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Necessary and Sufficient Conditions for the Existence of Rational Solutions to Homogeneous Difference Equations with Constant Coefficients

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Abstract. A necessary and a sufficient condition for solvability of homogeneous difference equations with constant coefficients in the class of rational functions are obtained. The necessary condition is a restriction on the Newton polyhedron of the characteristic polynomial. In the two-dimensional case, this condition is the existence of parallel sides on the polygon. The sufficient condition is the equality to zero of certain sums of the coefficients of the equation. If the sufficient condition is satisfied, the solution is the class of rational functions whose denominators form a subring in the ring of polynomials. This subring can be associated with an edge of the Newton polyhedron of the characteristic polynomial of the equation.

Keywords: difference equations, rational functions, Newton's polyhedron

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Научная статья

Необходимое и достаточное условия существования рациональных решений однородных разностных уравнений с

постоянными коэффициентами

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Аннотация. В работе получены необходимое условие и достаточное условие разрешимости однородных разностных уравнений с постоянными коэффициентами в классе рациональных функций. Необходимым условием является ограничение на многогранник Ньютона характеристического полинома. В двумерном случае это условие заключается в наличии параллельных сторон у многоугольника. Достаточным условием является равенство нулю определенных сумм коэффициентов уравнения. В случае выполнения достаточного условия решением является класс рациональных функций, знаменатели которых образуют подкольцо в кольце полиномов. Это подкольцо может быть ассоциировано с гранью многогранника Ньютона характеристического полинома уравнения.

Ключевые слова: разностные уравнения, рациональные функции, многогранник Ньютона

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1. Introduction

The problem of finding rational solutions to difference equations was posed more than 50 years ago. In the one-dimensional case for constant and polynomial coefficients the problem was solved by S. Abramov [1], [2].

Attempts to generalize Abramov's results to the multidimensional case led to significant difficulties. In 2013, the algorithmic unsolvability of checking the existence of rational solutions to difference equations with polynomial coefficients was proved [11].

However, failures in the search for a general algorithm do not mean that there cannot exist a particular algorithm for some classes of difference equations. Thus, from the class of difference equations one should select subclasses satisfying the conditions of necessity and sufficiency of existence of rational solutions and search for algorithms working in these subclasses.

The aim of the paper is to find necessary and sufficient conditions for solvability in the class of rational functions of homogeneous difference equations with constant coefficients.

Let $R(z_1,\dots,z_n) = \frac{N(z_1,\dots,z_n)}{D(z_1,\dots,z_n)}$ be a rational function, where $z \in \mathbb{C}^n$, N and D are coprime polynomials. The function R is analytic in the complement $\mathbb{C}^n \setminus \mathfrak{S}$, where $\mathfrak{S} = \{z \in \mathbb{C}^n : D(z) = 0\}$, $\#\mathfrak{S}$ is the number of irreducible components of $\mathfrak{S}$.

We write the difference equation as

$$P(\delta)R(z) = 0 \; , \tag{1.1}$$

where $A \subset \mathbb{Z}^n$, $P(\zeta) = \sum_{\alpha\in A} p_\alpha \zeta_1^{\alpha_1}\dots\zeta_n^{\alpha_n}$ is the characteristic polynomial (Laurent polynomial), $\delta = (\delta_1,\dots,\delta_n)$ is the vector of atomic shift operators $\delta_i R(z_1,\dots,z_n) = R(z_1,\dots,z_i+1,\dots,z_n)$. Let us assume that the dimension of the polyhedron $\mathbf{Ch}(A)$ (convex hull of the set A, which is called the *Newton polyhedron* of the polynomial P) is exactly n.

If R is an arbitrary rational function, then $R_P(z) := P(\delta)R(z)$ is also rational and holomorphic in $\mathbb{C}^n \setminus (\mathfrak{S} - A) := \{z \in \mathbb{C}^n : z+\alpha \notin \mathfrak{S},\ \forall \alpha \in A\}$.

We will denote R is the *rational solution to a difference equation* if it satisfies (1.1) for all z in the set $\mathbb{C}^n \setminus (\mathfrak{S} - A)$, $\mathfrak{S}$ is non-empty. Note that since N and D are coprime polynomials, R cannot be analytically continued to $\mathfrak{S}$. Due to the uniqueness theorem for holomorphic functions, R is a solution to (1.1) if $R_P(z) = 0$ in any subdomain of the domain $\mathbb{C}^n \setminus (\mathfrak{S} - A)$.

We note that in the case $n = 1$ for the homogeneous difference equation (1.1) there are no rational solutions. Indeed, a rational function of one variable has a finite number of poles and there is an extreme pole z_1^0 with minimal (or maximal) value of $\Re z_1$. Using the method of steps [10], we can construct an analytic continuation of the solution to the neighborhood of z_1^0. By identifying and listing all the poles, we are able to extend the analytic function to a larger region in the complex plane, known as a process of analytic continuation in the complex domain.

As an example of a rational solution in the case $n = 2$, let's consider the function

$$\frac{1}{(z_1 - z_2)} \; ,$$

which satisfies the equation

$$R(z_1,z_2) - R(z_1+1,z_2+1) + R(z_1+1,z_2) - R(z_1+2,z_2+1) = 0 \; .$$

The key difference between the univariate case (where $n = 1$) and the multivariate case (where $n > 1$) is that in the multivariate case, the poles of the rational function may be located in $\mathbb{C}^n$ (complex n-dimensional space) in such a way that the method of steps cannot be used to construct the analytic continuation of the solution.

In simpler terms, for $n > 1$, the poles of a rational function may be spread out in multiple dimensions, making it difficult to extend the solution using the usual method.

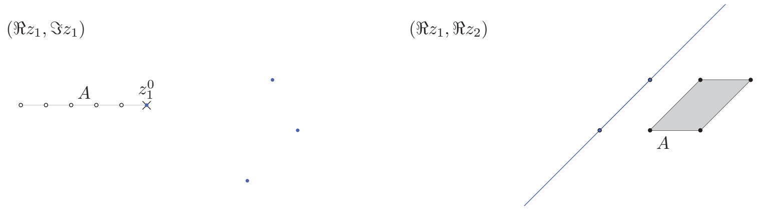

Figure 1. The difference between the one-dimensional case ($n = 1$) and the two-dimensional case ($n = 2$) lies in how the analytical continuation of the solution is achieved. In the one-dimensional case, the solution can be extended to the neighborhood of each individual pole using the method of steps. However, in the two-dimensional case, the rational solution's specific hyperplane, where $z_1 = z_2$, is positioned in a manner that prevents the analytical continuation from being executed through the method of steps.

In conclusion, another related challenge pertains to characterizing the solution space of the difference equation and determining the derivative functions of these solutions. When transitioning from the one-dimensional case to the multidimensional case, we encounter notable obstacles. The rationality of the derivative function is contingent upon the geometric properties of the Newton polyhedron associated with the characteristic polynomial. The recent publication, [8], delves deeper into this subject and provides additional references for further exploration.

The problem of computing the rational generating function is also a close problem. In [7] the two-dimensional case is considered and references to other works are given.

2. Necessary condition

The author discovered in [13] that in the two-dimensional case, Equation 1.1 can only possess rational solutions with denominators composed of linear factors of the form $\langle z, q\rangle - c$, where q represents the normal vector to the parallel sides of the polyhedron $\mathbf{Ch}(A)$. Only under such circumstances, the singularity of the rational solution cannot be eliminated by continued analytical continuation of the solution through the method of steps. This finding also implies a necessary condition for the existence of a rational solution, namely, the constraint on the Newton polyhedron of the equation's characteristic polynomial – it should have parallel sides.

Let's explore how this result can be extended to the general case of any dimension, denoted as n. It is important to note that the polyhedron $\mathbf{Ch}(A)$ and the singular set $\mathfrak{S}$ of the rational solution for (1.1) cannot intersect at only one vertex $\mathbf{Ch}(A)$. If such a scenario occurred, we could construct an analytic continuation of the solution on the irreducible component of the set $\mathfrak{S}$ that includes the tangent point, by solving Equation 1.1 for the corresponding summand related to that vertex.

When we refer to the relative positioning of the polyhedron $\mathbf{Ch}(A)$ and the algebraic hypersurface $\mathfrak{S} \subset \mathbb{C}^n$, we are describing their relationship in each real cross-section of the complex space, while ensuring that the polyhedron can be parallelized. All further considerations will be made for the case $A \subset \mathbb{Z}_+^n$, noting that replacing $z \to z + a$ extends all conclusions to the general case.

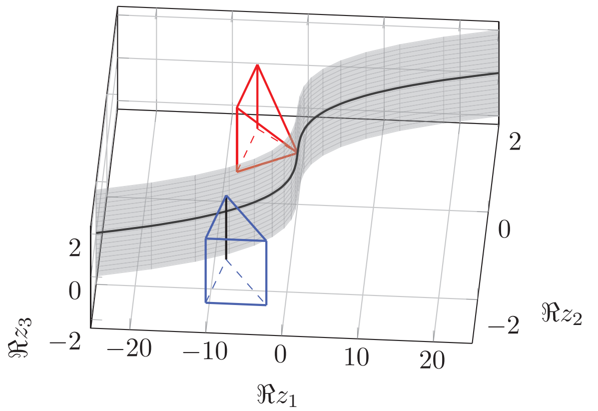

Figure 2. If an algebraic hypersurface is structured as a bundle and aligns with the blue polyhedron $\mathbf{Ch}(A)$, it hinders the possibility of analytic continuation of the solution. On the other hand, if the red polyhedron does not correspond with the bundle, it is considered inconsistent and does not obstruct the analytic continuation process.

A set of polynomials of the form

$$D_l(L_1(z), \ldots, L_{n-k}(z)) \tag{2.1}$$

can be associated with each k-dimensional plane $l = \{z \in \mathbb{C}^n : L_i(z) = c_i, i = 1, \ldots, n-k\}$. This set forms a subring within the ring of polynomials $\mathbb{C}[z]$, which can be denoted as $\mathbb{C}_l[z]$. It is important to highlight that $\mathbb{C}_l[z]$ remains consistent regardless of the specific homogeneous linear functions L_i used to define the plane l.

Consider a plane l that does not intersect the polyhedron $\mathbf{Ch}(A)$ solely at a vertex. In this scenario, the equation $D_l(L_1(z), \ldots, L_{n-k}(z)) = 0$ defines a hypersurface σ, which cannot be eliminated using the method of steps. Specifically, σ can be characterized as a vector bundle represented by the product $\sigma^{n-k-1} \times l$, where the base σ^{n-k-1} corresponds to a surface of dimension $n - k - 1$.

On the contrary, if an irreducible hypersurface $\sigma \in \mathfrak{S}$, intersects the set A at more than one point (at least two), it cannot be eliminated through the method of steps. Let's suppose these points are denoted as $a' \in A$, $a'' \in A \setminus a'$. In this case, for $z \in \sigma$ in the vicinity of a certain point $z^0 \in \sigma$, it follows that $z + \alpha \in \sigma$, where $\alpha = a'' - a'$. This is due to the fact that the neighborhood of a point acts as the uniqueness set for an irreducible analytic set σ [4]. Consequently, the polynomial D that defines σ satisfies the identity $D(z+\alpha) \equiv D(z)$.

Let's use a linear substitution $L : \mathbb{C}^n \to \mathbb{C}^n$ such that $L(\alpha) = (0, \dots, 0, 1)$ and $L(0) = 0$, where $L_n(z) = z_n/\alpha_n$ (let $\alpha_n \neq 0$) and $L_i(z)$ are homogeneous linear functions, $i = 1, \dots, n-1$. There is also a backward substitution $L^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ such that $L^{-1}(0, \dots, 0, 1) = \alpha$ and $L^{-1}(0) = 0$. Then the polynomial D can be written as $D(z) = D(L^{-1}L(z))$.

The polynomial $D'(w) = D(L^{-1}(w))$ satisfies the equality

$$D'(w + (0, \dots, 0, 1)) = D'(w) \ .$$

Since only constants can be periodic polynomials of one variable, $D'(w)$ does not depend on the variable w_n. This means $D'(w + (0, \dots, 0, t)) = D'(w)$ holds for any $t \in \mathbb{C}$. Let's now examin the behavior of the polynomial $D(z)$ on the layers $z + t\alpha$:

$$D(z + t\alpha) = D(L^{-1}L(z + t\alpha)) = D'(L(z + t\alpha)) = D'(L(z) + tL(\alpha))) =$$
$$= D'(L(z) + (0, \dots, 0, t)) = D'(L(z)) = D(L^{-1}L(z)) = D(z) \ .$$

Thus, the polynomial D can be represented as a composition

$$D(z) = D'(L_1(z), \dots, L_{n-1}(z))$$

of some polynomial in $n-1$ variable and homogeneous linear functions defining the line $t\alpha$. In essence, we have once again obtained a representation in the form (2.1) ($k = 1$) for σ and the line $t\alpha$ cannot intersect the polyhedron $\mathbf{Ch}(A)$ only at a vertex.

If $\mathfrak{S}$ contains several irreducible hypersurfaces $\mathfrak{S}' \subset \mathfrak{S}$, mutually impede analytic continuation, then they satisfy the equality $\sigma' + \alpha \equiv \sigma$, where $\{\sigma, \sigma'\} \subset \mathfrak{S}'$, α is some vector, connecting two points of A.

Then, the existence of an irreducible component σ in $\mathfrak{S}'$ is confirmed since it intersects only with a particular shift A. This relies on the notable property of an algebraic hypersurface that allows for any compact set to be accommodated within its complement, $\mathbb{C}^n \setminus \mathfrak{S}'$, as deduced from the amoeba structure of the set $\mathfrak{S}'$ [5]. This guarantees the presence of an "extreme" σ as well.

Considering σ as an irreducible singularity of the solution, the intersection of A and σ will entail at least two points, thereby validating the truth of representation (2.1) for σ. Consequently, for any other hypersurface $\sigma' \in \mathfrak{S}'$ connected to σ by the equation $\sigma' + \alpha \equiv \sigma$, representation (2.1) also holds true. Similarly, representation (2.1) can be derived for any other component $\sigma \in \mathfrak{S} \setminus \mathfrak{S}'$.

Considering the layer l of the curve σ, it is evident that it cannot solely intersect the polyhedron $\mathbf{Ch}(A)$ at a vertex. Thus, it must intersect the polyhedron along some edge Γ. When $\dim\Gamma = \dim l$, we can select the linear functions $L_i = \langle z, q_i\rangle$ as the defining functions for l. Here, q_i represents the normal vectors to the adjacent hyperfaces of the face Γ.

To determine the necessary normals from the entire set of normals to the hyperfaces of $\mathbf{Ch}(A)$, a condition is applied: they must lie within the normal cone to the face Γ.

The cone formed by the normals to point v of the polyhedron $\mathbf{Ch}(A)$ is referred to as the *normal cone*, denoted as C_v. It is defined as the set of points $x \in \mathbb{R}^n$ such that $\langle x, a - v\rangle \leqslant 0$, for all $a \in \mathbf{Ch}(A)$. The normal cone C_Γ to a face Γ is the normal cone C_v to any interior point v of this face. The dimension of the normal cone C_Γ is $n - \dim\Gamma$. This cone can contain more than $n-k$ normals, to construct linear functions, we can choose any $n-k$ of them.

That is, each face of the polyhedron $\mathbf{Ch}(A)$ can also be associated with a subring of polynomials $\mathbb{C}_\Gamma[z]$. If $\Gamma \subset l$ and $\dim l = \dim\Gamma$, then $\mathbb{C}_l[z] = \mathbb{C}_\Gamma[z]$. If $\dim l > \dim\Gamma$, then $\mathbb{C}_l[z] \subset \mathbb{C}_\Gamma[z]$.

Thus, polynomials from the subrings $\mathbb{C}_\Gamma[z]$ exhaust all factors of the denominator of rational solutions to (1.1).

We have shown that each irreducible component σ of the singular set $\mathfrak{S}$ of a rational solution to the Equation 1.1 is a vector bundle $\sigma = \sigma^{n-k-1} \times l$ whose fiber is some line l that intersects the polyhedron $\mathbf{Ch}(A)$ on a face. Now we can state the theorem.

Theorem 1. *If a rational function $R(z) = N(z)/D(z)$ is a solution to (1.1), then there is a non-empty set of planes $\{l_j\}_{j=1}^{\#\mathfrak{S}}$ and faces $\{\Gamma_j\}_{j=1}^{\#\mathfrak{S}}$ of the Newton polyhedron $\mathbf{Ch}(A)$ of the characteristic polynomial of the Equation 1.1, such that*

1) There are inclusions

$$\Gamma_j \subset \{x \in \mathbb{R}^n : \langle x, q_{ij}\rangle = c_{ij}, i = 1, \ldots, n-k(j)\} \subset l_j \ ,$$

where q_{ij} are the normals to the faces of the polyhedron $\mathbf{Ch}(A)$. These normals are located in the normal cone to the face Γ_j, $k(j) = \dim\Gamma_j \leqslant \dim l_j$;

2) For any $x \in \mathbb{R}^n$ and j the intersection $(l_j + x) \cap \mathbf{Ch}(A)$ is not a vertex of $\mathbf{Ch}(A)$;

3) The denominator $D(z)$ is represented as the product

$$D(z) = D_1(z) \cdot \ldots \cdot D_{\#\mathfrak{S}}(z) \ ,$$

of the factors of the form

$$D_j(z) = D_{\Gamma_j}(\langle z, q_{1j}\rangle, \ldots, \langle z, q_{n-k(j),j}\rangle) \ ,$$

where D_{Γ_j} is a polynomial in $n-k(j)$ variables. That is

$$R(z) = \frac{N(z)}{\prod_{j=1}^{\#\mathfrak{S}} D_{\Gamma_j}(\langle z, q_{1j}\rangle, \dots, \langle z, q_{n-k(j),j}\rangle)} \,. \tag{2.2}$$

Proof. 1) The set $\mathfrak{S}$ is not empty and consists of a finite number of irreducible algebraic hypersurfaces σ, each of which is a vector bundle $\sigma = \sigma^{n-k-1} \times l$, where the line l intersects the polyhedron $\mathbf{Ch}(A)$ in a face. From here it follows that a plane l and a face Γ of the polyhedron $\mathbf{Ch}(A)$ corresponds to each hypersurface σ.

Any k-dimensional face of a convex polyhedron can be represented as the intersection of $n-k$ hyperfaces adjacent to it. If x is the normal to the hyperface Γ' adjacent to Γ, v is an inner point of the face Γ and $a \in \Gamma'$ then $\langle x, a-v\rangle = 0$, as v belongs to all adjacent hyperfaces. If $\langle x, q_i\rangle = c_i$, $i = 1, \dots, n-k$ is the equations of adjacent hyperfaces, then a system of these equations defines the k-dimensional plane containing the face Γ. This plane, as we have seen, is contained in the layer l of the bundle σ.

2) If there exists such $x \in \mathbb{R}^n$ that the intersection $(l+x) \cap \mathbf{Ch}(A)$ is a vertex of $\mathbf{Ch}(A)$, the irreducible curve σ will admit an analytic continuation of the solution, which contradicts the definition of a rational solution.

3) Every irreducible hypersurface $\sigma_j \in \mathfrak{S}$ is a zero set of some analytic function $D_j(z)$, and $D(z) = D_j(z) \cdot D_{[j]}(z)$, $\{z : D_{[j]}(z) = 0\} = \mathfrak{S} \setminus \sigma_j$. Since $\mathfrak{S}$ is algebraic, $D_j(z)$ and $D_{[j]}(z)$ are polynomials, where $D_j(z)$ is irreducible.

The hypersurface σ_j is a vector bundle, whose layer is the k-dimensional line l. If $k = 1$, then, as we have seen, the polynomial $D_j(w)$ can be represented as a composition of the polynomial $D'_j(z)$ of $n-1$ variable and the homogeneous linear functions defining the line l.

If $k > 1$, then there is k linearly independent vectors $\{\alpha^i\}$ such that $D_j(z + t\alpha^i) = D_j(z)$, $\forall t \in \mathbb{C}$. Here again, using linear substitution such that $L(\alpha^i) = e_{n-i}$, $L(0) = 0$, where $L_{n-i}(z) = z_{n-i}/\alpha^i_{n-i}$ (by assuming $\alpha^i_{n-i} \neq 0$), $i = 0, \dots, k-1$ we can represent the polynomial D as a composition $D(L^{-1}L(z))$ and show that the polynomial $D'(w) = D(L^{-1}(w))$ is independent of k variables, i.e. D admits a representation in the form (2.1).

Hence, each irreducible component $\sigma_j \in \mathfrak{S}$, $j = 1, \dots, \#\mathfrak{S}$ is the zero of the irreducible polynomial D_j, which admits representation in the form (2.1) and divides the polynomial $D(z)$. □

The Theorem contains a necessary condition for the solvability of the Equation 1.1 in the class of rational functions. This condition is formulated as a restriction on the Newton polyhedron $\mathbf{Ch}(A)$ of the characteristic polynomial P and is a multidimensional generalization of the parallelism property of the sides of a polygon.

Necessary condition: *For the existence of a rational solution to Equation 1.1 it is necessary that there is at least one plane l, $1 \leqslant \dim l \leqslant n-1$ such that the shift $l+x$, for any $x \in \mathbb{R}^n$ does not intersect* $\mathbf{Ch}(A)$ *only at a vertex.*

3. Sufficient condition

The denominator of any rational solution to the difference equation consists of two type of factors – periodic and aperiodic polynomials [6].

A polynomial $\Pi \in \mathbb{C}[z_1, \ldots, z_n]$ is called *periodic* if the following set is infinite

$$\mathrm{Spread}(\Pi, \Pi) = \{\alpha \in \mathbb{Z}^n : \gcd(\Pi(z), \Pi(z+\alpha)) \neq 1\} ,$$

and *aperiodic* otherwise. If Γ is a face and $l_\Gamma \supset \Gamma$ is a plane, such that $\dim l_\Gamma = \dim \Gamma$, then $D_\Gamma(z+\alpha) = D_\Gamma(z)$ for all $\alpha \in l_\Gamma \cap \mathbb{Z}^n$ whence it follows that $\gcd(D(z), D(z+\alpha)) = D_\Gamma(z)$.

From this, it is clear that the set $\mathrm{Spread}(D, D) = \cup_{j=1}^{\#\mathfrak{S}} (l_{\Gamma_j} \cap \mathbb{Z}^n)$ is infinite. So

Lemma 1. *The denominator of a rational solution to Equation 1.1 is a periodic polynomial.*

The next idea is that the periodic factor in the denominator can be any element of the subring $\mathbb{C}_\Gamma[z]$ and to solve the Equation 1.1 one should look for a universal numerator $N_\Gamma(z)$ such that

$$R_\Gamma(z) = \frac{N_\Gamma(z)}{D_\Gamma(\langle z, q_1\rangle, \ldots, \langle z, q_{n-k}\rangle)} .$$

satisfies (1.1) for any denominator $D_\Gamma \in \mathbb{C}_\Gamma[z]$.

Therefore, we apply the rational function $R_\Gamma(z) = \frac{N_\Gamma(z)}{D_\Gamma(z)}$ to the equation and reduce the expression to a common denominator, in the numerator we get several terms of the form

$$\sum_{\alpha \in (l_\Gamma + x) \cap A} p_\alpha N_\Gamma(z+\alpha) \prod_\beta D_\Gamma(\langle z+\beta, q_1\rangle, \ldots, \langle z+\beta, q_{n-k}\rangle)$$

each of which should be equal to zero. We obtain a system of homogeneous difference equations with constant coefficients to the unknown polynomial $N_\Gamma(z)$:

$$\left\{ \sum_{\alpha \in (l_\Gamma + x) \cap A} p_\alpha N_\Gamma(z+\alpha) = 0 \text{ , where } x \in \mathbb{R}^n \; : \; (l_\Gamma + x) \cap A \neq \varnothing \right\} . \tag{3.1}$$

The existence of $N_\Gamma(z)$ depends on the compatibility of the obtained system and solvability of each equation in the class of polynomials. A necessary condition for this is the equality $\sum_{\alpha\in(l_\Gamma+x)\cap A} p_\alpha = 0$ [3]. If this condition is satisfied for all $x \in \mathbb{R}^n$, then at least there is a zero degree solution $N_\Gamma(z) \equiv const$. Solutions $N_\Gamma(z)$ of other degrees can be found by the method of undetermined coefficients and by reducing the finding of $N_\Gamma(z)$ to a system of polynomial equations for unknown coefficients, which can be solved using known algorithms [12]. The solution in the form (2.2) can be obtained by reducing to a common denominator the sum (over Γ faces) of all found solutions.

Please note that if the line $l_\Gamma + x$ and the set A intersect at one point (the necessary condition is not satisfied), then the sum $\sum_{\alpha\in(l_\Gamma+x)\cap A} p_\alpha$ has only one term, and it is not equal to zero. Thus, we can formulate a criterion for the existence of the solution with a denominator from the class $\mathbb{C}_\Gamma[z]$.

Theorem 2. *The set of functions* $\{\frac{N_\Gamma(z)}{D_\Gamma(z)}\}$*, where* $N_\Gamma(z)$ *is some polynomial and* $D_\Gamma(z)$ *is an arbitrary element from the subring* $\mathbb{C}_\Gamma[z]$ *satisfies (1.1) iff for any* $x \in \mathbb{R}^n$

$$\sum_{\alpha\in(l_\Gamma+x)\cap A} p_\alpha = 0\ ,$$

where $l_\Gamma \supset \Gamma$*,* $\dim l_\Gamma = \dim \Gamma$*.*

Proof. As we have seen, the class $\{\frac{N_\Gamma(z)}{D_\Gamma(z)}\}$, where $N_\Gamma(z)$ is some polynomial, and $D_\Gamma(z)$ an arbitrary element from the subring $\mathbb{C}_\Gamma[z]$ can be a solution to the Equation 1.1 if and only if the system (3.1) is joint and has a (polynomial) solution.

Each equation of the system (3.1) is a homogeneous difference equation with constant coefficients. According to [3], such an equation is solvable in the class of polynomials if and only if the sum of all coefficients of this equation is zero. At the same time, there is a solution (polynomial) of any degree, including the zero degree (constant). So, if all equations of the system (3.1) are solvable in the class of polynomials, then, at least, a polynomial of degree zero satisfies the whole system of equations (3.1).

If in at least one of the equations of the system the sum of the coefficients is not equal to zero, then this equation is not solvable in the class of polynomials, and the system 3.1 is inconsistent. Consequently, there is no such polynomial $N_\Gamma(z)$ that the whole class $\{\frac{N_\Gamma(z)}{D_\Gamma(z)}\}$ satisfies the Equation 1.1. □

Example 4 shows that the solvability criterion for the Equation 1.1 can be satisfied for l_Γ planes such that $\dim l_\Gamma > \dim \Gamma$ and fails if $\dim l_\Gamma = \dim \Gamma$. Therefore, the sufficient condition we formulate as follows.

Sufficient condition: *If there is a plane l such that for any $x \in \mathbb{R}^n$*

$$\sum_{\alpha \in (l+x) \cap A} p_\alpha = 0 \ ,$$

then Equation 1.1 is solvable in the class of rational functions.

If a sufficient condition is met, functions of the form $\frac{1}{D_l(z)}$, where $D_l(z) \in \mathbb{C}_l[z]$ are solutions to the Equation 1.1. Moreover, if $\dim l > \dim \Gamma$, where $\Gamma \subset l$, then $\mathbb{C}_l[z] \subsetneq \mathbb{C}_\Gamma[z]$.

4. Examples

Example 1. *Let the characteristic polynomial of the Equation 1.1 be $1 + \sum_{j=1}^n \zeta_j$, then (1.1) will be written in the form*

$$R(z) + \sum_{j=1}^{n} R(z_1, \ldots, z_{j-1}, z_j + 1, z_{j+1}, \ldots, z_n) = 0 \ . \tag{4.1}$$

In Example 1, the Newton polyhedron of the characteristic polynomial is a simplex. Since any plane is able to intersect the simplex at a vertex, the necessary condition for the existence of a rational solution, as obtained in the paper, is not fulfilled for the Equation 4.1. Consequently, this equation cannot be solved within the class of rational functions.

Furthermore, it is noteworthy that the necessary condition for solvability within the class of rational functions, derived in the paper, is also applicable to the inhomogeneous equation with an entire right part. Therefore, the inhomogeneous difference equations with a carrier A consisting of $n+1$ points, investigated in [9], also lack rational solutions.

Example 2. *Let's consider the equation*

$$\sum_{\alpha_i \in \{0,1\}} (-1)^{|\alpha|} R(z_1 + \alpha_1, \ldots, z_n + \alpha_n) = 0$$

with the characteristic polynomial $\sum_{\alpha_i \in \{0,1\}} (-1)^{|\alpha|} \zeta^\alpha$.

The Newton polyhedron of the characteristic polynomial is the hypercube. The hypercube has 2^n vertices, $2n$ hyperfaces, $2^{n-k} C_n^k$ k-dimensional faces, of which C_n^k are not parallel (adjacent to one vertex).

There are n one-dimensional faces at one vertex; they are not parallel. Each such face at vertex 0 can be defined by $(n-1)$ equations

$$\langle x, q_i \rangle = 0 \ , i = 1, \ldots, n, i \neq j \ ,$$

where $q_i = (0, \dots, 0, -1, 0, \dots, 0)$, $j = 1, \dots, n$.

Each such a face can be associated with a solution of the form

$$R^{n-1}(z) = \frac{N(z)}{D^{n-1}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)} ,$$

where D^{n-1} is a polynomial in $n-1$ variables. Substituting $R^{n-1}(z)$ into the equation we obtain that the polynomial $N(z)$ is independent of z_j.

Thus, a rational solution is associated with each one-dimensional face

$$R^{n-1}(z) = \frac{N(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)}{D^{n-1}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)} ,$$

where N and D^{n-1} are arbitrary polynomials in $(n-1)$ variables. And for each of the n hyperfaces at the vertex 0, we can associate a solution of the form

$$\frac{N(z_j)}{D^1(z_j)} , \ j = 1, \dots, n .$$

Finally, with each of the C_n^k faces of dimension k we can associate a solution of the form

$$R^{n-k}(z) = \frac{N(z[k])}{D^{n-k}(z[k])} ,$$

where N and D^{n-k} are arbitrary polynomials in $(n-k)$ variables $z[k]$ from the set $\{z_1, \dots, z_n\}$.

Example 3. *Let's consider the difference equation, which corresponds to the characteristic polynomial*

$$2 - \zeta_1^2 - \zeta_1^{-2} + 2\zeta_1\zeta_2^2 - \zeta_1^{-1}\zeta_2^2 + \zeta_1^{-1}\zeta_2\zeta_3 - 2\zeta_1\zeta_2\zeta_3 . \tag{4.2}$$

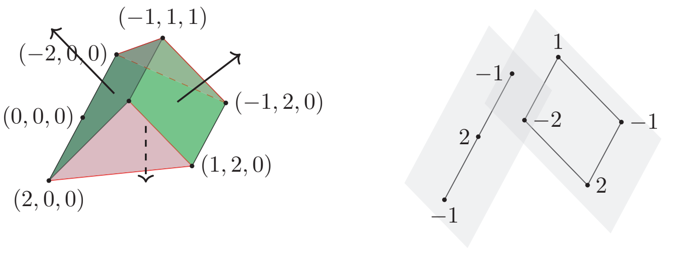

Figure 3. In Example 3 three hyperfaces and three edges satisfy the necessary condition. Only the plane $z_2 + z_3 = 0$ satisfies the sufficient condition.

In this example, the Newton polyhedron of the characteristic polynomial has three one-dimensional faces (edges) and three two-dimensional faces

that satisfy the necessary condition. They are defined by three normals $q_1 = (0, 0, -1)$, $q_2 = (0, 1, 1)$, $q_3 = (0, -1, 1)$.
Hyperfaces:

$$\langle x, q_1\rangle = 0, \langle x, q_2\rangle = 2, \langle x, q_3\rangle = 0 .$$

Edges:

$$\begin{cases} \langle x,q_1\rangle=0 \\ \langle x,q_2\rangle=2 \end{cases}, \begin{cases} \langle x,q_2\rangle=2 \\ \langle x,q_3\rangle=0 \end{cases}, \begin{cases} \langle x,q_1\rangle=0 \\ \langle x,q_3\rangle=0 \end{cases} .$$

The criterion of the Theorem 2 holds only for the plane $z_3 + z_2 = 0$, so the function

$$R(z) = \frac{N(z)}{D^1(z_3 + z_2)}$$

is the solution. By employing the equation $P(\delta)R(z) = 0$ and the method of undetermined coefficients, we obtain $N(z) = c_1(z_2 + z_3) \cdot z_1 + c_0(z_2 + z_3)$.

It means that the rational function

$$R(z) = \frac{c_1(z_2 + z_3) \cdot z_1 + c_0(z_2 + z_3)}{D^1(z_3 + z_2)} ,$$

where c_1, c_0, D^1 are arbitrary polynomials in one variable, satisfies the difference equation with the characteristic polynomial (4.2).

Example 4. *Let's consider the difference equation, which corresponds to the characteristic polynomial*

$$4 - \zeta_1\zeta_2 - \zeta_1^{-1}\zeta_2 - \zeta_1\zeta_2^{-1} - \zeta_1^{-1}\zeta_2^{-1} + \zeta_2\zeta_3 - \zeta_2^{-1}\zeta_3 + 2\zeta_1\zeta_3^{-1} - 2\zeta_1^{-1}\zeta_3^{-1} . \quad (4.3)$$

All the edges and four faces $\mathbf{Ch}(A) \cap \{z_1 = \pm 1\}$, $\mathbf{Ch}A \cap \{z_2 = \pm 1\}$ of the Newton polyhedron satisfy the necessary condition. Only the plane $z_3 = 0$ satisfies the sufficient condition:

$$\sum_{\alpha \in \{z_3=-1\}\cap A} p_\alpha = 2 - 2 = 0 , \quad \sum_{\alpha \in \{z_3=1\}\cap A} p_\alpha = 1 - 1 = 0 ,$$
$$\sum_{\alpha \in \{z_3=0\}\cap A} p_\alpha = 4 - 1 - 1 - 1 - 1 = 0 .$$

So the solution is the family of rational functions with the denominator $D(z_3)$. The numerator can also be an arbitrary polynomial in the variable z_3. The function $\frac{N(z_3)}{D(z_3)}$ is a rational solution to the difference equation with the characteristic polynomial (4.3).

The plane $z_3 = 0$ intersects (by some shifts) $\mathbf{Ch}(A)$ along two faces:

$$\Gamma_1 = [(0, 1, 1), (0, -1, 1)] ,$$
$$\Gamma_2 = [(1, 0, -1), (-1, 0, -1)] .$$

The subring $\mathbb{C}_{\Gamma_1}[z]$ is formed by polynomials of the form $D_{\Gamma_1}(z_3+z_1, z_3-z_1)$, the subring $\mathbb{C}_{\Gamma_2}[z]$ is formed by polynomials of the form $D_{\Gamma_2}(z_2-z_3, -z_2-z_3)$. The subring $\mathbb{C}_{\{z_3=0\}}[z] \subset \mathbb{C}_{\Gamma_1}[z]$ is formed by polynomials of the form $D_{\Gamma_1}(w_1, w_2) = D(\frac{w_1+w_2}{2}) = D(z_3)$, on the other hand, if we consider $\mathbb{C}_{\{z_3=0\}}[z]$ as a subring in $\mathbb{C}_{\Gamma_2}[z]$, then it will be formed by polynomials $D_{\Gamma_2}(w_1, w_2) = D(\frac{w_1+w_2}{-2}) = D(z_3)$.

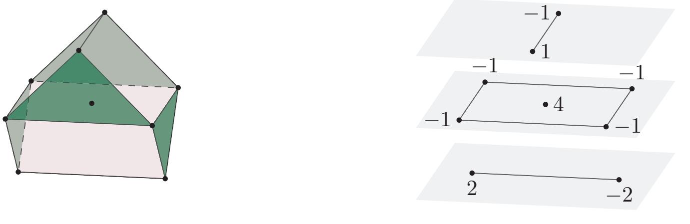

Figure 4. In Example 4, all edges and 4 faces satisfy the necessary condition; only the plane $z_3 = 0$ satisfies the sufficient condition.

5. Conclusion

In this paper, we have established necessary and sufficient conditions for finding the solution to Equation 1.1 within the rational functions class.

In other words, we have demonstrated that every multiplier located in the denominator of a rational solution can be represented as an element within the subring $\mathbb{C}_\Gamma[z]$, corresponding to a specific edge Γ in the Newton polyhedron of the characteristic polynomial of the difference equation.

This concept of the associated subring $\mathbb{C}_\Gamma[z]$ can be highly useful in addressing the challenging and currently unresolved problem of identifying periodic denominator multipliers for rational solutions to difference equations.

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