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About an Estimation Problem of a Linear System with Delay of Information

Boris I. Ananyev[1]✉, **Polina A. Yurovskikh**[1]

[1] N. N. Krasovskii Institute of Mathematics and Mechanics UB RAS, Yekaterinburg, Russian Federation
✉ abi@imm.uran.ru

Abstract. The problem of guaranteed estimation with geometrically bounded initial states and integrally limited disturbances is considered under delay information in the measurement equation. At the additional assumptions the problem is reduced to the creation of the reachable set of a special system. A discrete multistage system is specified for which the information set converges in Hausdorff's metric to the corresponding information set of the continuous system when the diameter of partition is reduced. A numerical example is given.

Keywords: guaranteed estimation, information set, reachable set

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Научная статья

О задаче оценивания линейной системы с запаздыванием информации

Б. И. Ананьев[1✉], **П. А. Юровских**[1]

[1] Институт математики и механики им. Н. Н. Красовского УрО РАН, Екатеринбург, Российская Федерация
✉ abi@imm.uran.ru

Аннотация. Рассмотрена задача гарантированного оценивания с геометрически ограниченными начальными состояниями и интегрально ограниченными возмущениями при запаздывании информации в уравнении измерения. При дополнительных предположениях проблема сведена к построению области достижимости специальной системы. Указана дискретная многошаговая система, для которой информационное множество сходится в метрике Хаусдорфа к соответствующему информационному множеству непрерывной системы при уменьшении диаметра разбиения отрезка наблюдения. Приведен численный пример.

Ключевые слова: гарантированное оценивание, информационное множество, область достижимости

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1. Introduction

The article is a continuation of works [1; 2] and follows the approach from the monograph [7]. A linear stationary system with an unknown state vector is considered. The measurement equation contains delays in the state, which is expressed by an integral with a function of bounded variation. In addition, both the system and the measurement equation additively include an uncertain and integrally bounded disturbance. It is assumed that the system state is bounded by the geometric inclusion at the initial instant. It is required to find an information set which with guarantee contains the true state of the system at a given instant. We note that the similar problem without delay were investigated in detail in the papers cited above. Guaranteed estimation problems for systems with delays were also considered in the works [3; 4; 6] and in many others. The difference of the work is that we do not use the space of infinite dimension and reduce the problem to the construction of reachable sets of a special finite-dimensional system. Various computational methods for the approximation of reachable sets for linear and nonlinear dynamic systems were investigated in [5; 10]. Note that we do not consider geometric restrictions on disturbances. For further researches of similar problems with geometric restrictions there can be useful results of works [8; 9].

We consider a linear autonomous system with delay in the measurement equation:

$$\dot{x}(t) = Ax(t) + Bv(t), \quad t \geq 0, \quad x(t) \in \mathbb{R}^n, \quad v(t) \in \mathbb{R}^l, \tag{1.1}$$

$$y(t) = \mathcal{G}x(t) + Hv(t), \quad y(t) \in \mathbb{R}^m, \quad \mathcal{G}x(t) = \int\limits_{-(h\wedge t)}^{0} dG(s)x(t+s), \tag{1.2}$$

where the given matrices A, B, H have the corresponding dimension, and the matrix function $G(s)$ of bounded variation in the operator $\mathcal{G}$ from (1.2), where $h > 0$, is left continuous to $[-h, 0]$, $G(s) = 0$ $\forall s > 0$, and $G(s) = G(-h)$ $\forall s \leq -h$; $h \wedge t = \min\{h, t\}$.

We assume that the uncertain function $v(\cdot) \in L_2^l[0, \infty)$ and the unknown initial state $x_0 \in \mathbb{R}^n$ are bounded by:

$$\int_0^\infty |v(t)|^2 dt \leq 1, \tag{1.3}$$

$$x_0 \in X_0, \tag{1.4}$$

where X_0 is a convex compact set; $|\cdot|$ is the Euclidean norm. We assume that the state $x(T)$ of system (1.1) can be realized for any $v(\cdot)$, x_0 satisfying the constraints (1.3) and (1.4) at any instant $T > h$.

2. System conversion and problem statement

We assume that the matrix $H \in \mathbb{R}^{m\times l}$ has $\operatorname{rank} H = m$, which implies the condition

$$HH' > 0, \tag{2.1}$$

where $'$ means transpose and the relation $Q > 0$ for square matrix Q means $x'Qx > 0$ for every $x \neq 0$. Let $C = (HH')^{-1}$ and let $C_1 = I_l - H'CH$ is orthogonal projection onto the subspace $\ker H$. Then $v(t) = H'CHv(t) + C_1 v(t)$ and $Hv(t) = y(t) - (\mathcal{G}x)(t)$. Therefore, constraint (1.3) takes the form

$$\int_0^\infty \left(|y(t) - \mathcal{G}x(t)|_C^2 + |v(t)|_{C_1}^2 \right) dt \leq 1 \tag{2.2}$$

due to orthogonality. Hereinafter, the symbol $|x|_P^2$ denotes the quadratic form $x'Px$, where the matrix P satisfies the condition $P' = P \geq 0$, $|x|_P = \sqrt{x'Px}$. We assume $|x|_I^2 = |x|^2$, where $P = I$ is the identity matrix. Substituting the decomposition of the function $v(t)$ into (1.1) we have

$$\dot{x}(t) = Ax(t) + B\left(C_1 v(t) + H'C\left(y(t) - \mathcal{G}x(t)\right)\right).$$

The resulting relation is a differential equation with a distributed delay. In the paper, we will not consider such equations and suppose

$$BH'C \int\limits_{[-(h\wedge t),0)} dG(s)x(t+s) = 0 \tag{2.3}$$

for all continuous n-dimensional functions $x(t)$, $t \geq 0$. Because of

$$\mathcal{G}x(t) = -G(0)x(t) + \int\limits_{[-(h\wedge t),0)} dG(s)x(t+s),$$

we come to the system

$$\begin{array}{c} \dot{x}(t) = \mathbf{A}x(t) + \mathbf{b}y(t) + BC_1v(t), \\ \mathbf{A} = A + \mathbf{b}G(0), \quad \mathbf{b} = BH'C, \end{array} \tag{2.4}$$

without delay.

Remark 1. Condition (2.3) is always satisfied if $BH' = 0$. In this case, the equation in (2.4) has the form $\dot{x}(t) = Ax(t) + BC_1v(t)$ and does not depend on the signal $y(t)$.

Remark 2. Similarly to [2], we lower the dimension of the uncertain disturbances $v(t)$ in relations (2.2), (2.4). Since $\ker H \oplus \operatorname{im} H' = \mathbb{R}^l$, $\operatorname{im} C_1 = \ker H$ and $\dim(\operatorname{im} H') = m$, then $\operatorname{rank} C_1 = l - m$. By a well-known linear algebra theorem, we represent C_1 as $C_1 = T\tilde{C}_1T'$, where T is an orthogonal matrix, $TT' = T'T = I_l$ and $\tilde{C}_1$ is a diagonal matrix with zeros and ones since C_1 is a projection matrix, $C_1^2 = C_1$. Removing m zero columns from $\tilde{C}_1$ and denoting the resulting matrix by $\tilde{D}_1$, we get $\tilde{C}_1 = \tilde{D}_1\tilde{D}_1'$ and $C_1 = D_1D_1'$, where $D_1 = T\tilde{D}_1$. Next, we set $u(t) = D_1'v(t) \in \mathbb{R}^{l-m}$, whence we have

$$C_1v(t) = D_1u(t), \quad D_1 \in \mathbb{R}^{l\times(l-m)}, \quad \operatorname{rank} D_1 = l - m.$$

In relations (2.2), (2.4), the Remark 1 equation and further, we will use the function $D_1u(t)$ instead of $C_1v(t)$. Note that $D_1'D_1 = I_{l-m}$. If $l = m$, then $C_1 = 0$ and the function $v(t) = H^{-1}(y(t) - \mathcal{G}x(t))$ becomes known. In that case we set $u(t) = 0$.

Let us introduce a definition.

Definition 1. *A family of state vectors $\mathcal{X}_T(y) = \{x_T\}$ is called an* information set *(IS) if, for any $x_T \in \mathcal{X}_T(y)$, there exists a function $v(\cdot)$ and an initial state x_0 satisfying constraints* (1.3), (1.4) *and such that equalities* (1.1), (1.2) *hold almost everywhere on an interval* $[0, T]$ *with a boundary condition $x(T) = x_T$.*

Considering (2.2) and the reasoning above we come to a statement.

Lemma 1. *Let conditions* (2.1) *and* (2.3) *be satisfied. Then IS* $\mathcal{X}_T(y)$ *is the reachable set of the equation*

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{b}y(t) + BD_1u(t) \tag{2.5}$$

at the instant $T > 0$ *over all disturbances* $u(\cdot)$ *and initial states* x_0 *satisfying constraints*

$$J(T, x_0, u, y) = \int_0^T \left(|y(t) - \mathcal{G}x(t)|_C^2 + |u(t)|^2 \right) dt \le 1 \tag{2.6}$$

and (1.4)*. Here the matrix* D_1 *and the disturbances* $u(t) \in \mathbb{R}^{l-m}$ *are defined according to the Remark* 2*.*

Our aim is to describe the IS $\mathcal{X}_T(y)$ using support function and also to construct an approximating multistage system whose IS converges to $\mathcal{X}_T(y)$ as the diameter of the partition of the segment $[0, T]$ decreases. In addition, the issues of ellipsoidal approximation of IS are considered, similar to how it is done in [2] for systems without information delay.

3. Support function for the information set

Since the IS by its construction is a convex compact set, it is uniquely described by its support function

$$\rho(l|\mathcal{X}_T(y)) = \max_{x \in \mathcal{X}_T(y)} l'x, \quad l \in \mathbb{R}^n. \tag{3.1}$$

Let us represent the solution of the system (2.5) as the sum $x(t) = \bar{x}(t) + \mathbf{x}(t)$, where

$$\dot{\bar{x}}(t) = \mathbf{A}\bar{x} + \mathbf{b}y(t), \quad x_0 \in X_0, \quad \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x} + BD_1u(t), \quad \mathbf{x}(0) = 0.$$

We put $\bar{y}(t) = y(t) - \mathcal{G}\bar{x}(t)$ on $[0, T]$. Then instead of (2.6) we have the inequality

$$\begin{aligned} J(T, x_0, u, y) = \bar{J}(T, x_0, y) + \int_0^T \big(|\mathcal{G}\mathbf{x}(t)|_C^2 + |u(t)|^2 - \\ 2\bar{y}'(t)C\mathcal{G}\mathbf{x}(t)\big) dt \le 1, \quad \text{where} \quad \bar{J}(T, x_0, y) = \int_0^T |\bar{y}(t)|_C^2 dt. \end{aligned} \tag{3.2}$$

We fix x_0 and select in (3.2) the full square in $u(\cdot)$, for which we introduce operator $\mathcal{K}$ according to the relation

$$\int_0^T \left(|\mathcal{G}\mathbf{x}(t)|_C^2 + |u(t)|^2\right) dt = \langle u, \mathcal{K}u \rangle,$$

where $u \in L_2^{l-m}[0, T]$ is an arbitrary function. Let us also introduce the matrix function

$$\mathbf{G}(t) = \int_{-(h \wedge t)}^{0} dG(s) e^{\mathbf{A}(t+s)}, \tag{3.3}$$

with which we represent $\mathcal{G}\mathbf{x}(t) = \mathbf{k}u(t) = \int_0^t \mathbf{G}(t-s)BD_1u(s)ds$. Therefore, the operator $\mathcal{K}$ is written as

$$\mathcal{K} = \mathrm{Id} + \mathbf{k}^*C\mathbf{k}, \quad \text{or}$$
$$\mathcal{K}u(t) = u(t) + D_1'B'\int_0^T\int_{t\vee\tau}^T \mathbf{G}'(s-t)C\mathbf{G}(s-\tau)dsBD_1u(\tau)d\tau,$$

where Id is the identity operator in $L_2^{l-m}[0,T]$ and $\mathbf{k}^*$ is the operator adjoint to $\mathbf{k}$. When choosing a full square in (3.2), it is necessary to solve the equation

$$\mathcal{K}u(\cdot) = \bar{f}(\cdot), \quad \text{where} \quad \bar{f}(t) = \mathbf{k}^*C\bar{y}(t) = D_1'B'\int_t^T \mathbf{G}'(s-t)C\bar{y}(s)ds, \tag{3.4}$$

which is a Fredholm integral equation of the 2-nd kind with non-negative symmetric kernel, which has a unique solution in space $L_2^{l-m}[0,T]$, as is well known. Therefore, the inequality (3.2) becomes

$$\bar{J}(T,x_0,y) + \left\langle u - \mathcal{K}^{-1}\bar{f}, \mathcal{K}\left(u - \mathcal{K}^{-1}\bar{f}\right)\right\rangle - \left\langle \bar{f}, \mathcal{K}^{-1}\bar{f}\right\rangle \le 1. \tag{3.5}$$

Hereinafter $\langle U, V\rangle = \int_0^T U'(t)V(t)dt \in \mathbb{R}^{q\times r}$, where $U(t) \in \mathbb{R}^{p\times q}$, $V(t) \in \mathbb{R}^{p\times r}$ are matrix functions with elements from $L_2[0,T]$. For a fixed vector $l \in \mathbb{R}^n$ and an initial state x_0, we find the value

$$\begin{gathered} R(l,T,x_0,y) = \max_{u(\cdot)} l'x(T) = l'\bar{x}(T) + \left\langle l_T, \mathcal{K}^{-1}\bar{f}\right\rangle + \sqrt{\langle l_T, \mathcal{K}^{-1}l_T\rangle} \\ \times\sqrt{1 + \left\langle \bar{f}, \mathcal{K}^{-1}\bar{f}\right\rangle - \bar{J}(T,x_0,y)}, \quad l_T(t) = D_1'B'e^{\mathbf{A}'(T-t)}l. \end{gathered} \tag{3.6}$$

Finally, we find the support function for the IS:

$$\rho(l \mid \mathcal{X}_T(y)) = \max_{x_0\in X_0,\, 1+\left\langle \bar{f},\mathcal{K}^{-1}\bar{f}\right\rangle \ge \bar{J}(T,x_0,y)} R(l,T,x_0,y). \tag{3.7}$$

Let us summarize.

Theorem 1. *The IS $\mathcal{X}_T(y)$ is a convex compact set under conditions* (2.1) *and* (2.3)*. The IS support function* (3.1) *is defined by formulas* (3.6), (3.7) *with parameters given by relations* (3.2)—(3.5).

4. Some ways of approximation of IS

4.1. Approximation by ellipsoids

Let us introduce the notation $E(Q,c) = \{x \in \mathbb{R}^n : |x - c|_Q \le 1\}$ for nondegenerate ellipsoids, where the matrix $Q = Q' > 0$, $c \in \mathbb{R}^n$. Ellipsoid support function $\rho(l|E(Q,c)) = l'c + \sqrt{l'Q^{-1}l}$. We choose an arbitrary ellipsoid $E(P_0, \bar{x}_0) \supset X_0$ for the initial set X_0 and consider the quadratic constraint

$$(1-\alpha)|x_0 - \bar{x}_0|^2_{P_0} + \alpha \int_0^T |v(t)|^2 dt \le 1, \quad \alpha \in (0,1). \tag{4.1}$$

Denote by $\mathcal{X}_T^{E,\alpha}(y)$ the IS for constraints (4.1). By definition, we have an inclusion for arbitrary specified ellipsoids:

$$\mathcal{X}_T^{E,\alpha}(y) \supset \mathcal{X}_T(y).$$

Repeating the reasoning of Section 2, we come to the inequality

$$\begin{gathered} J_\alpha(T, x_0, u, y) = (1-\alpha)|x_0 - \bar{x}_0|^2_{P_0} + \\ +\alpha \int_0^T \left(|y(t) - \mathcal{G}x(t)|^2_C + |u(t)|^2 \right) dt \le 1, \end{gathered} \tag{4.2}$$

which is similar to (2.6) for the equation (2.5). Using (3.3), the expansion $x(t) = \bar{x}(t) + \mathbf{x}(t)$ and denoting $\mathbf{y}(t) = \bar{y}(t) + \mathbf{G}(t)x_0$, rewrite (4.2)

$$(1-\alpha)|x_0 - \bar{x}_0|^2_{P_0} + \alpha \left(\|\mathbf{y} - \mathbf{G}x_0\|^2_C + \|u\|^2_{\mathcal{K}} - 2\left\langle \mathbf{y} - \mathbf{G}x_0, C\mathcal{G}\mathbf{x} \right\rangle \right) \le 1.$$

We introduce an operator in the space $\mathbb{R}^n \times L_2^{l-m}[0,T]$

$$\begin{gathered} \mathbb{K}_\alpha[x_0; u] = [\mathbf{P}_\alpha x_0 + \alpha \langle \mathbf{D}, u \rangle ; \alpha (\mathbf{D}(\cdot)x_0 + \mathcal{K}u)], \\ \mathbf{P}_\alpha = (1-\alpha)P_0 + \alpha \langle \mathbf{G}, C\mathbf{G} \rangle , \end{gathered} \tag{4.3}$$

where $\mathbf{D}(t) = \mathbf{k}^* C\mathbf{G}(t) = D_1' B' \int_t^T \mathbf{G}'(s-t)C\mathbf{G}(s)ds$, and with its help we transform the last inequality to

$$\begin{gathered} \|[x_0; u]\|^2_{\mathbb{K}_\alpha} + \bar{\mathbf{y}}_\alpha - 2\left\langle [\bar{\mathbf{x}}_\alpha; \alpha\mathbf{f}], [x_0; u] \right\rangle_0 \le 1, \quad \text{where} \\ \bar{\mathbf{y}}_\alpha = (1-\alpha)|\bar{x}_0|^2_{P_0} + \alpha\|\mathbf{y}\|^2_C, \quad \bar{\mathbf{x}}_\alpha = (1-\alpha)P_0\bar{x}_0 + \alpha \langle \mathbf{G}, C\mathbf{y} \rangle , \end{gathered} \tag{4.4}$$

and $\mathbf{f}(t) = \mathbf{k}^* C\mathbf{y}(t) = D_1' B' \int_t^T \mathbf{G}'(s-t)C\mathbf{y}(s)ds$. Here the symbol $\langle \cdot, \cdot \rangle_0$ denotes the inner product in $\mathbb{R}^n \times L_2^{l-m}[0,T]$.

Taking (4.4) into account, we find the support function

$$\begin{gathered} \rho(l|\mathcal{X}_T^{E,\alpha}(y)) = l' \int_0^T e^{\mathbf{A}(T-t)}\mathbf{b}y(t)dt + \left\langle [e^{\mathbf{A}'T}l; l_T], \mathbb{K}_\alpha^{-1}[\bar{\mathbf{x}}_\alpha; \alpha\mathbf{f}] \right\rangle_0 \\ + \sqrt{\|[e^{\mathbf{A}'T}l; l_T]\|^2_{\mathbb{K}_\alpha^{-1}} \left(1 + \|[\bar{\mathbf{x}}_\alpha; \alpha\mathbf{f}]\|^2_{\mathbb{K}_\alpha^{-1}} - \bar{\mathbf{y}}_\alpha \right)}, \end{gathered} \tag{4.5}$$

where the function $l_T(\cdot)$ is defined in (3.6).

To definite $\mathbb{K}_\alpha^{-1}$, it is necessary to minimize an expression of the form

$$|x|^2_{\mathbf{P}_\alpha} + \alpha\|u\|^2_{\mathcal{K}} + 2\alpha x' \langle \mathbf{D}, u \rangle - 2x'q - 2\langle u, v \rangle$$

with respect to $[x;u]$. The solution $[x^*;u^*]$ is $\mathbb{K}_\alpha^{-1}[q;v]$ and the minimum is $-\|[q;v]\|^2_{\mathbb{K}_\alpha^{-1}}$. We come to the relations

$$\begin{gathered} u^*(\cdot)=\mathcal{K}_\alpha^{-1}\tilde v(\cdot),\quad \tilde v(\cdot)=v(\cdot)-\alpha\mathbf{D}(\cdot)\mathbf{P}_\alpha^{-1}q,\\ x^*=\mathbf{P}_\alpha^{-1}\left(q-\alpha\left\langle\mathbf{D},u^*\right\rangle\right),\quad \mathcal{K}_\alpha=\alpha\mathcal{K}-\alpha^2\mathbf{D}\mathbf{P}_\alpha^{-1}\mathbf{D}'. \end{gathered}\tag{4.6}$$

The value $\|[q;v]\|^2_{\mathbb{K}_\alpha^{-1}}=|q|^2_{P_\alpha^{-1}}+\|\tilde v\|^2_{\mathcal{K}_\alpha^{-1}}$. The operator $\mathcal{K}_\alpha$ is coercive and invertible. We have $\mathcal{K}_\alpha=\alpha\mathrm{Id}+\mathbf{k}^*\left(\alpha^{-1}cc'+(1-\alpha)^{-1}\mathbf{G}P_0^{-1}\mathbf{G}^*\right)^{-1}\mathbf{k}$. Here $\mathbf{G}$ means the operator $\mathbf{G}:\mathbb{R}^n\to L_2^m[0,T]$ according to the equality $\mathbf{G}x=\mathbf{G}(\cdot)x$.

From the formula (4.5) it follows that IS $\mathcal{X}_T^{E,\alpha}(y)$ is a non-degenerate ellipsoid. Let $\mathbb{E}$ denotes the family of ellipsoids $E(P_0,\bar x_0)$ with property $E(P_0,\bar x_0)\supset X_0$. Similar to the reasoning in the proof of Theorem 1 in [2] we obtain the statement.

Lemma 2. *Let us define the set* $\mathcal{X}_T^E(y)=\bigcap_{\alpha\in(0,1)}\mathcal{X}_T^{E,\alpha}(y)$, *where* $E\in\mathbb{E}$. *Then*

$$\rho(l|\mathcal{X}_T^E(y))=\max_{x_0\in E,\,1+\left\langle\bar f,\mathcal{K}^{-1}\bar f\right\rangle\geq\bar J(T,x_0,y)}R(l,T,x_0,y)\quad\forall l\in\mathbb{R}^n.\tag{4.7}$$

The notation in the formula (4.7) *is the same as in* (3.7).

The next theorem describes the approximation by ellipsoids.

Theorem 2. *The equality* $\mathcal{X}_T(y)=\bigcap_{E\in\mathbb{E}}\mathcal{X}_T^E(y)$ *holds.*

Using Lemma 1 from [2], the Theorem 3 is proved similar to the proof of Theorem 1 from the cited paper.

4.2. Approximation with multistage systems

Exact and approximate methods for constructing IS using ellipsoids require solving integral equations of the form (3.4) and inverting operators of the form (4.3), (4.6). Let us propose a method for IS approximation with multistage systems. For simplicity, we assume the quantities T, h such that $h=r\Delta$, $T=N\Delta$, where r and N are natural numbers. Let $t_k=k\Delta$, $k\in 0:N$. Assuming the disturbances $u(t)=u_k$ to be constant on the half-intervals $[t_{k-1},t_k)$, $k\in 1:N$, of the partition, we obtain discrete system

$$\begin{gathered} x_k=\mathbf{a}x_{k-1}+\mathbf{Y}_k+\mathbf{B}u_k,\quad \mathbf{a}=e^{\mathbf{A}\Delta},\qquad k\in 1:N\\ \mathbf{Y}_k=\int_0^\Delta e^{\mathbf{A}t}\mathbf{b}y(t_k-t)dt,\quad \mathbf{B}=\int_0^\Delta e^{\mathbf{A}t}dtBD_1. \end{gathered}\tag{4.8}$$

In the inequality (2.6) we assume $\mathcal{G}x(t)=\mathcal{G}x(t_k)$, $t\in[t_{k-1},t_k)$, and

$$\begin{gathered} \mathcal{G}x(t_k)=\mathbf{G}(t)x_0+\mathbf{G}\mathbf{Y}_k+\textstyle\sum_{i=1}^k\mathbf{G}_i^ku_i,\quad\text{where}\\ \mathbf{G}\mathbf{Y}_k=\int_0^{t_k}\mathbf{G}(t_k-s)\mathbf{b}y(s)ds,\quad \mathbf{G}_i^k=\int_{t_{i-1}}^{t_i}\mathbf{G}(t_k-s)BD_1ds. \end{gathered}$$

Then instead of (2.6) we have an inequality with the sum on the left side:

$$J^N(x_0, y, u_{1:N}) = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \left(|y(t) - \mathcal{G}x(t_k)|_C^2 + |u_k|^2 \right) dt \leq 1.$$

We consider the solution of the system (4.8) as the sum $x_k = \bar{x}_k + \mathbf{x}_k$, where

$$\bar{x}_k = \mathbf{a}\bar{x}_{k-1} + \mathbf{Y}_k, \quad x_0 \in X_0, \quad \mathbf{x}_k = \mathbf{a}\mathbf{x}_{k-1} + \mathbf{B}u_k, \quad \mathbf{x}(0) = 0.$$

We put $\bar{y}(t) = y(t) - \mathbf{G}(t)x_0 - \mathbf{G}\mathbf{Y}_k$ on $[t_{k-1}, t_k]$. We come to the inequality

$$J^N(x_0, y, u_{1:N}) = \bar{J}^N(x_0, y) + \Delta \sum_{k=1}^{N} \big(|\sum_{i=1}^{k} \mathbf{G}_i^k u_i|_C^2 + |u_k|^2 \big) - \\ - 2 \sum_{k=1}^{N} \bar{y}_k' C \sum_{i=1}^{k} \mathbf{G}_i^k u_i \leq 1, \quad (4.9)$$

where $\bar{J}^N(x_0, y) = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} |\bar{y}(t)|_C^2 dt$, $\bar{y}_k = \int_{t_{k-1}}^{t_k} \bar{y}(t) dt$. Let us single out in (4.9) a full square on $u_{1:N}$, for which we introduce the matrix $\mathcal{K}^\Delta$ according to the relation

$$\Delta \sum_{k=1}^{N} \left(|\sum_{i=1}^{k} \mathbf{G}_i^k u_i|_C^2 + |u_k|^2 \right) = u_{1:N}' \mathcal{K}^\Delta u_{1:N}.$$

We assume that $u_{1:N} \in \mathbb{R}^{N(l-m)}$ is a column vector. We also introduce matrices according to relations

$$\mathbf{k}_k^\Delta u_{1:N} = \sum_{i=1}^{k} \mathbf{G}_i^k u_i,$$

with the help of which we obtain the representation for the matrix $\mathcal{K}^\Delta$:

$$\mathcal{K}^\Delta = \Delta \left(I_{N(l-m)} + \mathbf{k}_\Delta \right),$$

where $\mathbf{k}_\Delta = \sum_{k=1}^{N} (\mathbf{k}_k^\Delta)' C \mathbf{k}_k^\Delta \in \mathbb{R}^{N(l-m) \times N(l-m)}$ is a symmetrical matrix. When single out a full square in (4.9), it is necessary to decide algebraic equation

$$\mathcal{K}^\Delta u_{1:N} = \bar{f}, \quad \text{where} \quad \bar{f} = \sum_{i=1}^{N} (\mathbf{k}_i^\Delta)' C \bar{y}_i.$$

Therefore, the inequality (4.9) becomes

$$\bar{J}^N(x_0, y) + \left| u_{1:N} - (\mathcal{K}^\Delta)^{-1} \bar{f} \right|_{\mathcal{K}^\Delta}^2 - \left| \bar{f} \right|_{(\mathcal{K}^\Delta)^{-1}}^2 \leq 1.$$

For a fixed vector $l \in \mathbb{R}^n$ and an initial state x_0, we find the value

$$R^N(l, x_0, y) = \max_{u_{1:N}} l' x_N = l' \bar{x}_N + l_N' (\mathcal{K}^\Delta)^{-1} \bar{f} + \sqrt{l_N' (\mathcal{K}^\Delta)^{-1} l_N} \times \\ \sqrt{1 + \left| \bar{f} \right|_{(\mathcal{K}^\Delta)^{-1}}^2 - \bar{J}^N(x_0, y)}, \quad l_N = [\mathbf{a}^{N-1}\mathbf{B}, \ldots, \mathbf{B}]' l. \quad (4.10)$$

Finally, we find the support function for the IS of discrete system:

$$\rho(l \mid \mathcal{X}_N(y)) = \max_{x_0 \in X_0,\, 1+\left|\bar{f}\right|^2_{(\mathcal{K}^{\Delta})^{-1}} \geq \bar{J}^N(x_0,y)} R^N(l, x_0, y). \tag{4.11}$$

Formulas (4.10), (4.11) are discrete analog of formulas (3.6), (3.7). It is proved in [1] that $\mathcal{X}_N(y) \to \mathcal{X}_T(y)$ as $N \to \infty$ in the Hausdorff metric for systems without measurement delay. A similar theorem is also valid in the case under consideration. Its proof repeats the proof of Theorem 6 in [1] with minimal modifications.

Theorem 3. *Let conditions* (2.1), (2.3) *be satisfied and the quantities* T *and* h *be commensurable, that is,* T/h *is a rational number. Then the IS* $\mathcal{X}_N(y)$ *of the discrete system* (4.8) *for constraints* (4.9) *and* (1.4) *converges to IS* $\mathcal{X}_T(y)$ *in the Hausdorff metric as* $N \to \infty$.

5. Numerical example

We consider the motion of a material point along a straight line, subject to disturbances $v^1(t)$ and $v^2(t)$

$$\dot{x}^1 = x^2 + v^1(t) - v^2(t), \quad \dot{x}^2 = v^1(t) - v^2(t), \quad 0 \leq t \leq T.$$

Let the sum of the disturbances also have an additive effect on the measurement equation containing the delay

$$y(t) = x^1(t) + x^1(t-1) + x^2(t-1) + v^1(t) + v^2(t).$$

System parameters:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad G(s) = -[1,0]\chi(-s) - [1,1]\chi(-1-s),$$

$$H = [1,1], \quad C = (HH')^{-1} = 1/2, \quad C_1 = I_2 - H'CH = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}/2,$$

$$\chi(s) = \begin{cases} 1, & \text{if } s \geq 0, \\ 0, & \text{if } s < 0. \end{cases}$$

The condition (2.3) is met because $BH' = 0$. The dimension of the disturbance can be reduced, by setting $u(t) = (v^1(t) - v^2(t)/\sqrt{2}$. Then $D_1 = [1;-1]/\sqrt{2}$ and $bD_1 = [1;1]\sqrt{2}$. We have

$$C_1 v(t) = D_1 u(t), \quad |v(t)|^2_{C_1} = u^2(t).$$

The system (2.5) will take the form

$$\dot{x}^1 = x^2 + \sqrt{2}u(t), \quad \dot{x}^2 = \sqrt{2}u(t), \quad 0 \leq t \leq T,$$

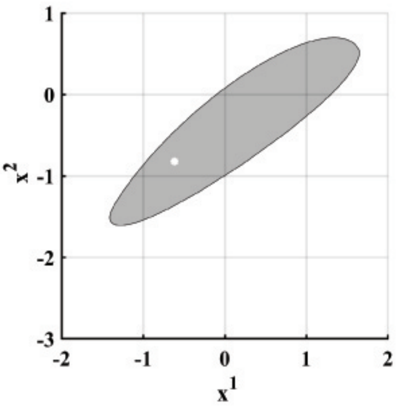

Figure 1. Discrete approximation of IS, $T = 2$.

with constraints (2.6)

$$J(T, x_0, u, y) = \int_0^T \Big(\big(y(t) - x^1(t) - x^1(t-1) - x^2(t-1)\big)^2 /2 + \\ + u^2(t)\Big) dt \le 1.$$

Here we put $x(t) = 0$ if $t < 0$. Let $X_0 = \big\{x \in \mathbb{R}^2 : |x_0^1| \le 1,\ |x_0^2| \le 1\big\}$, and the signal was realized at

$$x_0 = [0.5; -0.5], \quad v(t) = 0.9[\cos(t); \sin(t)]/\sqrt{T} \ \text{ and } \ T = 2.$$

The matrix function (3.3) has the form

$$\mathbf{G}(t) = [1, t](1 + \chi(t-1)).$$

Let us find the IS support function using discretization. For discretization we set $N = 20$, $\Delta = T/N = 0.1$. The formula (4.8) will take a form

$$x_k = \mathbf{a}x_{k-1} + \mathbf{B}u_k, \quad \mathbf{a} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}, \qquad k \in 1 : N,$$
$$\mathbf{B} = \begin{bmatrix} \Delta & \Delta^2/2 \\ 0 & \Delta \end{bmatrix} BD_1.$$

Further, using the formulas (4.10), (4.11), we find the support function of the discrete IS. The corresponding IS was obtained by intersection the support half-spaces and is shown in Fig. 1, where the star represents the true state at $T = 2$.

6. Conclusion

In this work, a problem of the guaranteed state estimation of a linear system is considered with geometrical restrictions for initial states and integral restrictions on perturbations with a delay of information in the measurement equation. Under additional assumptions on the system, the arising problem is reduced to the creation of reachable sets for a special system. Two ways of approximation of the information set for the original system are proposed: by means of ellipsoids and by a method with multistage systems. The information sets of discrete multistep systems converges in the Hausdorff metric to the corresponding information set of the continuous system when the partition diameter of the observation interval is reduced. A numerical example is given.

References

1. Ananyev B.I., Yurovskikh P.A. Approksimatsiya zadachi garantirovannogo otsenivaniya so smeshannymi ogranicheniyami. *Trudy IMM UrO RAN*, 2020, vol. 26, no. 4, pp. 48–63. https://doi.org/10.21538/0134-4889-2020-26-4-48-63 (in Russian)
2. Ananyev B.I., Yurovskikh P.A. O zadache otsenivaniya s razdel'nymi ogranicheniyami na nachalnyye sostoyaniya i vozmushcheniya. *Trudy IMM UrO RAN*, 2021, vol. 28, no. 1, pp. 27–39. https://doi.org/10.21538/0134-4889-2022-28-1-27-39 (in Russian)
3. Ananyev B.I. A Guaranteed State Estimation of Linear Retarded Neutral Type Systems. *AIP Conference Proceedings 1895*, 2017, 050001. https://doi.org/10.1063/1.5007373
4. Ahmedova N.K., Kolmanovskii V.B., Matasov A.I. Constructive filtering algorithms for delayed systems with uncertain statistics. *ASME Journal of Dynamic Systems, Measurement, and Control. Special Issue: Time Delayed Systems*, 2003, vol. 125, no. 2, pp. 229–235. https://doi.org/10.1115/1.1569951
5. Gusev M.I. O metode shtrafnich funktsii dlya upravlyaemich sistem c fazovimi ogranicheniyami pri integral'nich ogranicheniyach na upravleniye. *Trudy IMM UrO RAN*, 2021, vol. 27, no. 3, pp. 59–70. https://doi.org/10.21538/0134-4889-2021-27-3-59-70
6. Kolmanovskii V.B., Kopylova N.K., Matasov A.I. An approximate method for solving stochastic guaranteed estimation problem in hereditary systems. *Dynamic Systems and Applications*, 2001, vol. 10, no. 3, pp. 305–325.
7. Kurzhanski A.B., Varaiya P. *Dynamics and Control of Trajectory Tubes: Theory and Computation.* SCFA, Birkhäuser, 2014, 445 p.
8. Srochko V.A., Aksenyushkina E.V., Antonik V.G. Resolution of a Linear-quadratic Optimal Control Problem Based on Finite-dimensional Models. *The Bulletin of Irkutsk State University. Series Mathematics*, 2021, vol. 37, pp. 3–16. https://doi.org/10.26516/1997-7670.2021.37.3 (in Russian)
9. Srochko V.A., Aksenyushkina E.V. On Resolution of an Extremum Norm Problem for the Terminal State of a Linear System. *The Bulletin of Irkutsk State University. Series Mathematics*, 2020, vol. 34, pp. 3–17. https://doi.org/10.26516/1997-7670.2020.34.3

10. Filippova T.F. Control and estimation for a class of impulsive dynamical systems. *Ural Math. J.*, 2019, vol. 5, no. 2, pp. 21–30. https://doi.org/10.15826/umj.2019.2.003

Список источников

1. Ананьев Б. И., Юровских П. А. Аппроксимация задачи гарантированного оценивания со смешанными ограничениями // Труды ИММ УрО РАН. 2020. Т. 26, № 4. С. 48–63. https://doi.org/10.21538/0134-4889-2020-26-4-48-63
2. Ананьев Б. И., Юровских П. А. О задаче оценивания с раздельными ограничениями на начальные состояния и возмущения // Труды ИММ УрО РАН. 2021. Т. 28, № 1. С. 27–39. https://doi.org/10.21538/0134-4889-2022-28-1-27-39
3. Ananyev B. I. A Guaranteed State Estimation of Linear Retarded Neutral Type Systems // AIP Conference Proceedings 1895. 050001 (2017). https://doi.org/10.1063/1.5007373
4. Ahmedova N. K., Kolmanovskii V. B., Matasov A. I. Constructive filtering algorithms for delayed systems with uncertain statistics // ASME Journal of Dynamic Systems, Measurement, and Control. Special Issue: Time Delayed Systems. 2003. Vol. 125, N 2. P. 229–235. https://doi.org/10.1115/1.1569951
5. Гусев М. И. О методе штрафных функций для управляемых систем с фазовыми ограничениями при интегральных ограничениях на управление // Труды ИММ УрО РАН. 2021. Т. 27, № 3. С. 59–70. https://doi.org/10.21538/0134-4889-2021-27-3-59-70
6. Kolmanovskii V. B., Kopylova N. K., Matasov A. I. An approximate method for solving stochastic guaranteed estimation problem in hereditary systems // Dynamic Systems and Applications. 2001. Т. 10, № 3. С. 305–325.
7. Kurzhanski A. B., Varaiya P. Dynamics and Control of Trajectory Tubes: Theory and Computation. SCFA, Birkhäuser, 2014. 445 p.
8. Срочко В. А., Аксенюшкина Е. В., Антоник В. Г. Решение линейно-квадратичной задачи оптимального управления на основе конечномерных моделей // Известия Иркутского государственного университета. Серия Математика. 2021. Т. 37. С. 3–16. https://doi.org/10.26516/1997-7670.2021.37.3
9. Srochko V. A., Aksenyushkina E. V. On Resolution of an Extremum Norm Problem for the Terminal State of a Linear System // Известия Иркутского государственного университета. Серия Математика. 2020. Т. 34. С. 3–17. https://doi.org/10.26516/1997-7670.2020.34.3
10. Filippova T. F. Control and estimation for a class of impulsive dynamical Systems // Ural Math. J. 2019. Vol. 5, N 2. P. 21–30. https://doi.org/10.15826/umj.2019.2.003

Об авторах

Ананьев Борис Иванович, д-р физ.-мат. наук, с.н.с., Институт математики и механики им. Н. Н. Красовского УрО РАН, Российская Федерация, 620108, г. Екатеринбург, abi@imm.uran.ru, https://orcid.org/0000-0002-1378-0240

About the authors

Boris I. Ananyev, Dr. Sci. (Phys.–Math.), Senior Res., N. N. Krasovskii Institute of Mathematics and Mechanics UB RAS, Yekaterinburg, 620108, Russian Federation, abi@imm.uran.ru, https://orcid.org/0000-0002-1378-0240

Юровских Полина Александровна, аспирант, Институт математики и механики им. Н. Н. Красовского УрО РАН, Российская Федерация, 620108, г. Екатеринбург, polina2104@list.ru, https://orcid.org/0000-0001-8051-3428

Polina A. Yurovskikh, Postgraduate, N. N. Krasovskii Institute of Mathematics and Mechanics UB RAS, Yekaterinburg, 620108, Russian Federation, polina2104@list.ru, https://orcid.org/0000-0001-8051-3428

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