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On Control of Probability Flows with Incomplete Information

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Abstract. The mean-field type control problems with incomplete information are considered. There are several points of view that can be adopted to study the dynamics in probability space. Eulerian framework describes probability flows by the continuity equation. Kantorovich formulation describes each probability flows in terms of a single distribution on the set of admissible trajectories. The superposition principle connects these frameworks for uncontrolled dynamics. In this article, a probability flow in the both frameworks must be generated by a control that based on incomplete information about state and/or the probability at every time instance. This article presents some links between these frameworks in the case of incomplete information. In particular, besides the convexity condition, the assumptions are founded that guarantees the equivalence between the Kantorovich and Eulerian framework. This expands [6, Theorem 1] to mean-field type control problem with incomplete information.

Keywords: probability flow, continuity equation, incomplete information, mean-field optimal control

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Научная статья

Об управлении вероятностными потоками в условиях неопределенности

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Аннотация. Рассматриваются задачи управления средним полем в случае неполной информации. Имеется несколько подходов к описанию динамической системы в пространстве вероятностных мер. Подход, восходящий к Эйлеру, описывает поток заданных вероятностных мер как решение некоторого уравнения неразрывности. Подход, названный в [6] именем Канторовича, задает такой поток как поток образов одной и той же меры, заданной на множестве всех допустимых траекторий. Хорошо известный принцип суперпозиции связывает эти два подхода в случае отсутствия управления. В работе предполагается, что и в той, и в другой формулировке поток вероятностных мер должен быть порожден управлением, соблюдающим все ограничения, включая информационные. При этом неполной может оказаться как информация о позиции, так и информация о реализовавшейся вероятностной мере. Для таких задач управления средним полем исследуются взаимосвязи между указанными выше подходами, в частности найдены условия, помимо предположения выпуклости, гарантирующие эквивалентность этих подходов. Это развивает результат, показанный в [6, Theorem 1], в том числе для случая неполной информации.

Ключевые слова: потоки вероятностных мер, уравнение неразрывности, неполная информация, управление средним полем

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The article deals with a control of a dynamic system in a space of probabilistic measures in the case of incomplete information. The dynamics in a space of probability measures can be described in various ways [1], [6]. In this paper, the Eulerian and Kantorovich frameworks [6] are considered. Eulerian framework suggests that each velocity field sets the continuity equation, and a flow of probability measures is only its distributional solution. In Kantorovich framework we track all trajectories generated all velocity fields; then, we suggest that a flow at every time instance is the push-forward of some distribution on such trajectories by the evaluation map at this time instance. In this paper, based on the superposition principle [1], we investigate the conditions that guarantee the equivalence of these frameworks if the control is determined by a given observation function z and, in addition, the corresponding total resource g is assumed be finite. This result expands the result [6, Theorem 1] to mean-field type control system [4] with information constraints.

The rest of the paper is organised as follows. First, in Section 1, we introduce some general notations. The next two sections are devoted to the assumptions on dynamics f, total resource g, and observation z (Section 2) and its various interpretations (Section 3). The basic definitions of Eulerian and Kantorovich frameworks are given in Section 4. The links between them are investigated in Section 5 (the general case) and Section 6 (the convex case). The applications to the mean-field optimal control are considered in Section 7.

1. Preliminaries

Let $\mathcal{I}$ be a Polish space, i.e., a separable completely metrizable topological space. By $\mathcal{B}(\mathcal{I})$ denote the set of all Borel subsets of $\mathcal{I}$, the σ-algebra generated by all opens subsets of $\mathcal{I}$. Then, by $\mathcal{P}(\mathcal{I})$ denote the space of all Borel probabilities over $\mathcal{I}$. We also endow this probability space with the topology of narrow convergence [1]. Now, for every interval K, denote by $C(K, \mathcal{P}(\mathcal{I}))$ the set of all narrowly continuous functions from K to $\mathcal{P}(\mathcal{I})$.

By $B(\mathcal{I}_1, \mathcal{I}_2)$ denote the set of all Borel measurable maps from a Polish space $\mathcal{I}_1$ to a Polish space $\mathcal{I}_2$. For a Borel map $\phi \in B(\mathcal{I}_1, \mathcal{I}_2)$ and a probability $\nu \in \mathcal{P}(\mathcal{I}_1)$ the pushforward measure $\phi\sharp\nu \in \mathcal{P}(\mathcal{I}_2)$ is defined by the rule:

$$(\phi\sharp\nu)(A) = \nu\{x \in \mathcal{I}_1 \,|\, \phi(x) \in A\} \qquad \forall A \in \mathcal{B}(\mathcal{I}_2).$$

Further, for every Polish space $\mathcal{I}$, an interval K, and a time instance $t \in K$ denote by e_t the evaluation map $B(K, \mathcal{I}) \ni y \mapsto e_t(y) = y(t) \in \mathcal{I}$.

Let consider a real $p > 1$. Denote by $\mathcal{P}_p(Y)$ the set of all Borel measures over a Banach space Y with finite p-th moment. This space is endowed with the metric W_p defined by the rule: for all $m', m'' \in \mathcal{P}_p(Y)$

$$W_p^p(m', m'') \triangleq \inf \Big\{ \int_{Y \times Y} \|y' - y''\|^p \, \nu(y', y'') \,\Big|\, m', m'' \text{ are marginal measures of } \nu \in \mathcal{P}(Y^2) \Big\}.$$

Let A be a set and let B be its subset. The symbol $\imath_B$ denotes the indicator function of the subset B. This function from A to $\{0, +\infty\}$ has value 0 on B and $+\infty$ elsewhere.

2. The dynamics, observation, and resource

Let an Euclidean space $\mathbb{R}^d$ and a time interval $[0;T]$ be given. Let also $\mathbb{R}^d$ and the space $\Gamma \triangleq C([0;T],\mathbb{R}^d)$ are equipped the usual norms.

Let a control set U as well as a set of observation $\mathcal{Z}$ be given. Assume that U and $\mathcal{Z}$ are Polish spaces.

Denote by $\mathcal{U}$ the set of all LB-measurable maps from $[0;T]\times\mathcal{Z}$ to U. Recall that the LB-measurability is the measurability with respect to the σ-algebra generated by the products of Lebesgue measurable subsets in $[0;T]$ and Borel subsets in $\mathcal{Z}$. Every $u\in\mathcal{U}$ is called an admissible control. Notice that every $u\in U$ also is an admissible control.

Let a general dynamics function $f:[0;T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\times U\to\mathbb{R}^d$ be given; assume that this function is Borel.

Let an observation function z be a given Borel mapping from $\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)$ to $\mathcal{Z}$.

Notice that each admissible control $u\in\mathcal{U}$ generates the function f_u : $[0;T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\times\mathcal{U}$ by the following rule:

$$f_u(t,x,\nu)=f(t,x,\nu,u(t,z(x,\nu)))$$

whenever (t,x,ν). All these functions are also LB-measurable.

Let a total resource g be a tuple $(g_1,g_2,\ldots,g_r)$, here every function $g_k:[0;T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\times U\to\mathbb{R}\cup\{+\infty\}$ is Borel.

3. Cases of z and g

Now we will consider different information and resource constraints. These lists are not claiming to be full.

Let's list several classes for observation z.

complete information: $z(x,\nu)\equiv(x,\nu)$, here $\mathcal{Z}\triangleq\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)$;

no information: $z(x,\nu)=0$, here $\mathcal{Z}=\{0\}$; the programmed controls, corresponding to this case, is considered in [8].

only state $z(x,\nu)=x$ with $\mathcal{Z}\triangleq\mathbb{R}^d$;

only distribution $z(x,\nu)=\nu$, here $\mathcal{Z}\triangleq\mathcal{P}(\mathbb{R}^d)$;

only the support: $z(x,\nu)=\operatorname{supp}\nu$, here the Polish space $\mathcal{Z}$ is the set of all non-empty closed subsets of $\mathbb{R}^d$, equipped Wijsman convergence [5];

only barycenter: $z(x,\nu)=\int_{\mathbb{R}^d}y\,\nu(dy)$, here $\mathcal{Z}\triangleq\mathbb{R}^d$;

only some average: $z(x,\nu) = \int_{\mathbb{R}^d} \phi(y)\,\nu(dy)$ for a given Borel function ϕ from $\mathbb{R}^d$ to a Banach space $\mathcal{Z}$;

only some pushforward measure: $z(x,\nu) = \phi\sharp\nu$ for a given Borel function ϕ from $\mathbb{R}^d$ to a Polish space $\mathcal{I}$ and for $\mathcal{Z} \stackrel{\triangle}{=} \mathcal{P}(\mathcal{I})$;

only deviation of state: $z(x,\nu) = x - \int_{\mathbb{R}^d} y\,\nu(dy)$, here $\mathcal{Z} \stackrel{\triangle}{=} \mathbb{R}^d$;

only one observable: $z(x,\nu) = (\phi(x), \phi\sharp\nu)$ if a given Borel function ϕ from $\mathbb{R}^d$ to a Polish space $\mathcal{I}$ is observable with $\mathcal{Z} \stackrel{\triangle}{=} \mathcal{I} \times \mathcal{P}(\mathcal{I})$; e.g.

- $\mathcal{I} = \mathbb{R}$ and $\phi(x) = x_1$ is the first coordinate of x; in this case the probability $\phi\sharp\nu$ coincides with the first marginal distribution $x_1\sharp\nu$;
- $\mathcal{I} = \mathbb{R}^d$ and $\phi(x)$ is a nearest to x point of a given ε-net of $\mathbb{R}^d$.

Recall that the total resource is $g = (g_1, \dots, g_r)$. A such map

$$(t, x, \nu, u) \mapsto g_k(t, x, \nu, u)$$

may be

phase constraints: $\imath_{G(t)}(x,\nu)$ for a given multi-valued mapping G from $[0;T]$ to $\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ that is nonempty-valued and measurable;

control constraints: $\imath_{G(t)}(u)$ for a given multi-valued mapping G from $[0;T]$ to U that is nonempty-valued and measurable;

velocity constraints: $\imath_{G(t,\nu)}(f(t,x,\nu,u))$ for a given multi-valued mapping G from $[0;T] \times \mathcal{P}(\mathbb{R}^d)$ to $\mathbb{R}^d$ that is measurable;

mixed constraints: $\imath_{G(t)}(x,\nu,u)$ for a given multi-valued mapping G from $[0;T]$ to $\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times U$ that is nonempty-valued and measurable; e.g., the proximal normal cone condition in [3];

running cost: a lower semicontinuous function from $[0;T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times U$ to $\mathbb{R}$;

energy condition: either $\|u\|^p$, or $\|x\|^p$, or $\|f(t,x,\nu,u)\|^p$, or an interaction potential [7, (2.11)], or some its sums.

Thereinafter, we prescribe that g_1 is a runnung cost.

4. Eulerian and Kantorovich frameworks

In this section we formulate deterministic mean-field dynamics within Eulerian and Kantorovich frameworks. The Eulerian approach describes a probability flow as the evolution of distribution of agents by the controlled continuity equation. The Kantorovich approach identifies each agent by its trajectory in $\Gamma = C([0;T],\mathbb{R}^d)$ and allows to consider the probability flow as some distribution of a random process on Γ.

Definition 1. *For a certain Borel velocity field $v : [0;T]\times\mathbb{R}^d \to \mathbb{R}^d$ we say that a probability flow $\mu \in C([0;T],\mathcal{P}(\mathbb{R}^d))$ is a distributional solution of the continuity equation*

$$\partial_t\mu(t) + \mathrm{div}(v(t,x)\mu(t)) = 0 \tag{4.1}$$

if, for every smooth function ϕ in $C_c^\infty((0;T)\times\mathbb{R}^d)$, one has

$$\int_0^T\int_{\mathbb{R}^d}\big[\partial_t\phi(t) + \nabla\phi(t,x)v(t,x)\big]\,\mu(t,dx)\,dt = 0. \tag{4.2}$$

Here, $C_c^\infty((0;T)\times\mathbb{R}^d)$ is the set of all smooth compactly supported functions from $(0;T)\times\mathbb{R}^d$ to $\mathbb{R}$.

Definition 2. *We say that a pair $(\mu,u) \in C([0;T],\mathcal{P}(\mathbb{R}^d))\times\mathcal{U}$ is an Eulerian pair iff the flow μ is a distributional solution of continuity equation (4.1) coupled with the velocity field $(t,x)\mapsto v(t,x) \overset{\triangle}{=} f_u(t,x,\mu(t))$ that satisfies the resource constraints:*

$$\int_0^T\int_{\mathbb{R}^d} g\big(t,x,\mu(t),u\big(t,z(x,\mu(t))\big)\big)\,\mu(t,dx)\,dt \text{ is finite.} \tag{4.3}$$

Definition 3. *We say that a pair $(\eta,u_\Gamma)\in\mathcal{P}(\Gamma)\times B([0;T]\times\Gamma,U)$ is a Kantorovich pair iff the probability η is concentrated on the set of absolutely continuous curves, η-a.e. curves $\gamma\in\Gamma$ satisfying the differential equation*

$$\frac{d\gamma(t)}{dt} = f(t,\gamma(t),e_t\sharp\eta,u_\Gamma(t,\gamma)) \text{ a.e. on } [0;T];$$

furthermore,

$$\int_0^T\int_\Gamma g(t,\gamma(t),e_t\sharp\eta,u_\Gamma(t,\gamma))\,\eta(d\gamma)\,dt \text{ is finite,} \tag{4.4}$$

and one has the following implication: for η-almost all curves γ,γ'

$$\Big(z(\gamma(t),e_t\sharp\eta) = z(\gamma'(t),e_t\sharp\eta) \text{ a.e. on } [0;T]\Big) \Rightarrow \Big(u_\Gamma(t,\gamma) = u_\Gamma(t,\gamma') \text{ a.e. on } [0;T]\Big). \tag{4.5}$$

This implication guarantees that the coincidence of the observation on $[0;T]$ gives the coincidence of the controls on this interval.

Notice that resource conditions (4.3) as well as (4.4) may be phase, control and/or velocity constraints.

5. Links between Euler and Kantorovich pairs

The following statement generalizes [6, Proposition 7.4].

Proposition 1. *Let (μ, u) be an Eulerian pair and the flow μ lies in $(AC)^p([0;T], \mathcal{P}_p(\mathbb{R}^d))$.*

Then, there exists a Kantorovich pair (η, u_Γ) that satisfies $e_t\#\eta = \mu(t)$ for all $t \in [0;T]$ and

$$u_\Gamma(t,\gamma) = u(t, z(\gamma(t), \mu(t))) \tag{5.1}$$

for all $\gamma \in \Gamma$ and almost all $t \in [0;T]$; furthermore, one has

$$\int_0^T \int_\Gamma g(t, \gamma(t), e_t\sharp\eta, u_\Gamma(t,\gamma))\, \eta(d\gamma)\, dt \\ = \int_0^T \int_{\mathbb{R}^d} g\big(t, x, \mu(t), u\big(t, z(x, \mu(t))\big)\big)\, \mu(t, dx)\, dt. \tag{5.2}$$

In addition,

$$\int_0^T \int_{\mathbb{R}^d} \|f_u(t, x, \mu(t))\|^p\, \mu(t, dx)\, dt < +\infty \tag{5.3}$$

holds true.

Proof. Let a pair (μ, u) be Eulerian. Define the velocity field $[0;T] \times \mathbb{R}^d \ni (t, x) \mapsto v(t, x) \triangleq f_u(t, x, \mu(t))$. Now, for almost all t and $\mu(t)$-almost all x, one has

$$v(t, x) = f\big(t, x, \mu(t), u\big(t, z(x, \mu(t))\big)\big) \tag{5.4}$$

and the flow μ is a solution of continuity equation (4.1) with this field v.

Since $\mu \in (AC)^p([0;T], \mathcal{P}_p(\mathbb{R}^d))$, due to the superposition principle [1, Theorem 8.2.1], there exists a probability measure $\eta \in \mathcal{P}(\Gamma)$ that is concentrated on the set of all curves $\gamma \in (AC)^p([0;T], \mathbb{R}^d)$ solving

$$\frac{d\gamma(t)}{dt} = v(t, \gamma(t)) \text{ a.e. on } [0;T]$$

and $e_t\#\eta = \mu(t)$ for all $t \in [0;T]$.

Define the control u_Γ by the rule (5.1). Then, η is concentrated on the set of absolutely continuous curves, η-a.e. curves $\gamma \in \Gamma$ satisfying the differential equation $\frac{d\gamma(t)}{dt} = f(t,\gamma(t), e_t\sharp\eta, u_\Gamma(t,\gamma))$. Furthermore, the control u_Γ satisfies implication (4.5) by the definition. Finally, by the Tonelli–Fubini theorem, equality (5.2) as well as condition (4.4) are the direct consequences of (4.3). □

Remark 1. The requirement $\mu \in (AC)^p([0;T], \mathcal{P}_p(\mathbb{R}^d))$ for a given Eulerian pair (μ, u) is satisfied if, one find an $A \in \mathbb{R}$ that one has

$$\max(\|f(t,x,\nu,u)\|^p, \|x\|^p) \leq A\Big(1 + |g_1(t,x,\nu,u)| + \\ + |g_2(t,x,\nu,u)| + \ldots + |g_r(t,x,\nu,u)|\Big). \quad (5.5)$$

for all $(t,x,\nu,u) \in [0;T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times U$.

Indeed, in this case, from (4.3) and (5.5), it follows that $\mu(t)$ lies in $\mathcal{P}_p(\mathbb{R}^d)$ for almost all $t \in [0;T]$. In particular, one find $t' \in [0;T]$ such that $\mu(t')$ lies in $\mathcal{P}_p(\mathbb{R}^d)$. Then, similar to the proof of [6, (6.7)], one has

$$\int_{\mathbb{R}^d} \|x\|^p \,\mu(t',dx) \leq \int_{\mathbb{R}^d} \|x\|^p \,\mu(t'',dx) \\ + p\int_{\min(t',t'')}^{\max(t',t'')} \int_{\mathbb{R}^d} \|v(t,x)\| \cdot \|x\|^{p-1} \,\mu(t,dx)\,dt \\ \leq \int_{\mathbb{R}^d} \|x\|^p \,\mu(t'',dx) + Ap(1 + \int_0^T \int_{\mathbb{R}^d} \Big[\big|g_1(t,x,\mu(t),u)\big| \\ + \ldots + \big|g_r(t,x,\mu(t),u)\big|\Big]\,\mu(t,dx)\,dt.$$

So, all the $\mu(t)$ lie in $\mathcal{P}_p(\mathbb{R}^d)$; furthermore, $\mu \in C([0;T], \mathcal{P}_p(\mathbb{R}^d))$. Now, since from (5.5) it follows (5.3), the solution μ to the corresponding continuity equation (4.1) lies in $(AC)^p([0;T], \mathcal{P}_p(\mathbb{R}^d))$ due to [1, Theorem 8.3.1].

Notice that similar to (5.5) inequalities are typical assumptions for continuity equation. See [6, (3.3)], [3, (4)], and [2, (H3)]. Thereinafter, for simplicity's sake, we assume that (5.5) holds

The following proposition shows that if a Kantorovich pair applies only one admissible control, then one must generate some Eulerian pair.

Proposition 2. *Assume that (5.5) holds true. Let a pair (η, u_Γ) be a Kantorovich pair satisfying*

(U) *one finds a Borel subset $\Gamma' \in \mathcal{B}(\Gamma)$ with $\eta(\Gamma') = 1$ such that the images*

$$V(t,\zeta) \stackrel{\triangle}{=} \{u_\Gamma(t,\gamma) \,|\, \exists \gamma \in \Gamma' \; z(\gamma(t), e_t\sharp\eta) = \zeta\} \quad (5.6)$$

are singletons for all $\zeta \in \{\zeta \in \mathcal{Z} \,|\, \exists\gamma \in \Gamma' \; z(\gamma(t), e_t\sharp\eta) = \zeta\}$ and almost all $t \in [0;T]$.

Then, there exists an Eulerian pair (μ, u) *that satisfies* $\mu(t) = e_t\#\eta$ *for all* $t \in [0;T]$ *and (5.1) for* η*-almost all* $\gamma \in \Gamma$ *and almost all* $t \in [0;T]$. *Furthermore, (5.2) holds true.*

Proof. Let (η, u_Γ) be a Kantorovich pair. Define the flow $\mu : [0;T] \to \mathcal{P}(\mathbb{R}^d)$ by the following rule: $\mu(t) = e_t\#\eta$ for all $t \in [0;T]$.

Decreasing Γ' if it is needed, we can propose that (4.5) holds for all $\gamma, \gamma' \in \Gamma'$.

Since the images $V(t,\zeta)$ are singletons and u_Γ is Borel map, there exists a LB-measurable control $u : [0;T] \times \mathcal{Z} \to U$ satisfying

$$\{u(t, z(\gamma(t), e_t\sharp\eta))\} = \{u_\Gamma(t,\gamma') \,|\, \forall \gamma' \in \Gamma' \ z(\gamma'(t), e_t\sharp\eta) = z(\gamma(t), e_t\sharp\eta)\}$$

for all $\gamma \in \Gamma'$ and almost all $t \in [0;T]$. It means that (5.1) holds true and u is admissible.

Define the velocity field v by the rule (5.4). Further, by the Tonelli–Fubini theorem, condition (4.3) with equality (5.2) are the direct consequences of (4.4). Now, from (5.5) it follows that $\mu(t) \in \mathcal{P}_p(\mathbb{R}^d)$ for almost all $t \in [0;T]$; further, the map

$$\operatorname{supp}\eta \ni \gamma \mapsto R(\gamma) \triangleq \int_0^T \|v(t,\gamma(t))\|\,dt \tag{5.7}$$

is η-summable, in particular, this map is finite η-a.e.

Notice that η is concentrated on the set of all solutions to $\frac{d\gamma(t)}{dt} = v(t,\gamma(t))$, therefore, for all $\phi \in C_c^\infty((0;T)\times\mathbb{R}^d)$ and for η-almost all $\gamma \in \Gamma$, one has

$$0 = \phi(T_-, \gamma(T)) - \phi(0_+, \gamma(T)) = \int_0^T \left[\partial_t\phi(t,\gamma(t)) + \nabla\phi(t,\gamma(t))\frac{d\gamma(t)}{dt}\right] dt.$$

Since the norms of all derivatives of ϕ are bounded by a number N, we obtain

$$\int_0^T \left|\partial_t\phi(t,\gamma(t)) + \nabla\phi(t,\gamma(t))\frac{d\gamma(t)}{dt}\right| dt \le N(1 + R(\gamma))(d+1).$$

Hence, since R is η-summable, the Tonelli–Fubini theorem yields

$$\begin{aligned} 0 &= \int_\Gamma \int_0^T \left[\partial_t\phi(t,\gamma(t)) + \nabla\phi(t,\gamma(t))\frac{d\gamma(t)}{dt}\right] dt\,\eta(d\gamma) \\ &= \int_0^T \int_\Gamma \Big[\partial_t\phi(t,\gamma(t)) + \nabla\phi(t,\gamma(t))v(t,\gamma(t))\Big]\,\eta(d\gamma)\,dt \end{aligned}$$

for all $\phi \in C_c^\infty((0;T)\times\mathbb{R}^d)$. By the definition, we obtain that the flow μ is the distributional solution of (4.1) coupled with velocity field v.

Finally, fix a sequence of $t_n \in [0;T]$ converging to a time instance $\hat{t}$. Consider also a continuous and bounded real functions ϕ on $\mathbb{R}^d$. Then, $\phi(\gamma(t_n))$ converges to $\phi(\gamma(\hat{t}))$ for all $\gamma \in \Gamma$. Since ϕ is bounded, due to the Lebesgue dominated convergence theorem, we obtain that

$$\int_{\mathbb{R}^d} \phi(x)\,\mu(t_n,x) = \int_\Gamma \phi(\gamma(t_n))\,\eta(d\gamma) \to \int_\Gamma \phi(\gamma(\hat{t}))\,\eta(d\gamma) = \int_{\mathbb{R}^d} \phi(x)\,\mu(\hat{t},x)$$

for all continuous and bounded real function ϕ on $\mathbb{R}^d$. Then, by the definition, the sequence of $\mu(t_n)$ narrowly converges to $\mu\big(\lim_{n\to\infty} \hat{t}_n\big)$ for every converging sequence of $t_n \in [0;T]$. It means that μ is a narrowly continuous function, i.e. $\mu \in C([0;T], \mathcal{P}(\mathbb{R}^d))$. Thus, (μ,u) is an Eulerian pair. □

6. Convex case

In this section we assume that

(C1) the set U is convex;

(C2) the maps $\mathcal{U} \ni u \mapsto f_u(t,x,\nu)$ are affine for all $x \in \mathbb{R}^d$, $\nu \in \mathcal{P}(\mathbb{R}^d)$ and almost all $t \in [0;T]$;

(C3) the maps $U \ni u \mapsto g(t,x,\nu,u)$ are convex for all $x \in \mathbb{R}^d$, $\nu \in \mathcal{P}(\mathbb{R}^d)$ and almost all $t \in [0;T]$.

These assumptions are similar to [6, Assumption 3.4] and [2, (C1)–(C3)].

We also assume that

(Z) the set $\{x \in \mathbb{R}^d \,|\, z(x,\nu) = \zeta\}$ is a singleton for $(z(\cdot,\nu))\sharp\nu$-almost all ζ and all $\nu \in \mathcal{P}(\mathbb{R}^d)$.

This condition guarantees that state x may be reconstructed by $z(x,\nu)$ and ν. In particular, this condition is satisfied if either $z(x,\nu) \stackrel{\triangle}{=} (x,\nu)$, or $z(x,\nu) \stackrel{\triangle}{=} x$, or $z(x,\nu) \stackrel{\triangle}{=} x - \int_{\mathbb{R}^d} y\,\nu(dy)$.

The following proposition expands [6, Proposition 7.5].

Proposition 3. *Let inequality (5.5), conditions $(C1)$–$(C3)$ and (Z) hold.*

Then, for every Kantorovich pair (η, u_Γ) there exists an Eulerian pair (μ,u) that satisfies $\mu(t) = e_t\#\eta$ for all $t \in [0;T]$ and

$$\int_0^T \int_\Gamma g(t,\gamma(t), e_t\sharp\eta, u_\Gamma(t,\gamma))\,\eta(d\gamma)\,dt \le \int_0^T \int_{\mathbb{R}^d} g\big(t,x,\mu(t),u\big(t,z(x,\mu(t))\big)\big)\,\mu(t,dx)\,dt, \quad (6.1)$$

Proof. Let (η, u_Γ) be a Kantorovich pair. Define the flow $\mu : [0;T] \to \mathcal{P}(\mathbb{R}^d)$ by the following rule: $\mu(t) = e_t\#\eta$ for all $t \in [0;T]$.

Recall that the map z is Borel map from Polish space $\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ to $\mathcal{Z}$; it follows that the map $\mathbb{R}^d \ni x \mapsto z_t(x) \triangleq z(x, \mu(t))$ is also Borel. By assumption (Z), there exists a Borel function $X : [0;T] \times \mathcal{Z} \to \mathbb{R}^d$ that satisfies $\{X(t,\zeta)\} = \{x \in \mathbb{R}^d \,|\, z(x,\mu(t)) = \zeta\}$ for $z_t\sharp\mu(t)$-almost all ζ and almost all $t \in [0;T]$.

By the disintegration theorem [1, Theorem 5.3.1], for almost all $t \in [0;T]$, one find a Borel measurable family of probability measures $\mu_{t,x} \in \mathcal{P}(\Gamma)$, $x \in \mathbb{R}^d$, that satisfies $\mu_{t,x}\{\gamma \,|\, \gamma(t) = x\} = 1$ for all $\mu(t)$-a.e. $x \in \mathbb{R}^d$ and

$$\int_{\mathbb{R}^d} \int_\Gamma \phi(x)\, \mu_{t,x}(d\gamma)\, (e_t\sharp\eta)(dx) = \int_{\mathbb{R}^d} \phi(x)\, \mu(t, dx)$$

for all Borel bounded functions $\phi : \mathbb{R}^d \to \mathbb{R}$.

Define

$$u(t,\zeta) = \int_\Gamma u_\Gamma(t,\gamma)\, \mu_{t,X(t,\zeta)}(d\gamma)$$

for all $\zeta \in \mathcal{Z}$ and almost all $t \in [0;T]$. Since $\mathcal{U}$, as U, is convex, this control is well-defined; furthermore, by $z_t(X(t,\zeta)) = \zeta$ for $(z(\cdot,\nu))\sharp\nu$-almost all ζ, this control is admissible and

$$u(t, z_t(x)) = u(t, z(x,\mu(t))) = \int_\Gamma u_\Gamma(t,\gamma)\, \mu_{t,x}(d\gamma)$$

for $\mu(t)$-almost all x.

Define the velocity field v by the rule (5.4). Since, by the convexity of g, from the Jensen's inequality it follows

$$g(t, x, e_t\sharp\eta, u(t, z_t(x))) \leq \int_\Gamma g(t, x, e_t\sharp\eta, u_\Gamma(t,\gamma))\, \mu_{t,x}(d\gamma)$$

for $\mu(t)$-almost all x and almost all $t \in [0;T]$, the relations (4.4) and (6.1) are the direct consequences of (4.3). Now, (5.5) entails that the map R (5.7) is η-summable.

Let's consider a function $\phi \in C_c^\infty((0;T) \times \mathbb{R}^d)$. The support of ϕ is a compact subset of $(0;T) \times \mathbb{R}^d$, therefore $\partial_t\phi$ and $\nabla\phi$ are bounded. Hence, the function $(t,\gamma) \mapsto \partial_t\phi(t,\gamma(t)) + \nabla\phi(t,\gamma(t))\frac{d\gamma(t)}{dt}$ is $\lambda \otimes \eta$-summable. Integrating the identity

$$0 = \phi(T_-, \gamma(T)) - \phi(0_+, \gamma(0)) = \int_0^T \left[\partial_t\phi(t,\gamma(t)) + \nabla\phi(t,\gamma(t))\frac{d\gamma(t)}{dt}\right] dt$$

and using Fubini–Tonelli theorem, we have

$$\begin{aligned}0 &= \int_\Gamma \int_0^T \Big[\partial_t\phi(t,\gamma(t)) + \nabla\phi(t,\gamma(t))\frac{d\gamma(t)}{dt}\Big]\, dt\, \eta(d\gamma)\\ &= \int_0^T \int_\Gamma \Big[\partial_t\phi(t,\gamma(t)) + \nabla\phi(t,\gamma(t))\frac{d\gamma(t)}{dt}\Big]\, \eta(d\gamma)\, dt\\ &= \int_0^T \int_{\mathbb{R}^d} \Big[\partial_t\phi(t,x) + \int_\Gamma \nabla\phi(t,\gamma(t))\frac{d\gamma(t)}{dt}\, \mu_{t,x}(d\gamma)\Big]\, \mu(t,dx)\, dt.\end{aligned}$$

On the other hand, by $\mu_{t,x}\{\gamma \mid \gamma(t) = x\} = 1$ and $(C2)$, we also obtain that

$$\begin{aligned}\int_\Gamma \nabla\phi(t,\gamma(t))\frac{d\gamma(t)}{dt}\, \mu_{t,x}(d\gamma) &= \int_\Gamma \nabla\phi(t,x) f(t,x,e_t\sharp\eta, u_\Gamma(t,\gamma))\, \mu_{t,x}(d\gamma)\\ &= \nabla\phi(t,x) f\Big(t,x,e_t\sharp\eta, \int_\Gamma u_\Gamma(t,\gamma)\, \mu_{t,x}(d\gamma)\Big)\\ &= \nabla\phi(t,x) f\Big(t,x,e_t\sharp\eta, u(t,z(x,\mu(t)))\Big)\\ &= \nabla\phi(t,x) f_u(t,x,e_t\sharp\eta)\end{aligned}$$

for $\mu(t)$-almost all x. Thus, we obtain

$$0 = \int_0^T \int_{\mathbb{R}^d} \Big[\partial_t\phi(t,x) + \nabla\phi(t,x) f_u(t,x,e_t\sharp\eta)\Big]\, \mu(t,dx)\, dt$$

for all $\phi \in C_c^\infty((0;T)\times\mathbb{R}^d)$. By the definition, it means that the flow μ is a distributional solution of (4.1) coupled with the velocity field (5.4).

Repeating the last paragraph in the proof of Proposition 2 word-for-word, we obtain $\mu \in C([0;T], \mathcal{P}(\mathbb{R}^d))$ and (μ, u) is an Eulerian pair. □

The following example will show that assumption (Z) in Proposition 3 is essential and can not be omitted.

Example 1. Put $\mathbb{R}^d = \mathbb{R}^2$, $[0;T] \triangleq [0;2]$, $U \triangleq \mathbb{R}$, $\mathcal{Z} \triangleq \mathbb{R}$. Define

$$\begin{aligned}f_1(t,x_1,x_2,\nu,u) &\triangleq -3\sqrt[3]{x_1}/2, \qquad & f_2(t,x_1,x_2,\nu,u) &\triangleq u,\\ z(x_1,x_2,\nu) &\triangleq x_1, \qquad & g(t,x_1,x_2,\nu,u) &\triangleq \|(x_1,x_2)\|^p + |u|^p.\end{aligned}$$

for all $(t,x_1,x_2,\nu,u) \in [0;2]\times\mathbb{R}^2\times\mathcal{P}(\mathbb{R}^2)\times\mathbb{R}$. Notice that conditions $(C1)$–$(C3)$ hold.

For all $t \in [0;2]$ and $\gamma = (\alpha,\beta) \in C([0;2],\mathbb{R}^2)$ consider also

$$u_\Gamma(t,\gamma) = \begin{cases} 0 & t \in [0;1),\\ 5\alpha(t-1)/2, & t \in [1;2].\end{cases} \tag{6.2}$$

Now, for every $c \in [0;1]$ define the solution $\gamma_c = (\alpha_c, \beta_c)$ to the system

$$\frac{d\gamma(t)}{dt} = (f_1, f_2)(t, \gamma(t), \nu, u_\Gamma(t, \gamma)) = (-3\sqrt[3]{\alpha(t)}/2, 5\alpha(t-1)/2) \qquad (6.3)$$

that satisfies $\alpha_c(0) = c^{3/2}$ and $\beta_c(0) = 0$. It's easy to calculate that

$$\alpha_c(t) = \begin{cases} (c-t)^{3/2}, & t \in [0;c), \\ 0, & t \in [c;2]; \end{cases}$$

$$\beta_c(t) = \begin{cases} 0, & t \in [0;1), \\ c^{5/2} - (c+1-t)^{5/2}, & t \in [1;1+c), \\ c^{5/2}, & t \in [1+c;2]. \end{cases}$$

In particular, $(\alpha_c(1), \beta_c(1)) = (0,0)$ and $z(\alpha_c(t), \beta_c(t), \nu) = \alpha_c(t) = 0$ for all $t \geq 1$ and $c \in [0;1]$.

Let C be a random variable with the uniform (rectangular) distribution on $[0;1]$. Let η be the distribution of the random process $\gamma_C(\cdot) = (\alpha_C, \beta_C)(\cdot)$. We claim that (η, u_Γ) is a Kantorovich pair. Indeed, the probability η is concentrated on solutions of (6.3), the total resource is bounded; finally, since u_Γ is a function depend on the time instance and the map $t \mapsto z(\gamma(t), e_t\sharp\eta)$, we obtain (4.5). So, (η, u_Γ) is a Kantorovich pair.

Define the flow $\mu : [0;T] \to \mathcal{P}(\mathbb{R}^d)$ by the following rule: $\mu(t) = e_t\#\eta$ for all $t \in [0;2]$. Then, for all $t \in [1;2]$ the probability $\mu(t)$ is concentrated on $\{0\} \times [0; 1-(2-t)^{5/2}]$. This yields that $x_1 = 0$, and $z(x, \mu(t)) = 0$ for $\mu(t)$-almost all $x = (x_1, x_2) \in \mathbb{R}^2$ if $t \in [1;2]$. Now, $z(x, \mu(t)) \equiv 0$ entails that the map $x \mapsto f_2(t, x, \mu(t), u(t, z(x, \mu(t))) = u(t, z(x, \mu(t))$ is constant on $\operatorname{supp}\mu(t)$ for almost all $t \in [1;2]$ for all admissible controls $u \in \mathcal{U}$.

Let us prove that there no Eulerian pair (μ, u). Suppose it is false, there could be an Eulerian pair (μ, u) for some admissible control $u \in \mathcal{U}$. Then, by Proposition 1, there exists a Kantorovich pair (η', u'_Γ) that satisfies $\mu(t) = e_t\#\eta'$ and (5.1) for all $t \in [0;2]$. On the one hand, from $\operatorname{supp}\mu(t) \subset \{0\} \times \mathbb{R}$ for $t \in [1;2]$, it follows that $\alpha|_{[1;2]} = 0$ for η'-almost all $\gamma = (\alpha, \beta)$. Then, $\frac{d\alpha(t)}{dt} = 0$ for almost all $t \in [1;2]$ and η'-almost all $\gamma = (\alpha, \beta)$. On the other hand, the map $x \mapsto f_2(t, x, \mu(t), u(t, z(x, \mu(t)))$ is constant on $\operatorname{supp}\mu(t)$, therefore, by (5.1), $(\alpha, \beta) \mapsto \frac{d\beta(t)}{dt}$ is constant map η'-a.e. on Γ for almost all $t \in [1;2]$. Thus, the map $[1;2] \times \operatorname{supp}\eta' \ni (t, \gamma) \mapsto \frac{d\gamma(t)}{dt}$ is independent of γ. However, since $e_1\sharp\eta' = \mu(1)$ is concentrated on $\{(0,0)\}$, the probabilities $e_t\sharp\eta' = \mu(t)$, $t \in [1;2]$, would be atomic. This would contradict $\operatorname{supp}\mu(2) = \{0\} \times [0;1]$. Thus, there are no Eulerian pairs (μ, u).

7. Application to mean-field type optimal control problem

Recall that g_1 is a running cost.
Fix an endpoint cost $\sigma : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$.

Definition 4. *We say that an Eulerian pair $(\hat{\mu}, \hat{u})$ is an Eulerian minimizer (with cost (g_1, σ)) iff $\sigma(\hat{\mu}(0), \hat{\mu}(T))$ is finite and every Eulerian pair (μ, u) satisfies*

$$\int_0^T \int_{\mathbb{R}^d} g_1\big(t, x, \mu(t), \hat{u}\big(t, z(x, \hat{\mu}(t))\big)\big)\, \hat{\mu}(t, dx)\, dt + \sigma(\hat{\mu}(0), \hat{\mu}(T)) \\ \leq \int_0^T \int_{\mathbb{R}^d} g_1\big(t, x, \mu(t), u\big(t, z(x, \mu(t))\big)\big)\, \mu(t, dx)\, dt + \sigma(\mu(0), \mu(T)).$$

Let's consider the following mean-field type optimal control problem:

$$\text{minimize} \int_0^T \int_{\mathbb{R}^d} g_1\big(t, x, \mu(t), u\big(t, z(x, \mu(t))\big)\big)\, \mu(t, dx)\, dt + \sigma(\mu(0), \mu(T))$$
$$\text{subject to } \partial_t \mu(t) + \operatorname{div}\big(f(t, x, \mu(t), u(t, z(x, \mu(t))))\mu(t)\big) = 0, \quad u \in \mathcal{U},$$
$$\int_0^T \int_{\mathbb{R}^d} g\big(t, x, \mu(t), u\big(t, z(x, \mu(t))\big)\big)\, \mu(t, dx)\, dt \text{ is finite.}$$

Then, every Eulerian minimizer $(\hat{\mu}, \hat{u})$ (with cost (g_1, σ)) is its optimal solution.

Definition 5. *We say that a Kantorovich pair $(\hat{\eta}, \hat{u}_\Gamma)$ is a Kantorovich minimizer (with cost (g_1, σ)) iff $(e_0\sharp\hat{\eta}, e_T\sharp\hat{\eta})$ is finite and every Kantorovich pair (η, u_Γ) satisfies*

$$\int_0^T \int_\Gamma g_1(t, \gamma(t), e_t\sharp\hat{\eta}, \hat{u}_\Gamma(t, \gamma))\, \hat{\eta}(d\gamma)\, dt + \sigma(e_0\sharp\hat{\eta}, e_T\sharp\hat{\eta}) \\ \leq \int_0^T \int_\Gamma g_1(t, \gamma(t), e_t\sharp\eta, u_\Gamma(t, \gamma))\, \eta(d\gamma)\, dt + \sigma(e_0\sharp\eta, e_T\sharp\eta).$$

So, a Kantorovich minimizer $(\hat{\eta}, \hat{u}_\gamma)$ (with (g_1, σ)) is the optimal solution of the following mean-field type optimal control problem:

$$\text{minimize} \int_0^T \int_\Gamma g_1(t, \gamma(t), e_t\sharp\eta, u_\Gamma(t, \gamma))\, \eta(d\gamma)\, dt + \sigma(e_0\sharp\eta, e_T\sharp\eta)$$
$$\text{subject to } \frac{d\gamma(t)}{dt} = f(t, x, e_t\sharp\eta, u_\Gamma(t, \gamma)) \text{ for } \eta\text{-a.a. } \gamma \in \Gamma,$$
$$\eta \in \mathcal{P}(\Gamma), \quad u_\Gamma \in B([0; T] \times \Gamma, \mathbb{R}^d),$$
$$\int_0^T \int_\Gamma g(t, \gamma(t), e_t\sharp\eta, u_\Gamma(t, \gamma))\, \eta(d\gamma)\, dt \text{ is finite and (4.5) holds.}$$

The following corollary is the direct consequence of Propositions 1–3 and Remark 1.

Corollary 1. *Assume that (5.5) holds true. Then the following statements also hold true:*

($\imath$) *Let a Kantorovich pair* (η, u_Γ) *be a Kantorovich minimizer (with cost* (g_1, σ)*) and*

- *either* U*,* f*, and* g *satisfy convexity assumptions* $(C1)$–$(C3)$ *and* z *satisfies assumption* (Z)*;*
- *or* u_Γ *satisfies the condition* (U)*.*

Then, there exists an Eulerian minimizer (μ, u) *(with* (g_1, σ)*) that satisfies* $e_t \# \eta = \mu(t)$ *for all* $t \in [0; T]$*.*

($\imath\imath$) *Conversely, for an Eulerian minimizer* (μ, u) *(with* (g_1, σ)*), there exists a Kantorovich minimizer* (η, u_Γ) *(with to* (g_1, σ)*) that satisfies* $e_t \# \eta = \mu(t)$ *for all* $t \in [0; T]$ *and (5.1) for* η*-almost all* $\gamma \in \Gamma$ *and almost all* $t \in [0; T]$*.*

8. Conclusion

Corollary 1 demonstrates the conditions guaranteeing the equivalence between the Kantorovich and Eulerian frameworks in mean-field type optimal control problems. In the case of complete information this result expands [6] by the condition (U). Example 1 does not allow to transfer directly the conditions [6, Theorem 7.3] for probability flows with incomplete information. Is it due to non-Markovian strategy (6.2)? Is it possible to weaken the assumptions (Z) and (U)? This is another question the author does not know the answer to.

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