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Algebras of Binary Isolating Formulas for Strong Product Theories

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Abstract. Algebras of distributions of binary isolating and semi-isolating formulas are objects that are derived for a given theory, and they specify the relations between binary formulas of the theory. These algebras are useful for classifying theories and determining which algebras correspond to which theories. In the paper, we discuss algebras of binary formulas for strong products and provide Cayley tables for these algebras. On the basis of constructed tables we formulate a theorem describing all algebras of distributions of binary formulas for the theories of strong multiplications of regular polygons on an edge. In addition, we shows that these algebras can be absorbed by simplex algebras, which simplify the study of that theory and connect it with other algebraic structures. This concept is a useful tool for understanding the relationships between binary formulas of a theory.

Keywords: algebra of binary isolating formulas, strong product, model theory, Cayley tables

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Научная статья

Алгебры бинарных изолирующих формул для теорий сильных произведений

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Аннотация. Рассматриваются алгебры бинарных формул для сильных произведений и приводятся таблицы Кэли для этих алгебр. На основе построенных таблиц формулируется теорема, описывающая все алгебры распределений бинарных формул для теорий сильных умножений правильных многоугольников на ребро. Кроме того, показано, что эти алгебры могут быть поглощены симплекс-алгебрами, что упрощает изучение этой теории и связывает ее с другими алгебраическими структурами. Эта концепция является полезным инструментом для понимания отношений между бинарными формулами данной теории.

Ключевые слова: алгебра бинарных изолирующих формул, сильное произведение, теория моделей, таблицы Кэли

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1. Introduction

The present paper continues the study of algebras of distributions of binary isolation formulas [1–3; 8; 9; 11]. These algebras were described for theories of unars, Cartesian products of graphs, Archimedean bodies, polygonometric theories and some generalizations of them for semi-isolating formulas [4; 5; 10]. Algebras of binary isolating formulas serve as derived objects for these theories and reflect binary relationships between types and realizations. The general properties of these algebras are explored in [8], which characterizes a class of such algebras.

The paper provides multiplication tables for various examples and examines the interaction between these algebras and the structures of the initial theories. Describing the algebras for binary isolating formulas allows for the classification of theories in this class through derived algebraic objects. This paper proposes such a classification for structures of strong products of graphs bounded by sets of realizations of fixed type, with the theorem 1 showing the absorption of algebras by simplex algebras. These algebras can help in reconstructing the binary structure of a given 1-type to some extent.

2. Algebras of binary isolating formulas for strong product theories

Definition 1. [6] The *strong product* $G \boxtimes H$ of graphs G and H is a graph such that the vertex set of $G \boxtimes H$ is the Cartesian product $V(G) \times V(H)$; and distinct vertices (u, u') and (v, v') are adjacent in $G \boxtimes H$ if and only if:
$u = v$ and u' is adjacent to v', or $u' = v'$ and u is adjacent to v, or u is adjacent to v and u' is adjacent to v'.

The algebras for the strong product of graphs of edge to edge $H \boxtimes H$ and for the strong product of graphs of triangle and edge $T \boxtimes H$ will have the same set of labels $\rho_{\nu(p)} = \{0, 1\}$ and their tables will be the same:

$\cdot$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0,1\}$

Remark 1. It follows from the definition of strong products of graphs that such products of algebra for segment and regular polygons, give the presence of simplexes in graphs starting from multiplications of the form $H \boxtimes H$.

By strong product of the graph square and edge $Q \boxtimes H$ we obtain two identical algebras with a set of labels $\rho_{\nu(p)} = \{0, 1, 2\}$, which is given by the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Algebra for the strong product of graphs of a pentagon and an edge $P \boxtimes H$, with the set of labels $\rho_{\nu(p)} = \{0, 1, 2\}$ is given by the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

The number of labels directly depends on the diameter of the graph and if diameters of graphs obtained from strong products of n-gonals on an edge coincide, then the algebras of the binary isolation formulas for these products will be isomorphic.

The following algebras for strong products of an edge on a hexagon $Q \boxtimes H$ and an edge on a heptagon $Hp \boxtimes H$ have the same set of labels $\rho_{\nu(p)} = \{0, 1, 2, 3\}$ and will be set by an algebra isomorphic to the algebra $P \boxtimes H$.

Let us denote by HH a graph of two consecutive edges and call it a double edge. The algebra of binary isolating formulas for HH will have a Cayley table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Algebra for the strong product of graphs edge to double edge $H \boxtimes HH$, with the set of labels $\rho_{\nu(p)} = \{0, 1, 2\}$ is given by the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Algebra for the strong product of graphs of a triangle and an double edge $T \boxtimes HH$, with the set of labels $\rho_{\nu(p)} = \{0, 1, 2\}$ is given by the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Algebra for the strong product of graphs of a pentagon and an double edge $P \boxtimes HH$ and of the graph square and double edge $Q \boxtimes HH$, with the set of labels $\rho_{\nu(p)} = \{0, 1, 2\}$ is given by the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

By analogy with the double edge HH let us introduce a triple edge graph.

The algebra for the strong product of the triangle the triple edge graph $T \boxtimes HHH$, with the set of labels $\rho_{\nu(p)} = \{0, 1, 2, 3\}$ is given by the following table:

$\cdot$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
2	$\{2\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
3	$\{3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$

By strong product of the graph square and the triple edge $Q \boxtimes HHH$ we obtain two identical algebras with a set of labels $\rho_{\nu(p)} = \{0, 1, 2, 3\}$, which is given by the following table:

$\cdot$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
2	$\{2\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
3	$\{3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$

By strong product of the graph heptagon and the triple edge $Hp \boxtimes HHH$ we obtain two identical algebras with a set of labels $\rho_{\nu(p)} = \{0, 1, 2, 3\}$, which is given by the following table:

$\cdot$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
2	$\{2\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
3	$\{3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$

Definition 2. *Simplex* or *n-dimensional tetrahedron* (from the latin simplex) is a geometric figure that is a n-dimensional generalization triangle. In the simplest case, the simplex is a triangle.

Algebras for simplex theory [2] will be denoted by $\mathfrak{S}_d$, where d is the diameter of the graph.

We see that the previous algebras with the same graph diameters are isomorphic. So in the case of strong multiplications of an edge, double or triple edge on an n-gon, where algebras with the same diameter of graph were isomorphic, we can state the following:

Theorem 1. *If the strong product for n-gon and chain contains a simplex and has diameter d, then the algebra of binary isolation formulas of this product will coincide with the algebra of binary isolation formulas for simplexes* $\mathfrak{S}_d$.

3. Conclusion

As seen from Theorem 1, if at strong multiplication of graphs by regular n-gon, a simplex appears, then this simplex will absorb the whole algebra. It is also noticed that at strong multiplication of graphs the algebras will be isomorphic if the diameters of obtained graphs are equal.

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