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Variations of Rigidity for Ordered Theories

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Abstract. One of the important characteristics of structures is degrees of semantic and syntactic rigidity, as well as indices of rigidity, showing how much the given structure differs from semantically rigid structures, i.e., structures with one-element automorphism groups, as well as syntactically rigid structures, i.e., structures covered by definable closure of the empty set. Issues of describing the degrees and indices of rigidity represents interest both in a general context and in relation to ordering theories and their models. In the given paper, we study possibilities for semantic and syntactic rigidity for ordered theories, i.e., the rigidity with respect to automorphism group and with respect to definable closure. We describe values for indices and degrees of semantic and syntactic rigidity for well-ordered sets, for discrete, dense, and mixed orders and for countable models of $\aleph_0$-categorical weakly o-minimal theories. All possibilities for degrees of rigidity for countable linear orderings are described.

Keywords: definable closure, semantic rigidity, syntactic rigidity, degree of rigidity, ordered theory

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Научная статья

Вариации жесткости для упорядоченных теорий

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Аннотация. Утверждается, что одними из важных характеристик структур являются степени семантической и синтаксической жесткости, а также индексы жесткости, показывающие насколько данная структура отличается от семантически жесткой структуры, т.е. структуры с одноэлементной группой автоморфизмов, а также от синтаксически жесткой структуры, т. е. структуры, накрываемой определимым замыканием пустого множества. Вопросы описания степеней и индексов жесткости представляют интерес как в общем контексте, так и применительно к упорядоченным теориям и их моделям. Изучены возможности семантической и синтаксической жесткости упорядоченных теорий, т. е. жесткости по отношению к группе автоморфизмов и по отношению к определимому замыканию. Описаны значения индексов и степеней семантической и синтаксической жесткости для вполне упорядоченных множеств, для дискретных, плотных и смешанных порядков, а также для счетных моделей $\aleph_0$-категоричных слабо о-минимальных теорий. Отмечены все возможности степеней жесткости для счетных линейных порядков.

Ключевые слова: определимое замыкание, семантическая жесткость, синтаксическая жесткость, степень жесткости, упорядоченная теория

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We continue to study variations of algebraic closures [14; 15] considering and describing semantic and syntactic possibilities for definable closures. A general approach studying algebraic and definable characteristics, in particular, variations of rigidity is applied for ordered theories.

We use the standard model-theoretic terminology [6; 10; 11; 13; 16], notions and notations in [14; 15].

The paper is organized as follows. In Section 1, preliminary notions, notations and assertions are collected, as well as values for indexes of rigidity for linearly ordered structures are described. In Section 2, values of rigidity degrees for well-ordered sets and some their modifications are described. In Sections 3 and 4, we describe rigidity characteristics for discrete and dense orders, respectively. In Section 5, rigidity characteristics for countable models of $\aleph_0$-categorical weakly o-minimal theories are found. In Section 6, rigidity characteristics for mixed, discrete–dense orders are described, and Theorem describing possibilities for degrees of rigidity for countable linear orderings is proved.

1. Preliminaries and indices of rigidity for ordered structures

Let L be a countable first-order language. Throughout the paper we consider L-structures and their complete elementary theories, and assume that L contains a symbol of binary relation $<$, which is interpreted as a linear order in these structures.

Definition 1. [15]. For a set A in a structure $\mathcal{M}$, $\mathcal{M}$ is called *semantically A-rigid* or *automorphically A-rigid* if any A-automorphism $f \in \mathrm{Aut}(\mathcal{M})$ is identical. The structure $\mathcal{M}$ is called *syntactically A-rigid* if $M = \mathrm{dcl}(A)$.

Obviously, if $\mathcal{M}$ is an arbitrary structure, $\mathcal{M}$ is both semantically M-rigid and syntactically M-rigid. Also, $\mathcal{M}$ is syntactically A-rigid for any $A \subseteq M$ with $M \setminus \mathrm{dcl}(\emptyset) \subseteq A$. If $\mathcal{M}$ is an arbitrary infinite linearly ordered structure, $\mathcal{M}$ is semantically A-rigid for any co-finite $A \subseteq M$.

A structure $\mathcal{M}$ is called *$\forall$-semantically / $\forall$-syntactically n-rigid* (respectively, *$\exists$-semantically / $\exists$-syntactically n-rigid*), for $n \in \omega$, if $\mathcal{M}$ is semantically / syntactically A-rigid for any (some) $A \subseteq M$ with $|A| = n$.

The least n such that $\mathcal{M}$ is Q-semantically / Q-syntactically n-rigid, where $Q \in \{\forall, \exists\}$, is called the *$Q$-semantical / Q-syntactical degree of rigidity*, it is denoted by $\deg_{\mathrm{rig}}^{Q\text{-sem}}(\mathcal{M})$ and $\deg_{\mathrm{rig}}^{Q\text{-synt}}(\mathcal{M})$, respectively. Here if a set A produces the value of Q-semantical / Q-syntactical degree then we say that A *witnesses* that degree. If such n does not exist we put $\deg_{\mathrm{rig}}^{Q\text{-sem}}(\mathcal{M}) = \infty$ and $\deg_{\mathrm{rig}}^{Q\text{-synt}}(\mathcal{M}) = \infty$, respectively.

Definition 2. [15]. For a set A in $\mathcal{M}$ and an expansion $\mathcal{M}_A$ of $\mathcal{M}$ by constants in A, the least n such that $\mathcal{M}_A$ is Q-semantically / Q-syntactically n-rigid, where $Q \in \{\forall, \exists\}$, is called the *$(Q, A)$-semantical / (Q, A)-syntactical degree of rigidity*, it is denoted by $\deg_{\mathrm{rig},A}^{Q\text{-sem}}(\mathcal{M})$ and $\deg_{\mathrm{rig},A}^{Q\text{-synt}}(\mathcal{M})$, respectively. If such n does not exist we put $\deg_{\mathrm{rig},A}^{Q\text{-sem}}(\mathcal{M}) = \infty$ and

$$\deg_{\mathrm{rig},A}^{Q\text{-synt}}(\mathcal{M}) = \infty,$$

respectively.

Any expansion $\mathcal{M}_A$ of $\mathcal{M}$ with $\deg^{\exists\text{-}s}_{\rm rig}(\mathcal{M}_A)=0$, for $s\in\{\rm sem, synt\}$, is called a *s-rigiditization* or simply a *rigiditization* of $\mathcal{M}$.

Following [15] for a structure $\mathcal{M}$ we denote by $\deg_4(\mathcal{M})$ the tetrad

$$\left(\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M}), \deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M}), \deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{M}), \deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M})\right).$$

Remark 1. If $\mathcal{M}$ is a structure of pure linear order then for the dual structure $\mathcal{M}^*$, which is antiisomorphic to $\mathcal{M}$, $\deg_4(\mathcal{M}^*)=\deg_4(\mathcal{M})$. It is satisfied since the duality moves the automorphism group to the isomorphic one, and it preserves the set of formulae witnessing the definable closure.

Fact 1. [15]. *Let $\mathcal{M}$ be an arbitrary structure. Then*
1. $\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M})\le\deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{M})$.
2. $\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})\le\deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M})$.
3. $\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M})\le\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$.
4. $\deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{M})\le\deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M})$.
5. $\deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{M})=0$ *iff* $\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M})=0$.
6. $\deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M})=0$ *iff* $\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})=0$.

Definition 3. [15]. For a set A in a structure $\mathcal{M}$ the *index of rigidity* of $\mathcal{M}$ over A, denoted by $\mathrm{ind}_{\rm rig}(\mathcal{M}/A)$, is the supremum of cardinalities for the sets of solutions of algebraic types $\mathrm{tp}(a/A)$ for $a\in M$. We put $\mathrm{ind}_{\rm rig}(\mathcal{M})=\mathrm{ind}_{\rm rig}(\mathcal{M}/\emptyset)$. Here we assume that $\mathrm{ind}_{\rm rig}(\mathcal{M})=0$ if $\mathcal{M}$ does not have algebraic types $\mathrm{tp}(a)$ for $a\in M$.

Proposition 1. *For any linearly ordered structure $\mathcal{M}$, either* $\mathrm{ind}_{\rm rig}(\mathcal{M})=0$ *or* $\mathrm{ind}_{\rm rig}(\mathcal{M})=1$. *If $\emptyset\ne A\subseteq M$ then* $\mathrm{ind}_{\rm rig}(\mathcal{M}/A)=1$.

Proof. Let $\mathcal{M}$ be linearly ordered with the order $<$. Then any algebraic type has a unique solution. Thus, either $\mathrm{ind}_{\rm rig}(\mathcal{M})=0$, if $\mathcal{M}$ does not have algebraic types, or $\mathrm{ind}_{\rm rig}(\mathcal{M})=1$, if these types exists. If $\emptyset\ne A\subseteq M$ then an algebraic type $\mathrm{tp}(a/A)$ exists, taking arbitrary $a\in A$. Therefore, $\mathrm{ind}_{\rm rig}(\mathcal{M}/A)=1$.

2. Rigidity characteristics and their values for well-ordered sets and some their modifications

Lemma 1. *If $\mathcal{M}=\langle M,<\rangle$ is a well-ordered set, $A\subseteq M$ is $\emptyset$-definable then any its finite initial segment is contained in* $\mathrm{dcl}(\emptyset)$.

Proof. Let $\varphi(x)$ be a formula without parameters defining A. Since $\mathcal{M}$ is well-ordered, its restriction $\mathcal{M}|_A$ to the set A is well-ordered, too. Then each formula

$$\psi_n(x) = \exists x_1, \ldots, \exists x_n [\bigwedge_{i=1}^{n} \varphi(x_i) \wedge \forall y(\varphi(y) \to x_1 \le y) \wedge \bigwedge_{i=1}^{n-1} \{x_i < x_{i+1} \wedge$$

$$\wedge \forall y(\varphi(y) \wedge x_i \le y \le x_{i+1} \to y \approx x_i \vee y \approx x_{i+1}\} \wedge x_n \approx x]$$

expresses n-th element of A, $n \in \omega$, and its solution a_n, if it exists, is contained in $\mathrm{dcl}(\emptyset)$. Clearly, the sets $\{a_0, \ldots, a_n\}$ form finite initial segments, as required.

Notice that if $\mathcal{M}$ is isomorphic, by an isomorphism f, to a non-limit ordinal $\alpha + n$ then $\{f^{-1}(\alpha), f^{-1}(\alpha+1), \ldots, f^{-1}(\alpha + (n-1))\} \subseteq \mathrm{dcl}(\emptyset)$. Indeed, all elements $f^{-1}(\alpha + i)$ are defined by formulae describing the number of predecessors from the largest element $f^{-1}(\alpha + n)$. Since these elements are $\emptyset$-definable, by Lemma 1 we conclude:

Corollary 1. *If $\mathcal{M}$ is a well-ordered structure isomorphic to a non-limit ordinal then both the element corresponding to the largest limit ordinal in $\mathcal{M}$ and its successors belong to* $\mathrm{dcl}(\emptyset)$.

Lemma 2. *If $\mathcal{M} = \langle M, < \rangle$ is a well-ordered set then* $\mathrm{dcl}(\emptyset)$ *consists of all finite initial segments of $\emptyset$-definable subsets in $\mathcal{M}$.*

Proof. Let Z be the union of all finite initial segments of $\emptyset$-definable subsets in $\mathcal{M}$. By Lemma 1 we have $Z \subseteq \mathrm{dcl}(\emptyset)$. Conversely, any element $a \in \mathrm{dcl}(\emptyset)$ forms the $\emptyset$-definable singleton $\{a\}$ which is contained in Z by the definition. Thus, $Z = \mathrm{dcl}(\emptyset)$.

Corollary 2. *For any well-ordered set $\mathcal{M} = \langle M, < \rangle$ if $\mathcal{M}$ consists of finite initial segments of $\emptyset$-definable sets then* $\deg_4(\mathcal{M}) = (0, 0, 0, 0)$.

Proof. It is known [4] that well-ordered sets do not have non-identical automorphisms. Therefore, $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathcal{M}) = 0$. We have $\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathcal{M}) = 0$ in view of Lemma 2. Thus, $\deg_4(\mathcal{M}) = (0, 0, 0, 0)$.

Remark 2. Let $\mathcal{M}$ be a well-ordered L-structure with a well order $<$. For a L-formula $\varphi = \varphi(x)$ we define the formula

$$\psi_\varphi(x) = \varphi(x) \wedge \forall y\,(y < x \wedge \varphi(x) \to \exists z(y < z < x \wedge \varphi(z)))$$

saying that for any realization a of φ either a is a minimal element satisfying φ or it is not minimal and there are densely many smaller elements satisfying φ, i.e. a does not have predecessors with respect to φ. We also consider the L-formula

$$\theta(x) = \forall y(y < x \to \exists z(y < z < x))$$

defining the set of all elements without predecessors.

Now we define a sequence $(\varphi_n(x))_{n\in\omega}$ of formulae such that $\varphi_0(x) = \theta(x)$ and $\varphi_{n+1}(x) = \psi_{\varphi_n}(x)$, $n \in \omega$. Using these formulae we obtain that:

i) the first initial segment in $\mathcal{M}$ consisting of the least element and all its successors are contained in $\mathrm{dcl}(\emptyset)$;

ii) finite initial segments of $\varphi_n(\mathcal{M})$ are contained in $\mathrm{dcl}(\emptyset)$, $n \in \omega$.

In particular, the ordinals k, $\omega^l \cdot m$, for $k, l, m \in \omega$, and their well-ordered finite sums are contained in their $\mathrm{dcl}(\emptyset)$.

Theorem 1. *For any well-ordered set $\mathcal{M} = \langle M, <\rangle$ either $\deg_4(\mathcal{M}) = (0,0,0,0)$, if $\mathcal{M}$ is at most countable, or $\deg_4(\mathcal{M}) = (0,\infty,0,\infty)$, if $\mathcal{M}$ is uncountable.*

Proof. By the argument for Corollary 2 we have

$$\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathcal{M}) = 0$$

for any well-ordered set $\mathcal{M} = \langle M, <\rangle$.

Let $\mathcal{M}$ be at most countable. Then there exists $k < \omega$ such that $\mathcal{M}$ has the ordering type $\omega^k \cdot l_1 + \omega^{k-1} \cdot l_2 + \ldots + \omega \cdot l_k + m$ for some $l_1, l_2, \ldots, l_k, m \in \omega$ [8]. In view of Remark 2 we have $\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathcal{M}) = 0$ implying $\deg_4(\mathcal{M}) = (0,0,0,0)$.

If $\mathcal{M}$ is uncountable then $\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathcal{M}) = \infty$ since the definable closures of finite sets can not cover M as there are countably many formulae using finitely many fixed constants. Thus, in such a case $\deg_4(\mathcal{M}) = (0,\infty,0,\infty)$.

In view of Remark 1 the dichotomy in Theorem 1 is preserved under transformations of well-ordered sets $\mathcal{M}$ to dual ones $\mathcal{M}^*$. Moreover, it is preserved under the sum $\mathcal{M} + \mathcal{N}^*$ for well-ordered sets $\mathcal{M}$ and $\mathcal{N}$:

Corollary 3. *For any well-ordered sets $\mathcal{M}$ and $\mathcal{N}$ either $\deg_4(\mathcal{M}+\mathcal{N}^*) = (0,0,0,0)$, if $\mathcal{M}+\mathcal{N}^*$ is at most countable, or $\deg_4(\mathcal{M}+\mathcal{N}^*) = (0,\infty,0,\infty)$, if $\mathcal{M}+\mathcal{N}^*$ is uncountable.*

Example 1. By Theorem 1, Remark 1 and Corollary 3,

$$\deg_4(\omega) = \deg_4(\omega^*) = \deg_4(\omega + \omega^*) = (0,0,0,0).$$

At the same time, for $\mathbb{Z} = \omega^* + \omega$, $\deg_4(\mathbb{Z}) = (1,1,1,1)$, since $\mathrm{dcl}(\emptyset) = \emptyset$, the automorphism group $\mathrm{Aut}(\mathbb{Z})$ is transitive, $\mathrm{dcl}(\{a\}) = \mathbb{Z}$ and $\mathbb{Z}_{\{a\}}$ is semantically rigid for any $a \in \mathbb{Z}$.

3. Rigidity characteristics for discrete orders

In this section we consider some discrete orders different from well-ordered and dual ones.

Let $\mathcal{M} = \langle \mathbb{Z} \cdot n, < \rangle$ for $n \in \omega \setminus \{0\}$. We have

$$\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M}) = n$$

since $\mathcal{M}$ is rigid with respect to finite sets containing elements in each copy of $\mathbb{Z}$. Besides, as noticed above, $\deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathcal{M}) = 1$ for $n = 1$. At the same time, for $n \geq 2$, $\deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathcal{M}) = \infty$ since elements in a copy of $\mathbb{Z}$ do not define elements of other copies. Thus we obtain either $\deg_4(\mathcal{M}) = (1, 1, 1, 1)$ or $\deg_4(\mathcal{M}) = (n, n, \infty, \infty)$, for $n \in \omega \setminus \{0, 1\}$.

Taking a pure linearly ordered structure $\mathcal{M}$ as a sum of infinitely many copies of $\mathbb{Z}$ we obtain $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$ since finite sets A in $\mathcal{M}$ do not fix automorphisms for copies of $\mathbb{Z}$ which do not contain elements in A.

Thus we obtain the following:

Theorem 2. *For any disjoint sum $\mathcal{M}$ of copies of $\mathbb{Z}$ the following possibilities hold:*

1) $\deg_4(\mathcal{M}) = (1, 1, 1, 1)$, *if* $\mathcal{M} = \mathbb{Z}$;

2) $\deg_4(\mathcal{M}) = (n, n, \infty, \infty)$, *if* $\mathcal{M} = \mathbb{Z} \cdot n$ *for* $n \in \omega \setminus \{0, 1\}$;

3) $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$, *if* $\mathcal{M}$ *consists of infinitely many copies of* $\mathbb{Z}$.

Remark 3. The values of $\deg_4(\mathcal{M})$ in Theorem 2 are preserved if the set of components $\mathbb{Z}$ for $\mathcal{M}$ are extended by finitely many finite linear orders, say $\deg_4(\mathbb{Z} + m + \mathbb{Z}) = (2, 2, \infty, \infty)$ for any $m \in \omega$.

Remark 4. The characteristics $\deg_4(\mathcal{M})$ in Theorem 2 give the lower bounds for orders containing sums for copies of $\mathbb{Z}$. For instance, if $\mathcal{M} = \mathcal{M}_1 + \mathbb{Z} + \mathbb{Z} + \mathcal{M}_2$ for some linear orders $\mathcal{M}_1$, $\mathcal{M}_2$ then $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M}) \geq 2$, $\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M}) \geq 2$,$\deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathcal{M}) = \infty$, $\deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathcal{M}) = \infty$.

Lemma 3. *For any natural $m \geq 1$ there exists an infinite linear ordering $\mathcal{M} = \langle M, < \rangle$ such that* $\deg_4(\mathcal{M}) = (m, m, \infty, \infty)$.

Proof of Lemma 3. Consider the following infinite linear ordering for any natural $m \geq 1$:

$$\mathcal{M} = \langle \omega + \underbrace{\mathbb{Z} + \ldots + \mathbb{Z}}_{m \text{ times}}, < \rangle.$$

Obviously, $\deg_4(\mathcal{M}) = (m, m, \infty, \infty)$.

Lemma 4. *For any natural* $m \geq 1$ *there exists an infinite linear ordering* $\mathcal{M} = \langle M, < \rangle$ *such that* $\deg_4(\mathcal{M}) = (1, 1, m, m)$.

Proof of Lemma 4. If $\mathcal{M} = \langle \mathbb{Z}, < \rangle$ then by Theorem 2 $\deg_4(\mathcal{M}) = (1, 1, 1, 1)$. Consider the following infinite linear ordering for any natural $m \geq 1$:

$$\mathcal{M} = \langle m + \mathbb{Z}, < \rangle.$$

Obviously, $\deg_4(\mathcal{M}) = (1, 1, m + 1, m + 1)$.

4. Rigidity characteristics for dense orders

Let $\mathcal{M} = \langle \mathbb{Q}, < \rangle$. Clearly, for an arbitrary $A \subseteq \mathbb{Q}$ the structure $\mathcal{M}$ is syntactically A-rigid iff $A = \mathbb{Q}$. Moreover, for any finite $A \subset \mathbb{Q}$, $\mathrm{dcl}(A) = A \neq \mathbb{Q}$. Therefore we have the following:

Proposition 2. *For the structure* $\mathcal{M} = \langle \mathbb{Q}, < \rangle$,

$$\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathcal{M}) = \infty.$$

Now we consider values $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M})$ and $\deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathcal{M})$ based on the automorphism group $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$. Notice that this group and its properties are studied in [2; 3; 5; 17].

Taking a finite subset $A \subset \mathbb{Q}$ we have a dense part (in fact, infinitely many ones) in $\mathbb{Q} \setminus A$ producing many A-automorphisms $f \in \mathrm{Aut}(\mathcal{M})$. It implies that there are non-identical A-automorphisms. Thus we have the following:

Proposition 3. *For the structure* $\mathcal{M} = \langle \mathbb{Q}, < \rangle$,

$$\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathcal{M}) = \infty.$$

Propositions 2 and 3 immediately imply the following:

Corollary 4. $\deg_4(\langle \mathbb{Q}, < \rangle) = (\infty, \infty, \infty, \infty)$.

Since sums of linear orders with $\mathbb{Q}$ preserve the definable closures and automorphisms on $\mathbb{Q}$ we have:

Corollary 5. *If* $\mathcal{M} = \mathcal{M}_1 + \mathbb{Q} + \mathcal{M}_2$ *for some linear orders* $\mathcal{M}_1$, $\mathcal{M}_2$ *then* $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$.

Since there are many A-automorphisms if $\mathbb{Q} \setminus A$ has an infinite convex set, we have also the following:

Corollary 6. *If* $\mathbb{Q} \setminus A$ *has an infinite convex subset then* $\deg_4(\langle \mathbb{Q}, < \rangle_A) = (\infty, \infty, \infty, \infty)$.

In view of Corollary 6 the description of values $\deg_4(\langle\mathbb{Q},<\rangle_A)$ is reduced to the case when $\mathbb{Q}\setminus A$ does not have infinite convex sets, i.e., if all elements in $\mathbb{Q}\setminus A$ are isolated. In such a case $\mathbb{Q}\setminus A$ is either finite or countable and we obtain the following possibilities:

1. $\deg_4(\langle\mathbb{Q},<\rangle_A)=(0,n,0,\infty)$, if $|\mathbb{Q}\setminus A|=n\in\omega$ (since $\langle\mathbb{Q},<\rangle_A$ does not have non-identical automorphisms, and we obtain $\mathrm{dcl}(\mathbb{Q}\setminus A)=\mathbb{Q}$, whereas the definable closures of finite subsets of A do not cover $\mathbb{Q}$).

2. $\deg_4(\langle\mathbb{Q},<\rangle_A)=(0,\infty,0,\infty)$, if $|\mathbb{Q}\setminus A|=\omega$ (since $\langle\mathbb{Q},<\rangle_A$ does not have non-identical automorphisms, and the definable closures of finite sets do not cover $\mathbb{Q}$).

Collecting the described possibilities we obtain:

Theorem 3. *For any subset $A\subseteq\mathbb{Q}$ the following holds:*

(1) $\deg_4(\langle\mathbb{Q},<\rangle_A)=(\infty,\infty,\infty,\infty)$ *iff* $\mathbb{Q}\setminus A$ *has an infinite convex subset;*

(2) $\deg_4(\langle\mathbb{Q},<\rangle_A)=(0,n,0,\infty)$ *iff* $|\mathbb{Q}\setminus A|=n\in\omega\setminus\{0\}$*;*

(3) $\deg_4(\langle\mathbb{Q},<\rangle_A)=(0,\infty,0,\infty)$ *iff* $|\mathbb{Q}\setminus A|=\omega$ *and* $\mathbb{Q}\setminus A$ *has no infinite convex subsets;*

(4) $\deg_4(\langle\mathbb{Q},<\rangle_A)=(0,0,0,0)$ *iff* $\mathbb{Q}=A$.

5. Rigidity characteristics for $\aleph_0$-categorical weakly o-minimal structures

An *open interval* in a linearly ordered structure $\mathcal{M}$ is a parametrically definable subset of M of the form $I=\{c\in M:\mathcal{M}\models a<c<b\}$ for some $a,b\in M\cup\{-\infty,\infty\}$ with $a<b$. Similarly, we may define *closed, half open-half closed*, etc., *intervals* in $\mathcal{M}$. An arbitrary point $a\in M$ we can also represent as an interval $[a,a]$. By an *interval* in $\mathcal{M}$ we shall mean, ambiguously, any of the above types of intervals in $\mathcal{M}$. A subset A of a linearly ordered structure $\mathcal{M}$ is *convex* if for any $a,b\in A$ and $c\in M$ whenever $a<c<b$ we have $c\in A$.

This section deals with the notion of *weak o-minimality*, which initially deeply studied by H.D. Macpherson, D. Marker, and C. Steinhorn in [9]. A *weakly o-minimal structure* is a linearly ordered structure $\mathcal{M}=\langle M,=,<,\ldots\rangle$ such that any definable (with parameters) subset of the structure $\mathcal{M}$ is a finite union of convex sets in $\mathcal{M}$. Recall that such a structure $\mathcal{M}$ is said to be *o-minimal* if any definable (with parameters) subset of M is a union of finitely many intervals and points in $\mathcal{M}$. Thus, weak o-minimality generalizes the notion of o-minimality. Real closed fields with a proper convex valuation ring provide an important example of weakly o-minimal (not o-minimal) structures.

Let T be a weakly o-minimal theory, $\mathcal{M}\models T$, $A\subseteq M$, $p,q\in S_1(A)$ be non-algebraic. We say that p is not *weakly orthogonal* to q (denoting this by

$p \not\perp^w q$) if there exist an L_A-formula $H(x,y)$, $\alpha \in p(\mathcal{M})$ and $\beta_1, \beta_2 \in q(\mathcal{M})$ such that $\beta_1 \in H(\mathcal{M}, \alpha)$ and $\beta_2 \notin H(\mathcal{M}, \alpha)$.

In other words, p is *weakly orthogonal* to q (denoting this by $p \perp^w q$) if $p(x) \cup q(y)$ has a unique extension to a complete 2-type over A.

Lemma 5. [1] *Let T be a weakly o-minimal theory, $\mathcal{M} \models T$, $A \subseteq M$. Then the relation of non-weak orthogonality $\not\perp^w$ is an equivalence relation on $S_1(A)$.*

Proposition 4. *Let T be an $\aleph_0$-categorical o-minimal theory, $\mathcal{M} \models T$. Suppose that $p \not\perp^w q$ for any non-algebraic $p, q \in S_1(\emptyset)$. Then $\mathcal{M}$ is syntactically A-rigid for any $A \subseteq M$ containing the set of realizations of an arbitrary non-algebraic $p \in S_1(\emptyset)$.*

Proof of Proposition 4. Firstly, by the $\aleph_0$-categoricity of T the definable closure of the empty set is finite, and there are only finitely many non-algebraic 1-types over $\emptyset$. Also, if $p \not\perp^w q$ for some $p, q \in S_1(\emptyset)$, there is a unique $\emptyset$-definable strictly monotonic bijection between $p(\mathcal{M})$ and $q(\mathcal{M})$, whence $\mathrm{dcl}(A) = M$.

The following example shows that Proposition 4 is not true for $\aleph_0$-categorical weakly o-minimal theories in general.

Example 2. [9] Let $\mathcal{M} = \langle M; <, P_1^1, P_2^1, f^1 \rangle$ be a linearly ordered structure such that M is a disjoint union of the interpretations of unary predicates P_1 and P_2, where $P_1(\mathcal{M}) < P_2(\mathcal{M})$. We identify the interpretation of P_2 with the set of rational numbers $\mathbb{Q}$, ordered as usual, and the interpretation of P_1 with $\mathbb{Q} \times \mathbb{Q}$, lexicographically ordered. The symbol f is interpreted by a partial unary function with $Dom(f) = P_1(\mathcal{M})$ and $Range(f) = P_2(\mathcal{M})$ and defines by the equality $f((n,m)) = n$ for all $(n,m) \in \mathbb{Q} \times \mathbb{Q}$.

It can be proved that $\mathrm{Th}(\mathcal{M})$ is a weakly o-minimal (not o-minimal) theory. Let $p(x) := \{P_1(x)\}$, $q(x) := \{P_2(x)\}$. Obviously, $p, q \in S_1(\emptyset)$, $p \not\perp^w q$, and there are no other non-algebraic 1-types over $\emptyset$, i.e., the hypothesis that $p \not\perp^w q$ for any non-algebraic $p, q \in S_1(\emptyset)$ holds. But if we take the set A as the set of realizations of q, we have that $\mathcal{M}$ is not syntactically A-rigid.

Proposition 5. *There exists an $\aleph_0$-categorical weakly o-minimal theory T such that $p \not\perp^w q$ for any non-algebraic $p, q \in S_1(\emptyset)$ and for any $M \models T$ there are $A \subseteq M$ and $p \in S_1(\emptyset)$ with $p(\mathcal{M}) \subseteq A$ so that $\mathcal{M}$ is not syntactically A-rigid.*

Proposition 6. *Let T be an $\aleph_0$-categorical o-minimal theory, $\mathcal{M} \models T$. Then there exist $k < \omega$ and pairwise weakly orthogonal non-algebraic $p_1, p_2, \ldots, p_k \in S_1(\emptyset)$ such that $A = p_1(\mathcal{M}) \cup p_2(\mathcal{M}) \cup \ldots \cup p_k(\mathcal{M})$ and $\mathcal{M}$ is syntactically A-rigid.*

Proof of Proposition 6. Since there are only finitely many non-algebraic 1-types over $\emptyset$, any family of pairwise weakly orthogonal non-algebraic 1-types over $\emptyset$ is also finite.

Theorem 4. *Let T be an $\aleph_0$-categorical weakly o-minimal theory. Then* $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$ *for any countable* $\mathcal{M} \models T$.

Proof of Theorem 4. For any $\mathcal{M} \models T$ and any finite set $A \subset M$ there exists at least one non-algebraic $p \in S_1(A)$ such that $p(\mathcal{M})$ is a densely ordered convex set. It implies that neither A can cover M by $\mathrm{dcl}(A)$ nor produce a singleton $\mathrm{Aut}(\mathcal{M}_A)$. Thus $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$.

6. Rigidity characteristics for mixed orders

In this section we consider rigidity characteristics for *mixed* orders, i.e., dense orders composed by discrete parts, where discrete parts replace elements of dense orders.

In view of Theorem 2 if a linearly ordered set $\mathcal{M}$ has infinitely many copies of $\mathbb{Z}$ then $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$. Besides, each additional copy of $\mathbb{Z}$ increases finite values of both $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M})$ and $\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M})$ by one, such that $\mathbb{Z}$ with infinite complement and positive $\deg_{\mathrm{rig}}^{\forall\text{-sem}}$ and $\deg_{\mathrm{rig}}^{\forall\text{-synt}}$ gives $\deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathcal{M}) = \infty$. So describing $\deg_4(\mathcal{M})$ it suffices to consider mixed orders without copies of $\mathbb{Z}$, i.e., discrete parts consisting of finite linear orders only.

Proposition 7. *If $\mathcal{M}$ is a countable mixed ordered set without parts $\mathbb{Z}$ and with maximal finite discrete parts of bounded lengths then* $\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathcal{M}) = \infty$. *If additionally $\mathcal{M}$ is homogeneous then* $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$.

Proof. Since $\mathcal{M}$ does not have parts $\mathbb{Z}$ and maximal finite discrete parts have bounded lengths $\mathcal{M}$ contains a dense suborder S whose maximal finite parts have same lengths and the quotient by these parts is isomorphic to $\mathbb{Q}$. It implies that no finite family A of finite discrete parts can not define all elements of $\mathcal{M}$. Indeed, any finite family A defines elements in finite parts such that these parts are situated distinctly with respect to other parts. Since these parts are finite, it means that there are distinct finite possibilities of mixtures of these parts. But by the conjecture there are finitely many isomorphism types for finite parts and S is dense, configurations for the mixtures describing distinct parts should be repeated for distinct parts $P_1, P_2 \in S$ with respect to A, i.e., $\mathrm{tp}(P_1/A) = \mathrm{tp}(P_2/A)$, implying $P_1 \cup P_2 \not\subseteq \mathrm{dcl}(A)$.

It is checked by routine considerations of cases. For instance, let A consists of two parts P_0, P_0' and $R_1, R_2 \in S$ with $P_0, < R_1 < R_2 < P_0'$ and there are k_i pairwise isomorphic parts $U_j \not\simeq R_i$ between P_0 and R_i, $i = 1, 2$, $k_1 < k_2$, and k_i' pairwise isomorphic parts $U'_{j'} \not\simeq R_i$ between R_i and P_0', $i = 1, 2$, $k_1' > k_2'$, $U_j \simeq U'_{j'}$. Taking elements V of S between R_1 and R_2 we have both finitely many possibilities for k_1, k_2, k_1', k_2' and infinitely many V. Thus S contains distinct parts P_1, P_2 between R_1, R_2 with same number of copies of U_j and $U'_{j'}$ with respect to A. Considering similar finiteness conditions we choose distinct parts $P_1, P_2 \in S$ with $\mathrm{tp}(P_1/A) = \mathrm{tp}(P_2/A)$.

Thus we obtain $\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M}) = \infty$. By Fact 1 we have $\deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathcal{M}) = \infty$, too.

If $\mathcal{M}$ is homogeneous then for any finite $A \subset M$ there are many A-automorphisms for S implying $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M}) = \deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathcal{M}) = \infty$ and $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$.

The following example shows that maximal finite discrete parts of unbounded lengths can produce rigid mixed linearly ordered sets $\mathcal{M}$, i.e., with $\deg_4(\mathcal{M}) = (0, 0, 0, 0)$.

Example 3. Let C_n, $n \in \omega$, be disjoint finite linear orders of pairwise distinct lengths. Now we enumerate the order $\mathbb{Q}$: $\mathbb{Q} = \{a_n \mid n \in \omega\}$ and replace each a_n by C_n. The obtained linearly ordered set $\mathcal{M}$ is required. Indeed, we have both $\mathrm{dcl}(\emptyset) = M$ since each part C_n is defined by its length. We have also $|\mathrm{Aut}(\mathcal{M})| = 1$ since no elements in $\mathcal{M}$ can not be moved into another one: any two distinct elements in $\mathcal{M}$ have distinct types, as two distinct elements in one part C_n have distinct distances from the least element of C_n, and elements in distinct parts C_m and C_n defines their distinct cardinalities. Thus, in view of Fact 1, $\deg_4(\mathcal{M}) = (0, 0, 0, 0)$.

The $\aleph_0$-categorical linear orders were classified by Joseph Rosenstein in [12], where he constructed them from finite linear orders using two operations.

Definition 4. $\langle \mathbb{Q}_n, <_{\mathbb{Q}_n}, C_1^1, \ldots, C_n^1 \rangle$ is *the Fraïssé generic n-colored linear order*, i.e. the countable dense linear order with n colors which occur interdensely (for all x and y there are $z_1, \ldots, z_n$ between x and y such that $C_i(z_i)$ holds for each i).

Definition 5. Let $\langle L_1, <_1 \rangle, \ldots, \langle L_n, <_n \rangle$ be linear orders. For each $q \in \mathbb{Q}_n$ we define $L(q)$ to be a copy of $\langle L_i, <_i \rangle$ if $\mathbb{Q}_n \models C_i(q)$. The $\mathbb{Q}_n$-*shuffle* of $\langle L_1, <_1 \rangle, \ldots, \langle L_n, <_n \rangle$, denoted by $\mathbb{Q}_n(L_1, \ldots, L_n)$, is the linear order $\langle \bigcup_{q \in \mathbb{Q}_n} L(q), < \rangle$, where

$$a < b \text{ iff } ([a, b \in L(q) \wedge a <_i b] \text{ or } [a \in L(q), b \in L(p) \wedge q <_{\mathbb{Q}_n} p])$$

For example, $\mathbb{Q}_1(1)$ is the set of rational numbers $\mathbb{Q}$, $\mathbb{Q}_1(2)$ is the set of duplets ordered by the order type $\mathbb{Q}$, $\mathbb{Q}_2(2, 3)$ is the set of duplets and triplets ordered by the order type $\mathbb{Q}$, and et cetera. Obviously, if $\mathcal{M} = \mathbb{Q}_n(t_1, \ldots, t_n)$ for some linear orders $t_1, \ldots, t_n$ then $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$.

Theorem 5. [12] *$\mathcal{M}$ is an $\aleph_0$-categorical linear order iff $\mathcal{M}$ can be constructed from singletons by a finite number of concatenations or shuffles.*

Also, in [7] a criterion for Ehrenfeuchtness of P-combinations of countably many copies of an $\aleph_0$-categorical structure of pure linear order in terms of shuffles was obtained.

The following theorem describes all the possibilities for degrees of semantical and syntactical rigidity for an infinite countable linear ordering.

Theorem 6. *Let $\mathcal{M} = \langle M, < \rangle$ be an infinite countable linear ordering. Then only the following values for $\deg_4(\mathcal{M})$ are possible:*
(1) $(0, 0, 0, 0)$;
(2) $(1, 1, m, m)$, *where* $m \in \omega \setminus \{0\}$;
(3) (m, m, ∞, ∞), *where* $m \in \omega \setminus \{0\}$;
(4) $(\infty, \infty, \infty, \infty)$.

Proof of Theorem 6. The case (1) is guarantied by Theorem 1. The case (2) is guarantied by Lemma 4. The case (3) is guarantied by Lemma 3. The case (4) is guarantied by Theorem 2 and Corollary 4.

Prove now that there is no other values for $\deg_4(\mathcal{M})$. Obviously, if $\mathcal{M}$ contains at least one copy of $\mathbb{Q}$ then $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$. Obviously, if $\mathcal{M}$ contains at least one shuffle $\mathbb{Q}_n(t_1, \ldots, t_n)$ for some $1 \leq n < \omega$ and some linear orders $t_1, \ldots, t_n$ then we also have $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$. Also, by Theorem 2 if $\mathcal{M}$ contains infinitely many copies of $\mathbb{Z}$ then $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$.

Therefore, further we suppose that $\mathcal{M}$ contains only finitely many copies of $\mathbb{Z}$ and does not have neither copies of $\mathbb{Q}$ nor copies of $\mathbb{Q}_n(t_1, \ldots, t_n)$.

Suppose now that $\mathcal{M}$ contains infinitely many copies of ω. Such a set of copies of ω can be ordered by the order type ω, ω^*, $\mathbb{Z}$ or their mixed variations.

If $\mathcal{M} = \langle \omega^k \cdot l_1 + \omega^{k-1} \cdot l_2 + \ldots + \omega \cdot l_k + m, < \rangle$ for some $k, l_1, \ldots, l_k, m \in \omega$ then by Theorem 1 $\deg_4(\mathcal{M}) = (0, 0, 0, 0)$.

If $\mathcal{M} = \langle \omega^k \cdot \mathbb{Z}, < \rangle$ for some natural $k \geq 1$ then $\deg_4(\mathcal{M}) = (1, 1, 1, 1)$.

If $\mathcal{M}$ contains finitely many copies of kind $\omega^k \cdot \mathbb{Z}$, for example:

$$\mathcal{M} = \langle \omega^{k_1} \cdot \mathbb{Z} + \omega^{k_2} \cdot \mathbb{Z} + \ldots + \omega^{k_m} \cdot \mathbb{Z}, < \rangle$$

for some natural $k_1, \ldots, k_m \in \omega$, $m \geq 2$ and $k_{i_1}^2 + k_{i_2}^2 \neq 0$ for some $1 \leq i_1 < i_2 \leq m$ then $\deg_4(\mathcal{M}) = (m, m, \infty, \infty)$. If there exist infinitely many copies of kind $\omega^k \cdot \mathbb{Z}$ in $\mathcal{M}$, we have $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$.

If $\mathcal{M} = \langle \omega^{k_1} \cdot \omega^* + \omega^{k_2} \cdot \omega^* + \ldots + \omega^{k_m} \cdot \omega^*, < \rangle$ for some $m, k_1, \ldots, k_m \in \omega$ then we can also prove that $\deg_4(\mathcal{M}) = (0, 0, 0, 0)$.

We have similar reasonings for copies of ω^* ordered by ω, ω^* or $\mathbb{Z}$. We also have the same degrees of rigidity for the case when $\mathcal{M}$ contains infinitely many copies of both ω and ω^*.

If $\mathcal{M}$ contains m_1 copies of ω, m_2 copies of ω^* and finitely many finite linear orderings for some $m_1, m_2 \in \omega$ with $m_1^2 + m_2^2 \neq 0$, then $\mathrm{dcl}(\emptyset) = M$ and we have $\deg_4(\mathcal{M}) = (0, 0, 0, 0)$. If $\mathcal{M}$ contains m_1 copies of ω, m_2 copies of ω^*, m_3 copies of $\mathbb{Z}$ and finitely many finite linear orderings for some $m_1, m_2, m_3 \in \omega$ with $m_1^2 + m_2^2 \neq 0$ and $m_3 \geq 1$ then $\deg_4(\mathcal{M}) = (m_3, m_3, \infty, \infty)$.

7. Conclusion

We described possibilities for the semantical and syntactical degrees of rigidity and indices for various ordered theories including well-ordered sets, discrete, dense, and mixed orders, and for countable models of $\aleph_0$-categorical weakly o-minimal theories. All possibilities for degrees of rigidity for countable linear orderings are described. It would be natural to describe basic characteristics of rigidity for uncountable ordered structures, circularly and spherically ordered structures.

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