]$ *can be defined as a function:*

1) *p is normal which is* $\exists x_0 \in \mathbb{R}; p(x_0) = 1$,
2) *p is fuzzy convex set* $p(\gamma x + (1-\gamma)y) \geq \min\{p(x), p(y)\}, \forall x, y \in \mathbb{R}, \gamma \in [0,1]$.
3) *p is upper semi-continuous on* $\mathbb{R}$,
4) $\{x \in \mathbb{R} : p(x) > 0\}$ *is a compact set.*

$\mathbb{R}_F$ shows all fuzzy numbers sets.

Definition 2. $(\underline{p}(\varepsilon), \overline{p}(\varepsilon)), 0 \leq \varepsilon \leq 1$ *is the parametric form of an arbitrary fuzzy number satisfying the following conditions:*

1) $\underline{p}(\varepsilon)$ *is a bounded left continuous non-decreasing function over* $[0,1]$,
2) $\overline{p}(\varepsilon)$ *is a bounded left continuous non-increasing function over* $[0,1]$,
3) $\underline{p}(\varepsilon) \leq \overline{p}(\varepsilon), 0 \leq \varepsilon \leq 1$

We show the scalar multiplication and addition of fuzzy numbers as:

1) $(p \oplus p_1)(\varepsilon) = (\underline{p}(\varepsilon) + \underline{p_1}(\varepsilon), \overline{p}(\varepsilon) + \overline{p_1}(\varepsilon))$,
2) $(\gamma \odot p)(\varepsilon) = \begin{cases} (\gamma\underline{p}(\varepsilon), \gamma\overline{p}(\varepsilon)) & \gamma \geq 0, \\ (\gamma\overline{p}(\varepsilon), \gamma\underline{p}(\varepsilon)) & \gamma < 0. \end{cases}$

Definition 3. *Let $p = (\underline{p}(\varepsilon), \overline{p}(\varepsilon)), p_1 = (\underline{p_1}(\varepsilon), \overline{p_1}(\varepsilon))$ be two fuzzy numbers then the distance can be defined as*

$$\mathcal{D}(p, p_1) = \sup_{\varepsilon \in [0,1]} \max\{|\underline{p}(\varepsilon) - \underline{p}(\varepsilon)|, |\overline{p}(\varepsilon) - \overline{p_1}(\varepsilon)|\}.$$

We have the following properties for distance $\mathcal{D}$.

Theorem 1. *1) $(\mathbb{R}_F, \mathcal{D})$ is a complete metric space,*

2) $\mathcal{D}(p \oplus p_2, p_1 \oplus p_2) = \mathcal{D}(p, p_1) \forall p, p_1, p_2 \in \mathbb{R}_F$,

3) $\mathcal{D}(k \odot p, k \odot p_1) = |k|\mathcal{D}(p, p_1), \forall p, p_1 \in \mathbb{R}_F \forall k \in \mathbb{R}$,

4) $\mathcal{D}(p \oplus p_1, p_2 \oplus p_3) \leq \mathcal{D}(p, p_2) + \mathcal{D}(p_1, p_3) \forall p, p_1, p_2, p_3 \in \mathbb{R}_F$.

Theorem 2. *1) We have a commutative semigroup for $(\mathbb{R}_F, \oplus)$ with the zero element $(\mathbb{R}_F, \oplus)$.*

2) There is no opposite element if there are fuzzy numbers which are not crisp ($(\mathbb{R}_F, \oplus)$ cannot be a group).

3) $\forall b_1, b_2 \in \mathbb{R}$ with $b_1, b_2 \geq 0$ or $b_1, b_2 \leq 0$ and $\forall p \in \mathbb{R}_F$, one get $(b_1 + b_2) \odot p = b_1 \odot p \oplus b_2 \odot u$.

4) $\forall \gamma \in \mathbb{R}$ and $p, p_1 \in \mathbb{R}_F$, one get $\gamma \odot (p \oplus p_1) = \gamma \odot p \oplus \gamma \odot p_1$

5) $\forall \gamma, \varepsilon \in \mathbb{R}$ and $p \in \mathbb{R}_F$, one get $\gamma \odot (\varepsilon \odot p) = (\gamma\varepsilon) \odot p$.

6) There is the general attributes of the norm for of $\|.\|_F : \mathbb{R}_F \to \mathbb{R}$ by $\|p\|_F = \mathcal{D}(p, \tilde{0})$ which is $\|p\|_F = 0 \Leftrightarrow p = \tilde{0}$, $\|\gamma \odot p\|_F = |\gamma|\|p\|_F$ and $\|p \oplus p_1\|_F \leq \|p\|_F + \|p_1\|_F$

7) $|\|p\|_F + \|p_1\|_F| \leq \mathcal{D}(p, p_1)$ and $\mathcal{D}(p, p_1) \leq \|p\|_F + \|p_1\|_F$ for any $p, p_1 \in \mathbb{R}_F$.

Definition 4. *Continuity of a fuzzy real number valued function $H : [b_1, b_2] \to \mathbb{R}_F$ can be defined in $x_0 \in [b_1, b_2]$ as $\forall \varepsilon > 0$, $\exists \rho > 0$; $\mathcal{D}(H(x), H(x_0)) < \varepsilon$, whenever $x \in [b_1, b_2]$ and $|x - x_0| < \rho$.*

Definition 5. *Assume that $H : [b_1, b_2] \to \mathbb{R}_F$ is a bounded mapping. The modulus of continuity $\omega_{[b_1,b_2]}(H, .) : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ is defined as*

$$\omega_{[b_1,b_2]}(H, \rho) = \sup\{\mathcal{D}(H(x), H(y)) : x, y \in [b_1, b_2], |x - y| \leq \rho\}. \quad (2.1)$$

Also $\omega_{[b_1,b_2]}(H, \rho)$ is the uniform modulus of continuity of H if

$$H \in C_F[b_1, b_2].$$

Theorem 3. *We have the following properties for the modulus of continuity:*

1) $\mathcal{D}(H(x), H(y)) \leq \omega_{[b_1,b_2]}(H, |x-y|)$ *for any* $x, y \in [b_1, b_2]$.

2) $\omega_{[b_1,b_2]}(H, \rho)$ *is increasing function of* ρ,

3) $\omega_{[b_1,b_2]}(H, 0) = 0$,

4) $\omega_{[b_1,b_2]}(H, \rho_1 + \rho_2) \leq \omega_{[b_1,b_2]}(H, \rho_1) + \omega_{[b_1,b_2]}(H, \rho_2), \ \rho_1, \rho_2 \geq 0$

5) $\omega_{[b_1,b_2]}(H, n\rho) \leq n\omega_{[b_1,b_2]}(H, \rho)$ *for any* $\rho \geq 0$ *and* $n \in N$,

6) $\omega_{[b_1,b_2]}(H, \gamma\rho) \leq (\gamma + 1)\omega_{[b_1,b_2]}(H, \rho), \ \rho, \gamma \geq 0$,

7) For $[b_3, b_4] \subseteq [b_1, b_2]$ *one get* $\omega_{[b_3,b_4]}(H, \rho) \leq \omega_{[b_1,b_2]}(H, \rho)$.

Definition 6. *Assume that* $H : [b_1, b_2] \to \mathbb{R}_{\mathcal{F}}$. H *is a Riemann integrable of fuzzy type to* $I(H) \in \mathbb{R}_{\mathcal{F}}$ *if* $\forall \varepsilon > 0, \ \exists \rho > 0$; $\forall$ *division* $P = \{[p, p_1] : \xi\}$ *of* $[b_1, b_2]$ *with the norms* $\Delta(p) < \rho$, *it holds*

$$\mathcal{D}\left(\sum_{p}^{*}(p_1 - p) \odot H(\xi), I(H)\right) < \varepsilon; \tag{2.2}$$

where $\sum^*$ *shows the fuzzy summation. Then*

$$I(H) = (\mathcal{FR})\int_{b_1}^{b_2} H(x)dx.$$

And for $H \in C_{\mathcal{F}}[b_1, b_2]$ *it follows*

$$\underline{(\mathcal{FR})\textstyle\int_{b_1}^{b_2} H(t;r)dt} = \textstyle\int_{b_1}^{b_2} \underline{H}(t;r)dt,$$

$$\overline{(\mathcal{FR})\textstyle\int_{b_1}^{b_2} H(t;r)dt} = \textstyle\int_{b_1}^{b_2} \overline{H}(t;r)dt$$

Lemma 1. *If* $H, V : [b_1, b_2] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ *are fuzzy and continuous functions, then* $H : [b_1, b_2] \to \mathbb{R}_+$ *by* $F(x) = \mathcal{D}(H(x), V(x))$ *is continuous on* $[b_1, b_2]$ *and*

$$\mathcal{D}\left((\mathcal{FR})\int_{b_1}^{b_2} H(x)dx, (\mathcal{FR})\int_{b_1}^{b_2} V(x)dx\right) \leq \int_{b_1}^{b_2} \mathcal{D}(H(x), V(x))dx. \tag{2.3}$$

Theorem 4. *Assume that* $H : [b_1, b_2] \to \mathbb{R}_{\mathcal{F}}$ *is a Henstock integrable and a bounded function. Then for* $b_1 = x_0 < x_1 < ... < x_n = b_2$ *and* $\xi_i \in [x_{i-1}, x_i]$ *it gives:*

$$\mathcal{D}\left((\mathcal{FH})\int_{b_1}^{b_2} H(t)dt, \sum_{i=1}^{n}{}^{*}(x_i - x_{i-1}) \odot H(\xi_i)\right) \leq$$

$$\leq \sum_{i=1}^{n}(x_i - x_{i-1})\omega_{[x_i,x_{i-1}]}(H, x_i - x_{i-1})$$

Corollary 1. *Let $H : [b_1, b_2] \to \mathbb{R}_{\mathcal{F}}$ be a bounded and Henstock integrable function. Then*

1) $\mathcal{D}\Big((\mathcal{FH})\int_{b_1}^{b_2} H(t)dt, (b_2-b_1)\odot H(\frac{b_1+b_2}{2})\Big)\leq\frac{b_2-b_1}{2}\omega_{[b_1,b_2]}(H, \frac{b_2-b_1}{2})$

2) $\mathcal{D}\Big((\mathcal{FH})\int_{b_1}^{b_2} H(t)dt, \frac{b_2-b_1}{2}\odot(H(b_1)\oplus H(b_2))\Big)\leq\frac{b_2-b_1}{2}\omega_{[b_1,b_2]}(H, \frac{b_2-b_1}{2})$

3) $\mathcal{D}\Big((\mathcal{FH})\int_{b_1}^{b_2} H(t)dt, \frac{b_2-b_1}{6}\odot(H(b_1)\oplus 4\odot H(\frac{b_1+b_2}{2})\oplus H(b_2))\Big)\leq$
$\leq 2(b_2-b_1)\omega_{[b_1,b_2]}(H, \frac{b_2-b_1}{6})$.

3. Main Idea

In this section, we will examine the existence and uniqueness of the solution to problem (1.1) using the method of successive approximations. Let us define the space of continuous functions as $X = \{H : [b_1, b_2] \to \mathbb{R}_{\mathcal{F}} \mid H \text{ is continuous}\}$, equipped with the fuzzy distance given by $\mathcal{D}^*(H, p) = \sup_{b_1\leq p\leq b_2} \mathcal{D}(H(p), V(p))$. Consider $A : X \to X$ as a nonlinear integral operator. The application of A to problem (1.1) results in

$$AS(p)=H(p)\oplus(\mathcal{FR})\sum_{t=1}^{m'}\int_{\delta_{t-1}(p)}^{\delta_t(p)}W_t(q,p)\odot G(S(q))dq, \forall q,p\in[b_1,b_2], \forall F\in X.$$

Theorem 5. *Assume that the kernels $W_1(q, p), W_2(q, p), \ldots, W_{m'}(q, p)$ are positive and continuous for $b_1 \leq q, p \leq T \leq b_2$. Let the function $H(p)$ be fuzzy continuous with respect to p in the interval $b_1 \leq p \leq T \leq b_2$. Furthermore,*

$$\exists L > 0;\ \mathcal{D}(G(Z_1(p)), G(Z_2(p_1))) \leq L\mathcal{D}(Z_1(p), Z_2(p_1)), \quad \forall p, p_1 \in [b_1, b_2].$$

If $c = \sum_{t=1}^{m'} M_t L(\delta_t - \delta_{t-1}) < 1$, then there exists a unique solution $F^* \in X$ for the FVIE (1.1) using the method of successive approximations as follows:

$$\begin{cases} Z_0(p) = H(p), \\ Z_m(p) = H(p) \oplus (\mathcal{FR})\displaystyle\sum_{t=1}^{m'}\int_{\delta_{t-1}(p)}^{\delta_t(p)} W_t(q, p)\odot G(Z_{m-1}(q))dq, \\ b_1 \leq q, p \leq T \leq b_2, \quad m \geq 1, \end{cases} \tag{3.1}$$

which is convergent to F^*. And the error bound is

$$\mathcal{D}(U^*(p), Z_m(p)) \leq \frac{c^{m+1}}{L(1-c)}M_0, \quad \forall t \in [b_1, b_2], \quad m \geq 1 \tag{3.2}$$

for $M_0 = \sup_{b_1 \le p \le b_2} \|G(H(p))\|_F$.

Now we can introduce the following numerical method to find the approximate solution of (1.1). As

$$b_1 = p_0 < p_1 < ... < p_{n-1} < p_n = b_2$$

where $p_i = b_1 + ih$ and $h = \frac{b_2 - b_1}{n}$ and one have the following iterative procedure as

$$\begin{cases} y_0(p) = H(p), \\ y_m(p) = H(p) \oplus \sum_{t=1}^{m'} \frac{h}{2} \odot \\ \qquad \odot \Big[W_t(p_0, p) \odot G(y_{m-1}(p_0)) \oplus W_t(p_n, p) \odot G(y_{m-1}(p_n)) \oplus \\ \qquad \oplus 2 \sum_{l=1}^{n-1} W_t(p_l, p) \odot G(y_{m-1}(p_l)) \Big], \quad m \ge 1. \end{cases}$$

Also the compact form of the relation is

$$\begin{cases} y_0(p) = H(p), \\ y_m(p) = H(p) \oplus \sum_{t=1}^{m'} \sum_{l=1}^{n-1} \frac{h}{2} \odot \Big[W_t(p_l, p) \odot \\ \qquad \odot G(y_{m-1}(p_l)) \oplus W_t(p_l, p) \odot G(y_{m-1}(p_l)) \Big], \quad m \ge 1. \end{cases} \tag{3.3}$$

4. Error Estimation

Theorem 6. *Assume the nonlinear functional Volterra integral equation (1.1) with the kernel* $W_t(q,p)$ *is defined along continuous curves* $\delta_t(p)$ *for* $t = 1, 2, \ldots, m$*, where the kernel is positive on the domain* $[b_1, b_2] \times [b_1, b_2]$*. Additionally, let* G *be a continuous function on* $\mathbb{R}_F$ *and* H *be continuous on the interval* $[b_1, b_2]$*. Furthermore, there exists a constant* $L > 0$ *such that*

$$\mathcal{D}(G(Z_1(p)), G(Z_2(p_1))) \le L.\mathcal{D}(Z_1(p), Z_2(p_1)), \quad \forall p, p_1 \in [b_1, b_2].$$

For the condition $C_t = M_t L(b_2 - b_1) < 1$*, where*

$$M_t = \max_{b_1 \le q, p \le T \le b_2} |W_t(q,p)|,$$

the iterative scheme (referenced as (3.1)) converges to the unique solution of (1.1), denoted as F*. Additionally, an error estimation can be derived as* *follows:*

$$\mathcal{D}^*(U, y_m) \le \sum_{t=1}^{m'} \frac{C_t}{2(1-C_t)}\omega_{\delta_{t-1},\delta_t}(H, h) + \sum_{t=1}^{m'} \frac{C_t^{m+1}L_1}{L(1-C_t)}$$

$$+\sum_{t=1}^{m'} \frac{C_t^2+2C_t}{2LM_t(1-C_t)}(L_1\omega_s(W_t, h) + L_2\omega_t(W_t, h))$$

where

$$\omega_s(W_t, h) = \sup_{b_1 \le p \le T \le b_2} \{\sup |W_t(x, p) - W_t(y, p)| : |x-y| \le h\},\ t = 1, 2, ..., m',$$

and

$$\omega_t(W_t, h) = \sup_{b_1 \le q \le T \le b_2} \{\sup |W_t(q, p_1) - W_t(q, p_2)| : |p_1 - p_2| \le h\}, t = 1, 2, ..., m'.$$

Remark 1. As we know $C_t < 1, t = 1, 2, ..., m'$ and it shows

$$\lim_{m\to\infty} C_t^{m+1} = 0, t = 1, 2, ..., m'.$$

And we have

$$\lim_{h\to 0} \omega_{[\delta_{t-1},\delta_t]}(H, h) = 0, \lim_{h\to 0} \omega_s(W_t, h) = 0, \lim_{h\to 0} \omega_t(W_t, h) = 0, t = 1, 2, ..., m'.$$

The convergence of this scheme can be obtained by

$$\lim_{m\to\infty, h\to 0} \mathcal{D}^*(U, y_m) = 0.$$

5. Numerical Results

In this section some examples are presented. We apply the mentioned method for solving the problems. This is the first time that the problem (1.1) has been solved and there are no other methods to compare with. But in order to show the accuracy of the method we compare to the homotopy analysis method. All the mentioned examples are simulated problems.

Example 1. We examine the issue presented in (1.1) with the following definitions: $W_1(q, p) = 1+p-r$, $W_2(q, p) = p-1$, $m' = 2$, and $a = \delta_0(p) = 0$, $\delta_1(p) = \frac{p}{3}$, and $\delta_2(p) = p$, where

$$\underline{H} = (-2+\varepsilon)(-1+p^2) - \frac{2}{81}(-2+\varepsilon)(-1+p)p(-27+13p^2)$$
$$-\tfrac{1}{324}(-2+\varepsilon)p(-108-90p+4p^2+3p^3),$$

$$\overline{H} = (-2+\varepsilon)(1+p^2) - \frac{2}{81}(-2+\varepsilon)(-1+p)p(27+13p^2)$$
$$-\tfrac{1}{324}(-2+\varepsilon)p(108+90p+4p^2+3p^3).$$

The exact solution is given by $(\underline{U}(p), \overline{U}(p)) = ((\varepsilon-2)(p^2-1), (\varepsilon-2)(p^2+1))$. A comparison between the exact solution $(\underline{U}(p), \overline{U}(p))$ and the approximate solution $(\underline{U}_{10}(p), \overline{U}_{10}(p))$ for $\varepsilon = 0.5$ is illustrated in Fig. 1. The absolute errors for $m = 10$ are shown in Fig. 2. Additionally, Fig. 3 presents the graphs of the obtained solutions for different values of q. Table 1 provides a comparison of the absolute errors between the successive approximation and the HAM.

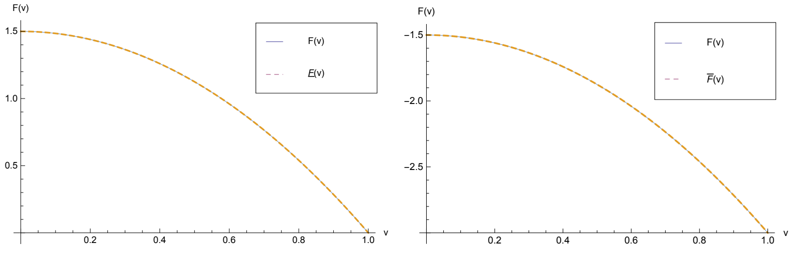

Figure 1. Comparison between the exact solution $(\underline{U}(p), \overline{U}(p))$ and approximate solution $(\underline{U}_{10}(p), \overline{U}_{10}(p))$ for $\varepsilon = 0.5$.

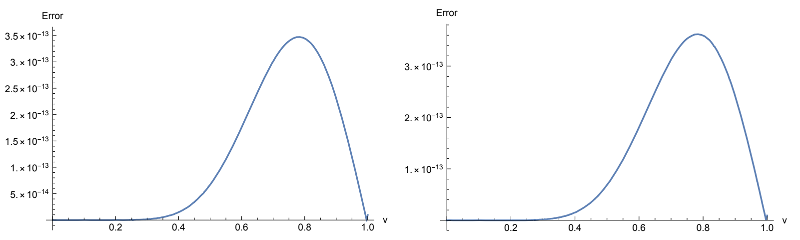

Figure 2. The absolute error for $(\underline{U}_{10}(p), \overline{U}_{10}(p))$ and $\varepsilon = 0.5$.

Table 1

The error results of Example 1

p	$\lvert\underline{U}(p) - \underline{U}_{10}(p)\rvert$	$\lvert\overline{U}(p) - \overline{U}_{10}(p)\rvert$	$\lvert\underline{U}(p) - \underline{U}_{HAM}(p)\rvert$	$\lvert\overline{U}(p) - \overline{U}_{HAM}(p)\rvert$
0.00	0	0	0	0
0.30	1.55431×10^{-15}	1.77636×10^{-15}	1.48935×10^{-14}	1.35746×10^{-14}
0.60	1.75859×10^{-13}	1.81188×10^{-13}	1.768177×10^{-12}	1.64527×10^{-12}
0.90	2.30593×10^{-13}	2.41585×10^{-13}	7.86275×10^{-12}	7.24517×10^{-12}

Example 2. We have $W_1(q,p) = p, W_2(q,p) = p-1, W_3(q,p) = r-p, m' = 3$, $b_1 = \delta_0(p) = 0, \delta_1(p) = \frac{p}{8}, \delta_2(p) = \frac{2p}{8}$ and $\delta_3(p) = p$, with nonlinear term

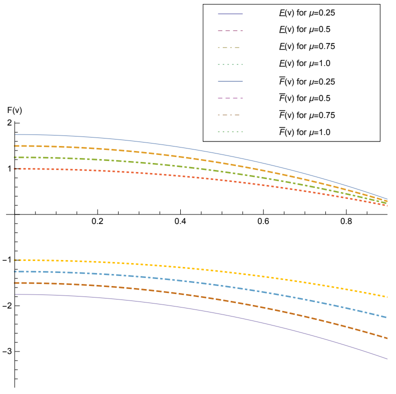

Figure 3. Fuzzy approximate solution for various ε.

$G(S(q)) = F^3(q)$ where

$$\underline{H} = (1-\varepsilon)p^3 + \frac{(1023(-1+\varepsilon)^3(-1+p)p^{10})}{10737418240} - \frac{(1073733109(-1+\varepsilon)^3 p^{11})}{118111600640},$$

$$\overline{H} = (2+\varepsilon)p^3 - \frac{1023(2+\varepsilon)^3(-1+p)p^{10}}{10737418240} + \frac{1073733109(2+\varepsilon)^3 p^{11}}{118111600640},$$

and the exact solution $(\underline{U}(p), \overline{U}(p)) = ((1-\varepsilon)p^3, (\varepsilon+2)p^3)$. The comparative graphs between the exact and approximate solutions $(\underline{U}_{20}(p), \overline{U}_{20}(p))$ have been presented in Fig. 4 for $\varepsilon = 0.5$. Fig. 5 shows the absolute errors for both underline and overline cases. Fig. 6 demonstrates the approximate solutions $(\underline{U}_{20}(p), \overline{U}_{20}(p))$ for various ε. In order to show efficiency and accuracy of the method, we have compared the successive approximation method with the traditional homotopy analysis method. The results have been shown in Table 2.

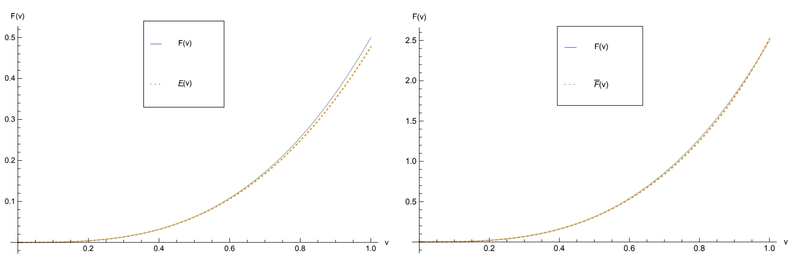

Figure 4. Comparison between the exact solution $(\underline{U}(p), \overline{U}(p))$ and approximate solution $(\underline{U}_{20}(p), \overline{U}_{20}(p))$ for $\varepsilon = 0.5$.

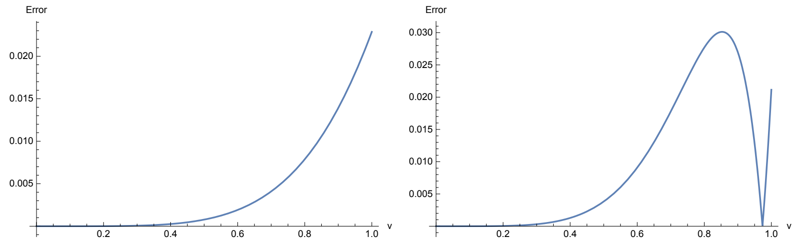

Figure 5. The absolute error for $(\underline{U}_{20}(p), \overline{U}_{20}(p))$ and $\varepsilon = 0.5$.

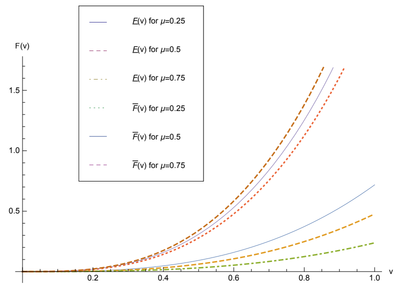

Figure 6. Fuzzy approximate solution for various ε.

Table 2

The error results of Example 2.

p	$\lvert\underline{U}(p) - \underline{U}_{20}(p)\rvert$	$\lvert\overline{U}(p) - \overline{U}_{20}(p)\rvert$	$\lvert\underline{U}(p) - \underline{U}_{HAM}(p)\rvert$	$\lvert\overline{U}(p) - \overline{U}_{HAM}(p)\rvert$
0.00	0	0	0	0
0.25	0.0000253047	0.000126491	0.0000521732	0.000358749
0.50	0.00077713	0.00381917	0.00084267	0.00417518
0.75	0.00574238	0.0229733	0.0079563421	0.05269427
1.00	0.0228692	0.0211465	0.024871277	0.02565382

6. Conclusion

In this work, the fuzzy Volterra integral equation of the second kind with piecewise kernel was studied.We applied the successive approximation scheme. This is the first time that the method has been implemented for solving this problem. The existence of an unique solution with the error bound and also the error estimation theorems were discussed. Some examples have been solved. Plotting the graphs of fuzzy approximate solutions for various ε and error functions we showed the accuracy of the method. Also the method has been compared with the traditional homotopy analysis method and we can see that the method is more accurate than the HAM. As the limitations of the method, generally the iterative methods are not fast thus when we need to make more iterations we need more time. Also for solving nonlinear problems if we have special and complicated nonlinear terms, applying the successive approximation method will not be easy. As our future works, we will combine the method with the CESTAC-CADNA strategy to find the numerical optimality results and optimal distance.

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