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On the Existence of f-local Subgroups in a Group with Finite Involution

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Abstract. An f-local subgroup of an infinite group is each its infinite subgroup with a nontrivial locally finite radical. An involution is said to be finite in a group if it generates a finite subgroup with each conjugate involution. An involution is called isolated if it does not commute with any conjugate involution. We study the group G with a finite non-isolated involution i, which includes infinitely many elements of finite order. It is proved that G has an f-local subgroup containing with i infinitely many elements of finite order. The proof essentially uses the notion of a commuting graph.

Keywords: group, f-local subgroup, finite involution, commuting graph

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Научная статья

О существовании в группе с конечной инволюцией f-локальных подгрупп

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Аннотация. f-локальной подгруппой бесконечной группы называется каждая её бесконечная подгруппа с нетривиальным локально конечным радикалом. Инволюция называется конечной в группе, если она с каждой сопряженной инволюцией порождает конечную подгруппу. Инволюция называется изолированной, если она не перестановочна ни с одной сопряженной с ней инволюцией. Изучается группа G с конечной не изолированной инволюцией i, содержащая бесконечно много элементов конечных порядков. Доказано, что в G есть f-локальная подгруппа, содержащая вместе с инволюцией i бесконечно много элементов конечных порядков. Доказательство существенно использует понятие коммутирующего графа.

Ключевые слова: группа, f-локальная подгруппа, конечная инволюция, коммутирующий граф

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1. Introduction

Many studies in infinite group theory are devoted to finding in a group of infinite "good" subgroups. Thus, according to the well-known Kargapolov-Hall-Kulatilaki theorem [3; 4] any infinite locally infinite finite group contains an infinite abelian subgroup, and any element of the group is contained in the infinite an infinite abelian subgroup when its centralizer is infinite. There is a certain relationship between the subgroups of Silov in finite groups and good subgroups in infinite groups, which allows one to apply local and binary methods in their study. In contrast to locally finite groups, in groups with weak finiteness conditions, the search for good subgroups is considerably more branched. A far from complete list of studies on search for good subgroups is given in the bibliography (reference list) [12] – [11]. In this note the research involves the notion of commuting, well known in the theory of in the theory of finite groups the notion of a commuting graph [2, p.144] is well known.

Note that f-local subgroup of an infinite group G is each of its infinite subgroups with a nontrivial locally finite radical. A mixed group G is said to have *periodic part*, if all its elements of finite order compose a subgroup. An involution i is called *finite in* G if for any $g \in G$ the order of the product ii^g is finite, and *is isolated in* G if the orders of all such products of ii^g are odd.

Theorem 1. *Let the group G contain infinitely many elements of finite orders and a finite non-isolated involution i. Then G has a f-local subgroup containing, together with the involution i infinitely many elements of finite orders.*

Note that the condition of non-isolated involution i in the theorem is essential. In [9, page 7] examples of groups with isolated involution are given, that satisfy all the other conditions of the theorem, but not its conclusion.

2. Proof of the theorem

Dietzmann's lemma is used for above theorem [5, page 338]:

Proposition 1 (Dietzmann lemma)**.** *In an arbitrary group a finite invariant set of elements of finite order generates a finite subgroup.*

Proposition 2. *A group which is finite over its centre is a group with finite classes of of conjugate elements. A group with finite classes of conjugate elements has locally finite periodic part.*

The following sentence is the main part of V.V. Belyaev's theorem. [1].

Proposition 3. *The group G with finite almost regular involution is locally finite.*

Further, let the group G and its involution i satisfy the condition of Theorem 1, but not its conclusion. Directly from the Dietzmann lemma follows such properties of the group G

Lemma 1. *For any element x of finite order from G the class x^G is infinite. The centralizer $C = C_G(i)$ has a finite periodic part T and for any finite subgroup K of C $N_G(K)$ has a finite periodic part.*

Denote by J the set of involutions of the group G.

Lemma 2. *For almost all involutions $k \in J$ the orders of products of ik are finite and odd, all involutions in G are conjugate and $1 < |J \cap C| < \infty$.*

Proof. By virtue of finiteness of involution i for any involution $k \in G$ the dihedral group $D_k = \langle i, k \rangle$ is finite and either i and k are conjugate in D_k or $Z(D_k)$ has involution z_k. By the lemma 1, the set of involutions in T is finite, and for every involution $z \in T$ the equality $z = z_k$ is possible only for a finite number of involutions $k \in J$. Hence, for almost all involutions $k \in J$ the orders of products ik are odd and due to infinity of classes k^G (lemma 1) we conclude that $k \in i^G$ and $J = i^G$. Since i is not isolated in G by the terms of the theorem, then $|C \cap J| > 1$. The lemma is proved. □

Consider a commuting graph (permutation graph) $\Gamma = C(G, J)$ [2, page 144] with set J of vertices and a set of edges (j, k), where $j, k \in J$ and $jk = kj$. In view of the lemma 2, the group G acts by conjugations transitive on the set J of vertices of the graph Γ, $G/Z(G) \leq \mathrm{Aut}\,\Gamma$, in this case, due to the choice of group G either $Z(G) = 1$ or $Z(G)$ is a torsionless group. By virtue of the lemmas 1, 2 the graph Γ is locally finite and the valence of each of its vertices (number of adjacent vertices) is $|C \cap J| - 1 \geq 1$.

Let Γ_0 be a connected component of the graph Γ, containing vertex i.

Lemma 3. *The connectivity components of the graph* Γ *are finite, the subgroup* L *generated by the set* J_0 *of vertices of the subgraph* Γ_0 *is finite, and* $L \cap J = J \cap B = J_0$*, where* $B = N_G(L) = N_G(\Gamma_0)$

Proof. For a real r we denote by J_r the set of vertices of a sphere of radius r with centre at vertex i:

$$J_r = \{\, j \in \Gamma_0 \,|\, d(i, j) \leq r \,\}.$$

Due to the local finiteness of the graph Γ the set J_r is finite. Since C is a stabilizer of vertex i of graph Γ, then the subgroup $C_r = C_G(J_r)$ is contained in C with a finite index. Suppose that the subgroup $L_r = \langle J_r \rangle$ is infinite. In this case, by Dietzmann's lemma the set of elements of finite orders in L_r is infinite. Consider the group $G_r = L_r C_r$. Since $C_r \leq C_G(L_r)$, then C_r is normal in G_r, and in view of the assumptions C_r is a torsionless group. According to Dietzmann's lemma, any finite subgroup K of L_r is a periodic part of the subgroup $C_r K$. In particular, $\langle i \rangle$ is– a periodic part of the subgroup $\langle i, C_r \rangle$ and in the factor group $\overline{G}_r = G_r / C_r$ we have $C_{\overline{G}_r}(\bar{i}) \leq C/C_r$. Hence the centralizer $C_{\overline{G}_r}(\bar{i})$ in the factor-group $\overline{G}_r$ is finite. By 3's suggestion, the factor-group $\overline{G}_r = G_r / C_r \simeq L_r / (C_r \cap L_r)$ is locally finite. But then $L_r / Z(L_r)$ is a finite group (due to finite generativity of L_r), L_r is a group with finite classes of conjugate elements and by Dietzmann lemma L_r is finite. The resulting contradiction with the assumption made means that L_r is finite.

Considering spheres with centres at vertex i of radii $r = 1, 2, ...,$ $J_r \subseteq J_{r+1}$, $L_r \leq L_{r+1}$, $J_0 = \cup_r J_r$, and the set J_0 of all involutions which are vertices of component Γ_0 generates locally finite subgroup $L = \cup_r L_r$ of group G. Due to the choice of G and inclusion of $i \in L$, the subgroup L is finite and Γ_0 is a finite graph. Since any node-involution v of Γ_0 belongs to $N_G(\Gamma_0)$, then $L \leq N_G(\Gamma_0)$. Obviously, that all involutions of the power 2-is subgroup of the group L, containing involution i belong to Γ_0. From the Silow theorem and inclusion of $L \leq N_G(\Gamma_0)$ it follows that all involutions of L belong to J_0 and $J \cap L = J_0$. It is well known that the number of involutions in a finite group of even order is odd. Therefore the external involutions not belonging to J_0, the subgroup $B = N_G(L) = N_G(\Gamma_0)$ does not contain any, so $J \cap B = J_0$. The lemma is proved. □

Let's finish the proof of the theorem If $j, k \in J$ and $|jk|$ is even, then $D = \langle j, k \rangle$ has a central involution and hence all the involutions of D belong to the same component of connectivity of Γ. Therefore for any $j \in J_0$ and $k \in J \setminus J_0$ the order of product jk is odd and $j^v = k$, $k^v = j$ for some involution $v \in \langle j, k \rangle$.

In view of the lemma 2 $|J_0| \geq 3$, and let i, j are different involutions of J_0. Let the involution t be forced to run the set $J \setminus J_0$, and let $k = i^t$, $k^v = j$, where $v \in \langle j, k \rangle$. Then $i^{tv} = j$ and $tv \in B = N_G(\Gamma_0)$. By the lemma 2 the order of the element tv is finite. According to the lemma 3, the subgroup B is f-local, and since $i \in B$, the number of finite order elements in B is finite due to the choice of group G. Hence, some element $b \neq 1$ of B can be represented as $b = tv$ for an infinite set of involutions t of $J \setminus J_0$. Since each such involution inverts an element b, the f-local subgroup $N_G(\langle b \rangle)$ contains infinitely many involutions conjugate to i (lemma 2). However, this contradicts the choice of group G and the theorem is proved.

3. Conclusion

Up to now it is not known whether there exist infinite locally finite subgroups in Shunkov groups containing a fixed element of prime order (question 14.100 of the Kourov notebook). The result obtained is one of the steps towards the solution of this question.

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