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On Covering of Cylindrical and Conical Surfaces with Equal Balls

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Abstract. The article concerns the problem of covering the lateral surface of a right circular cylinder or a cone with equal balls. The surface is required to belong to their union, and the balls' radius is minimal. The centers of the balls must lie on the covered surface. The problem is relevant for mathematics and for applications since it arises in security and communications. We develop heuristic algorithms for covering construction based on a geodesic Voronoi diagram. The construction of a covering is a non-trivial task since the line of intersection of a cylinder or a cone with a sphere is a closed curve of the fourth order. To compare the numerical results with the known ones, we unroll the surface of revolution onto a plane. Another feature is that, we use both Euclidean distance and a special non-Euclidean metric, which can describe the speed of signal propagation in a heterogeneous medium. We also perform a numerical experiment and discuss its results. Meanwhile, it is shown that with a small number of circles covering a planification of the cylindrical surface, their radius is significantly less than for a similar rectangle.

Keywords: covering problem, surface of revolution, equal balls, Voronoi diagram

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Научная статья

О покрытии поверхности цилиндра и конуса равными шарами

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Аннотация. Рассматривается задача о покрытии равными шарами боковой поверхности прямого кругового цилиндра или конуса. Требуется, чтобы поверхность лежала в их объединении при минимальном радиусе. Центры шаров должны находиться на покрываемой поверхности. Задача представляет интерес как с точки зрения математики, так и для приложений, поскольку возникает в области безопасности и связи. Разработаны эвристические алгоритмы отыскания искомых покрытий, основанные на геодезических диаграммах Вороного. Построение покрытия является нетривиальной задачей, поскольку линией пересечения цилиндра или конуса со сферой является замкнутая кривая 4-го порядка. Для того чтобы сравнить результаты с известными, предложен метод развертывания криволинейных поверхностей на плоскость. Помимо обычного евклидового расстояния, применяется также специальная неевклидовая метрика, которая может характеризовать скорость распространения сигнала в неоднородной среде. Выполнена серия вычислительных экспериментов, по результатам которых удалось сделать некоторые содержательные выводы.

Ключевые слова: задача покрытия, поверхность вращения, равные шары, диаграмма Вороного

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1. Introduction

Constructing minimal (thinnest) coverings and maximum (dense) packings belong to the classical formulations of computational geometry [24]. Such problems have been studied, without exaggeration, for a century but remain relevant [2; 23]. Most often, coverings of a plane shape with circles [21] in different variants are considered [4; 16; 17]. The problem of optimal covering of a curved surface with a given number of equal balls is much less studied. The most obvious application of such statements is the placement of the same type of wireless sensors and the design of global navigation and communication systems [20]. In all cases, the covered surface is a surface of revolution [5; 9]. Another application is the image transfer from a curved surface onto a plane for laser dimensional processing of surfaces of revolution through a plane mask [10]. Let us note that signal distortion can occur in all of these applied tasks, leading to a violation of the spherical shape of the sensor's or transmitter's range of action. As a rule, this property is omitted. Previously, we proposed using a special non-Euclidean metric to take into account such effects. It reflects the properties of the environment by replacing the physical distance with the time required to pass it [12; 13].

The simplest and most natural form of a surface of revolution is a sphere. Papers [7; 9] concern the construction of coverings of a sphere in three-dimensional space, present the thinnest coverings with a given number of balls, and estimate their redundancy.

Some scientists also consider coverings of spheres in spaces of arbitrary dimension d. For instance, paper [5] designs a covering that gives the covering density of order $(d \ln d)/2$ for a sphere of any radius $r > 1$ for growing dimension d. In [1], the authors show that for a sufficiently large number n, it is possible to arrange n equal covering figures so that no point belongs to more than a constant number of them, which depends only on d and the size of the covering figure. In addition, [11] claims that for $d \geq 3$, the unit sphere S^d can be covered in such a way that each of its points is covered at most $400d \ln d$ times. Moreover, if the sphere S^d is covered by $d + 3$ equal spheres, then the found arrangement is optimal. We also deal with a similar formulation [14; 25] using geometric methods and the optical-geometric approach [12].

The subjects of this study are the cylinder and cone, the other two commonly occurring surfaces of revolution. Note that the lines along which the cylinder and cone intersect with the sphere are spatial curves of the fourth order [15; 19]. This fact significantly complicates the construction of coverings, even in comparison with a sphere. We have managed to find only one paper concerning the cylinder covering problem [3], which, in particular, proves that the highest density of the thinnest covering of a cylinder with balls is equal to $\pi/2$. This short article is theoretical and does not present numerical calculations.

This study aims to fill the mentioned gap. We consider the problem of the thinnest covering of a cylinder and a cone by equal balls and propose heuristic algorithms based on a geodesic Voronoi diagram and optical-geometrical approach. We perform a numerical experiment and discuss it.

2. Formulation

Let we are given a metric space X, a surface $S(x, y, z) \subset X$, defined parametrically as $\{x = x(\alpha, u), y = y(\alpha, u), z = z(\alpha, u) \mid \alpha \in [0, 2\pi]; u \in (-\infty; +\infty)\}$. Let we also have a continuous function $0 \leqslant f(x, y, z) \leqslant \beta$, which is instantaneous speed of movement at each point $(x, y, z) \in X$. If $f(x_i, y_i, z_i) = 0$, then the point (x_i, y_i, z_i) is impassible. Then, the minimum time of movement between two points $a, b \in X$ can be determined as a solution to the problem

$$\rho(a, b) = \min_{\Gamma \in G(a,b)} \int_{\Gamma} \frac{d\Gamma}{f(x, y, z)}, \tag{2.1}$$

where $G(a, b)$ is the set of continuous curves that belong to X and connect points a and b. In other words, we will consider the shortest path between two points to be the curve that takes the least time to move along [14].

The ball covering problem is to locate n balls $C_i(O_i)$ with centers $O_i = (x_i, y_i, z_i)$ and the equal radii R so that the surface S belongs to the union of the balls, and the radius is minimal. Radius R is called 'coverage radius'. Then we have the following optimization problem

$$R \to \min, \tag{2.2}$$

$$\forall p \in S, \exists i\colon \rho(O_i, p) \leqslant R, \tag{2.3}$$

$$O_i \in S, i = \overline{1, n}, \tag{2.4}$$

Objective function (2.2) minimizes the coverage radius. Constraint (2.3) provides that any point of the surface S belongs to at least one covering ball, and condition (2.4) means that all balls' centers are placed on S.

3. Solution method

To solve problem (2.1)–(2.4), we apply our traditional approach [12–14], based on the analogy between the propagation of a light wave in an optically inhomogeneous medium and the search for a global extremum of functional (2.1). This optical-geometric analogue follows from physical laws of Fermat and Huygens [6]. According to Fermat law, a photon moves from the starting point to the target using a route that minimizes the travel time. Huygens's principle claims that each point reached by any photon becomes a new source of waves. Therefore, we can consider the envelope of the wave fronts of all secondary sources as the wave front at the certain moment. An algorithm that describes the propagation of light waves in homogeneous and inhomogeneous media and the construction of a generalized Voronoi diagram on its basis is presented in more detail in [12].

In order to reduce the problem of covering with n balls to a series of problems of covering with a ball, we partition the surface S into n cell, known as Dirichlet cells, using the Voronoi diagram. For each cell we need to determine edge points, which are the points simultaneously reached by more than two waves. Next, we initiate light waves from all edge points for every cell and find their Hausdorff center [14]. This center is also the center of covering ball having the minimum radius.

Voronoi diagram for cylindrical and conical surfaces. For a set of n points $O_i = (x_i, y_i, z_i) \in S, i = \overline{1,n}$, Voronoi cells V_i with centers O_i forming a Voronoi diagram are defined as follows: $V_i = \{a \in S\colon \rho(a, O_i) \leqslant \rho(a, O_j), \forall j \neq i\}$. The most famous algorithm for constructing a Voronoi diagram is Fortune's algorithm [8], but this algorithm works only on a plane. In this section, we propose modified versions for the surfaces considered.

The lateral surface of a right cylinder with the base radius r and the height h can be described as

$$\{x = r\cos\alpha, y = r\sin\alpha, z = u : \alpha \in [0, 2\pi); u \in [0, h]\}, \tag{3.1}$$

where α is the rotation angle, and u is the height.

Then, the length of the shortest curve connecting two points $a(\alpha_1, u_1)$ and $b(\alpha_2, u_2)$ on the cylindrical surface is following:

$$d_{cyl}(a, b) = \sqrt{\pi^2(\alpha_1 - \alpha_2)^2 + (u_1 - u_2)^2}. \tag{3.2}$$

The conical surface with the same parameters takes the form

$$\left\{x = u\cos\alpha, y = u\sin\alpha, z = h\left(1 - \frac{u}{r}\right) : \alpha \in [0, 2\pi); u \in [0, r]\right\}, \tag{3.3}$$

where α is the rotation angle, and u is the radius to height ratio.

In this case, the length of the shortest curve connecting points $a(\alpha_1, u_1)$ and $b(\alpha_2, u_2)$ is calculated as

$$d_{con}(a, b) = \sqrt{\left(1 + \frac{h^2}{r^2}\right)\left(u_1^2 + u_2^2 - 2u_1u_2\cos\frac{r|\alpha_1 - \alpha_2|}{\sqrt{h^2 + r^2}}\right)}. \tag{3.4}$$

Algorithm 1 for constructing a 'geodesic' Voronoi diagram

Let we are given n points $O_i,\ i = \overline{1,n}$.

Step 1: Introduce a uniform mesh with an angle step $s\alpha$ and a height step su: $S(s\alpha, su) \subset S$. This mesh is the same for a cylinder and a cone.

Step 2: Initiate a light wave from each point $O_i \in S(s\alpha, su),\ i = \overline{1,n}$. In contrast to the traditional optical-geometric method [12], the geodesic distance (3.2) is used here if we cover the cylinder and (3.4) – for the cone. As a result, we calculate the time $T_i(s)$ spending to reach all points $s(s\alpha, su) \in S(s\alpha, su)$, which allows us to find the vector $T(s) = \left\{T_i(s), i = \overline{1,n}\right\}$.

Step 3: For each point $s \in S(s\alpha, su)$, determine indexes of waves that reached this point first. The indexes form the set $D(s) = \{k\colon T_k(s) = \min_i T_i(s)\}$. For each s, $D(s)$ contains at least one element.

Step 4: Voronoi cells V_i with respect to $O_i,\ i = \overline{1,n}$, are constructed as

$$V_i = \{s \in S(s\alpha, su)\colon i \in D(s)\}\,.$$

Construction of coverings. The idea of the algorithm is to find the fastest path between the two most distant points of the Voronoi cell. Its middle is the center of the covering circle of minimum radius.

Algorithm 2 for constructing a covering of Voronoi cells

Step 1: Randomly generate points $O_i \in S(s\alpha, su),\ i = \overline{1,n}$, which are the initial centers of covering balls.

Step 2: Construct a Voronoi diagram by determining the cells V_i with respect to $O_i, i = \overline{1,n}$ using Algorithm 1.

Step 3: Find a boundary ∂V_i of the cells V_i and approximate it by a closed polyline with nodes at points $v_{i,k}, k = \overline{1,m}$.

Step 4: Each point $v_{i,k}$ initiates a light wave, which propagates according to the algorithm proposed in [12]. It allows us for each $s(s\alpha, su) \in V_i$ to figure out what wave reached it first and calculate the time spent as

$$T(\alpha, u) = \min_{k=\overline{1,m}} T_k(\alpha, u),$$

where $T_k(\alpha, u)$ is the propagation time of a light wave from $v_{i,k}$ till $s(\alpha, u) \in V_i$.

Step 5: The radius and center of the covering balls of the area are determined as

$$R_i = \max_{s(\alpha,u)\in V_i} T(\alpha, u),\ O_i^* = \arg \max_{s(\alpha,u)\in V_i} T(\alpha, u).$$

Steps 3–5 are carried out independently for each cell V_i.

Step 6: To guarantee complete coverage of the set S, the maximum radius of the covering balls is chosen: $R = \max_{i=\overline{1,n}} R_i$.

Steps 2–6 are being carried out as long as $\rho(O_i, O_i^*) \geq \delta, i = \overline{1,n}$, where δ is given in advance.

Step 7: If the radius found at the current iteration is less than the previous one, the solution found is memorized as the best. A new generation of initial positions is performed. The algorithm terminates when the specified number of generations is reached.

As a result, we get a set of equal balls, which union completely covers the surface S, and the balls' radius is minimal.

Planification of lateral surfaces. It is necessary to unroll the lateral surface to compare our results with the best-of-known results of solving the covering problem for a unit square. Besides, this procedure helps visualize the coverings found.

The lateral surface and its planification (unrolling) are two geometric figures having a one-to-one correspondence between their points. Therefore, the straight line on the planification corresponds to the shortest path on the surface.

The intersection of a surface of revolution with a sphere is a fourth-order spatial curve. Therefore, when considering an unrolled surface of revolution, the covering elements cannot be circles as for the classical 2-D covering problem. This section proposes a procedure for constructing such peculiar curves for a cylinder and a cone.

First, let us consider cylinder (3.1) and a sphere having the center $O(\alpha_O, u_O)$ and the radii R. In order to determine all points belonging to the intersection of the sphere and the cylinder, we introduce the circle I, which is the directrix passing through the point O. Next, from each point $M(\alpha_M, u_M) \in I$, we draw a line on a cylindrical surface perpendicular to the plane of the circle I until it intersects with the sphere at the point $N(\alpha_N, u_N)$ (see Fig. 1). By construction, we have

$$MN = \sqrt{ON^2 - OM^2} = \sqrt{R^2 - 4r^2 \sin^2 \frac{|\alpha_O - \alpha_M|}{2}}.$$

It is easy to see that $u_O = u_M$ since O and M belong the same directrix; $\alpha_M = \alpha_N$ because M and N lie on the same generatrix. Hence,

$$u_N = u_O \pm \sqrt{R^2 - 4r^2 \sin^2 \frac{|\alpha_O - \alpha_M|}{2}}.$$

When the point M runs through the arc ℓ of the circle I satisfying the constraint

$$|\alpha_O - \alpha_M| \leq 2 \left| \arcsin \frac{R}{2r} \right|,$$

we get a set of intersection points:

$$J_{cyl}(\alpha_O, u_O) = \left\{ (\alpha, u) : \alpha \in \ell; u = u_O \pm \sqrt{R^2 - 4r^2 \sin^2 \frac{|\alpha_O - \alpha|}{2}} \right\}. \tag{3.5}$$

Second, let us consider conical surface (3.3) and a sphere having the center $O(\alpha_O, u_O)$ and the radii R. Here we also the circle I, which is the directrix passing through the point O and for each point $M(\alpha_M, u_M) \in I$ draw a line passing through it and the cone apex D until it intersects with the sphere at the point $N(\alpha_N, u_N)$ (see Fig. 1).

Let us introduce the notation $L = \sqrt{h^2 + r^2}$. By construction, we have

$$DM = DO = u_O \frac{L}{r}, MO = 2u_O \sin \frac{|\alpha_O - \alpha_M|}{2}.$$

Consequently, $\cos(\angle MDO) = 1 - \frac{MO^2}{2DO^2}$. Thus, according to the cosine theorem, we get the following quadratic equation to calculate DN:

$$DN^2 - 2DNu_O\frac{L}{r}\left(1 - \frac{2r^2\sin^2(\frac{|\alpha_O - \alpha_M|}{2})}{L^2}\right) + \left(u_O\frac{L}{r}\right)^2 - R^2 = 0. \quad (3.6)$$

To solve (3.6), let us find its discriminant

$$\Delta = 4R^2 - 8u_O^2\sin^2\frac{|\alpha_O - \alpha_M|}{2}.$$

When the point M runs through the arc ℓ of the circle I satisfying the constraint $\Delta \geq 0$, we get a set of intersection points:

$$J_{con}(O(\alpha_O, u_O)) = \left\{(\alpha, u) : \alpha \in \ell; u = \frac{r}{L}k_{\pm}; L = \sqrt{h^2 + r^2}\right\}, \quad (3.7)$$

where $k_{\pm}$ are the roots of (3.6).

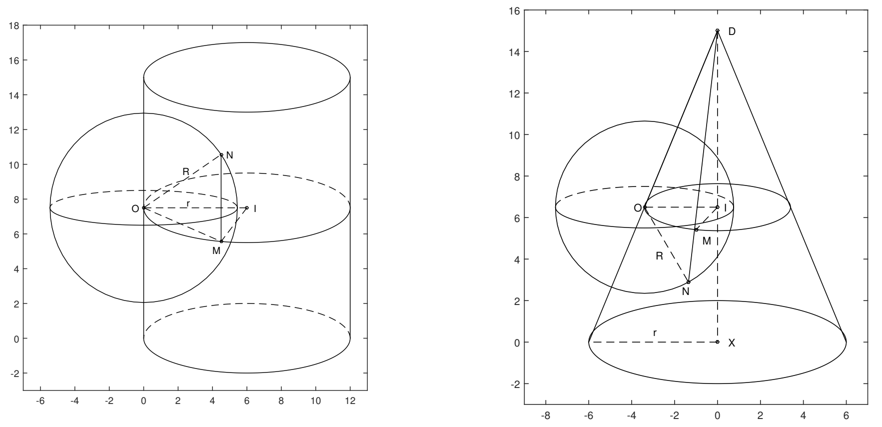

Figure 1. Illustration for constructing the set of intersection points of a ball with a cylinder and a cone

4. Computational experiment

In this section, we present some numerical results. The experiment is carried out on a personal computer with Intel (R) Core(TM) i5-3337U (1.8 GHz, 4CPUs, 6 GB RAM) configuration and operating system Windows 10. The algorithms are implemented in C# programming language using Visual Studio 2021.

Example 1. This example presents the best solutions to the covering problem of a cylinder with radius $r = \frac{1}{2\pi}$ and height $h = 1$ with $n = 3, ..., 20$ balls in the case of Euclidean metric, i.e., $f(x, y, z) = 1$.

Assuming the lower base of the cylinder belongs to the Oxy plane and its center is the origin, then the centers of covering balls ($n = 20$) are following:

$$(0.1271, 0.0958, 0.9000), (-0.1580, 0.0194, 0.5200),\\ (0.1065, -0.1183, 0.2800), (-0.1271, -0.0958, 0.2800),\\ (0.0465, 0.1522, 0.6400), (0.1430, -0.0698, 0.0400),\\ (-0.0166, -0.1583, 0.4800), (-0.0958, -0.1271, 0.0800),\\ (0.0111, 0.1588, 0.4200), (0.0772, -0.1392, 0.9600),\\ (0.1576, -0.0222, 0.8400), (0.1530, -0.0439, 0.6000),\\ (0.0980, 0.1254, 0.1400), (-0.0331, -0.1557, 0.7000),\\ (-0.1023, 0.1219, 0.0200), (-0.1288, 0.0935, 0.2400),\\ (-0.0698, 0.1430, 0.9200), (-0.1465, 0.0622, 0.7200),\\ (-0.1454, -0.0647, 0.9000), (0.1572, 0.0249, 0.4000)$$

The radius of the balls is $R = 0.1535$. Figure 2 shows the thinnest covering with 20 balls.

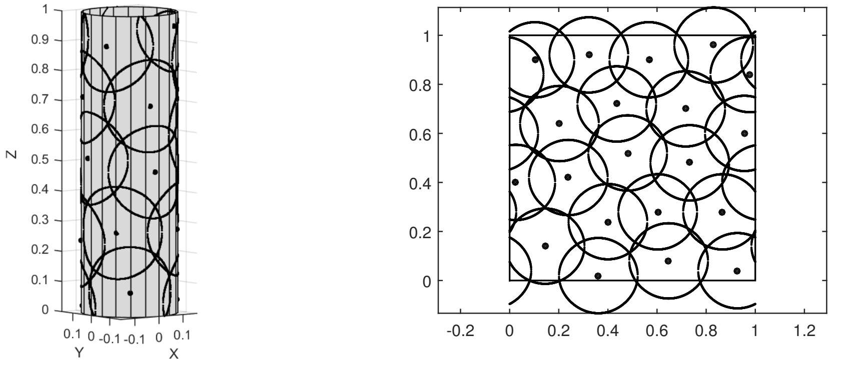

Figure 2. Covering of the cylinder (left) and its unrolled surface (right) with 20 equal balls and circles, respectively.

Unrolling cylindrical surface into a plane, we obtain a unit square. Next, we compare the results obtained with the best coverings of the unit square with equal circles [18; 22] (see Table 1).

Here n is the number of covering balls or circles, respectively; R is the best covering radius found using the algorithm proposed, R^* is the best-of-known radius from [18; 22] for the corresponding number of circles; $\Delta R(\%) = \frac{R-R^*}{R^*} \times 100$.

Table 1 shows that compared with [18; 22], our results are better for $n \leq 11$. The reason is that the cylindrical surface is transformed into a square when unfolding, and one circle can cover two sides of the square. Such a situation is impossible for classical covering problem. As the number

Table 1
Covering of the cylindrical surface the unit square in the Euclidean metric

n	R	R^*	$\Delta R(\%)$	n	R	R^*	$\Delta R(\%)$
3	0.4997	0.5039	-0.83	12	0.2032	0.2023	0.44
4	0.3397	0.3535	-3.90	13	0.1952	0.1943	0.46
5	0.3117	0.3261	-4.41	14	0.1862	0.1855	0.38
6	0.2900	0.2989	-2.98	15	0.1805	0.1797	0.45
7	0.2736	0.2742	-2.22	16	0.1705	0.1694	0.65
8	0.2595	0.2605	-0.38	17	0.1668	0.1657	0.66
9	0.2300	0.2306	-0.26	18	0.1617	0.1606	0.69
10	0.2180	0.2182	-0.09	19	0.1592	0.1578	0.89
11	0.2129	0.2125	-0.19	20	0.1535	0.1522	0.85

of circles increases, this advantage disappears. For $n > 11$, deviation from the best-known radii increases but does not exceed 0.89%. Note, that it directly depends on the mesh step.

Example 2. This example considers the same cylinder having $r = \frac{1}{2\pi}$ and $h = 1$ in the case of non-Euclidean metric with $f(x, y, z) = \frac{1}{1+2z}$. The wave propagation velocity here decreases in the vertical direction when moving from the lower base of the cylinder. If the ball's center is located closer to the lower base, then visually this ball looks large (see Fig. 3). However, in the given metric, all balls have the same radius.

Table 2 shows the best coverings found. Here n is the number of covering balls, r is the best radius, t is the executing time of the algorithm (in seconds).

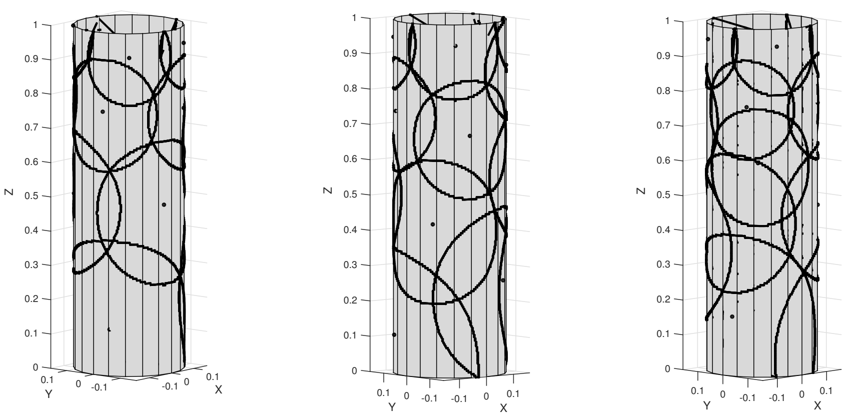

Figure 3. Covering of the cylinder with equal balls in Example 2 for $n = 13, 14, 15$.

Table 2

The best radii in Example 2

n	R	t	n	R	t
3	0.8638	87.32	12	0.5003	113.23
4	0.7734	90.77	13	0.4902	119.02
5	0.6758	94.64	14	0.4624	126.24
6	0.6225	95.70	15	0.4211	127.23
7	0.5994	99.94	16	0.3744	129.83
8	0.5519	105.19	17	0.3704	132.74
9	0.5331	108.31	18	0.3629	134.85
10	0.5222	110.22	19	0.3616	135.36
11	0.5158	111.58	20	0.3585	137.03

Example 3. This example considers a cone with the base radius $r = 1$ and the height $h = 3$. Let $f(x, y, z) = 1/(1 + 0.5z)$. The wave propagation velocity here also decreases with moving from the lower base of the cone to its apex. If the ball's center is located closer to the base, then visually this ball looks large (see Fig. 4). However, in the given metric, all balls have the same radius. Table 3 shows the best coverings found.

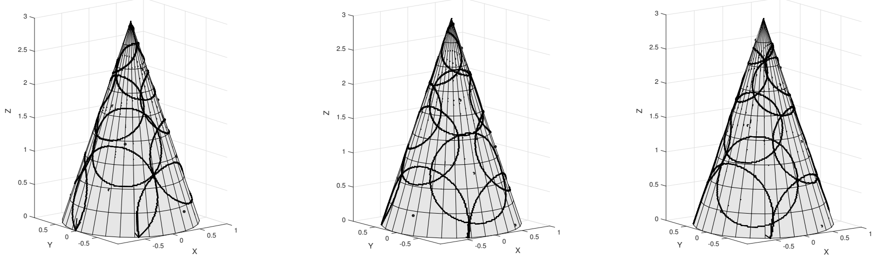

Figure 4. Covering a conical surface with $n = 13, 14, 15$ balls.

Table 3

Covering a conical surface with a non-Euclidean metric

n	R	t	n	R	t
3	1.9337	50.64	12	1.0139	103.25
4	1.7885	54.34	13	0.9777	119.54
5	1.5209	66.04	14	0.9649	136.49
6	1.4429	70.70	15	0.9197	147.23
7	1.3116	73.59	16	0.8654	159.53
8	1.1952	80.98	17	0.8587	172.47
9	1.1652	84.31	18	0.8313	188.87
10	1.0659	87.22	19	0.7863	196.11
11	1.0478	92.58	20	0.7834	204.82

5. Conclusion

The article contributes to research on covering surfaces of revolution with equal balls. It is required that the surface lies inside the union of a specified number of balls, the radii of which must be minimal. We consider a particular non-Euclidean metric as a measure of distance between points. In applied problems, such a metric means signal propagation time in ananisotropic medium.

Previously, we dealt with the case when the covered surface was a sphere, and then its intersection with a covering element was a spherical cap. This paper concerns the coverings of cylinders and cones. Their construction is a much more time-consuming problem since the sections of the covering elements have a complex shape. As for a cone, their shape depends significantly on the distance to the top. Therefore we couldn't transfer directly the approaches previously used for a sphere. In particular, constructing an analogue of the Voronoi diagram, a key element of the algorithm proposed, has become significantly more complicated. Nevertheless, all the difficulties have been successfully overcome. A heuristic algorithm has been developed that allows solving the considered problems in a wide range of parameters. The algorithm is based on the modified optical-geometrical approach, which takes into account the features of the formulations under consideration.

We have performed numerical calculations which have shown the applicability of the proposed approach. Unfortunately, we have not managed to find a material for comparison. However, judging by indirect signs, such as covering density, the results are promising.

Further development of the research may be related to the complication of the shape of the covered set. It seems advisable from the point of view of security applications to consider torus and disks of various types.

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