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Topologies and Ranks for Families of Theories in Various Languages

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Abstract. Topological properties and characteristics of families of theories reflect possibilities of separation of theories and a complexity both for theories and their neighbourhoods. Previously, topologies were studied for families of complete theories, in general case and for a series of natural classes, and for various families of incomplete theories in a fixed language. The ranks were defined and described for complete theories in a given language, for a hierarchy of theories, for families of incomplete theories, for formulae and for a series of natural families of theories, including families of ordered theories, families of theories of permutations and families of theories of abelian groups.

In this paper, we study properties and characteristics for topologies and ranks for families of theories in various languages. It is based on special relations connecting formulae in a given language. These relations are used to define and describe kinds of separations with respect to T_0-topologies, T_1-topologies and Hausdorff topologies. Besides special relations are used to define and study ranks for families of theories in various languages. Possibilities of values for the rank are described, and these possibilities are characterized in topological terms.

Keywords: topology, rank, family of theories, language

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Научная статья

Топологии и ранги для семейств теорий различных сигнатур

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Аннотация. Топологические свойства и характеристики семейств теорий отражают возможности отделимости теорий и сложность как самих теорий, так и их окрестностей. Ранее топологии изучались для семейств полных теорий в общем случае и для ряда естественных классов, а также для различных семейств неполных теорий фиксированной сигнатуры. Были определены и описаны ранги для полных теорий данной сигнатуры, иерархии теорий, семейств неполных теорий, формул и ряда естественных семейств теорий, включая семейства упорядоченных теорий, семейства теорий подстановок и семейства теорий абелевых групп.

В этой статье мы изучаем свойства и характеристики топологий и рангов семейств теорий, имеющих различные сигнатуры. Рассмотрение основано на специальных отношениях, связывающих формулы данной сигнатуры. Эти соотношения используются для определения и описания видов отделимости относительно T_0-топологий, T_1-топологий и хаусдорфовых топологий. Кроме того, специальные отношения используются для определения и изучения рангов для семейств теорий различных сигнатур. Описаны возможные значения ранга, и эти возможности охарактеризованы в топологических терминах.

Ключевые слова: топология, ранг, семейство теорий, сигнатура

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1. Introduction

Topologies for families of theories produce possibilities of various natural links between theories and constructions of new theories with desirable properties. Properties of topologies [1] reflect ones for families of theories and for theories in these families [4; 5; 12].

Ranks for families of theories are important characteristics producing measures of these families. These ranks are natural modifications for Cantor–Bendixson rank and Morley rank [8; 11]. They are defined and their properties are studied for families of complete theories in a given language [6; 13], for a hierarchy of theories [14], for families of incomplete theories [4; 5], for formulae [9; 15] and for a series of natural families of theories [2; 3; 10].

At the present paper, on a base of special relations for formulae [16], we study topological properties for families of theories in various languages. The rank for families of theories is spread for families of theories in this general case. Properties of topologies and ranks in this general case are described.

2. Preliminaries

Following [13] we define the *rank* $\mathrm{RS}(\cdot)$ for families $\mathcal{T}$ of first-order theories in a language $\Sigma = \Sigma(\mathcal{T})$, similar to Morley rank for a fixed theory, and a hierarchy with respect to these ranks in the following way.

By $F(\Sigma)$ we denote the set of all formulas in the language Σ and by $\mathrm{Sent}(\Sigma)$ the set of all sentences in $F(\Sigma)$.

For a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ we denote by $\mathcal{T}_\varphi$ the set of all theories $T \in \mathcal{T}$ with $\varphi \in T$.

Any set $\mathcal{T}_\varphi$ is called the *φ-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$, or the (φ-)*definable* subset of $\mathcal{T}$. The set $\mathcal{T}_\varphi$ is also called (*formula*- or *sentence*-)*definable* (by the sentence φ) with respect to $\mathcal{T}$, or (*sentence*-)$\mathcal{T}$-*definable*, or simply *s-definable*.

Definition [13]. For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}(\mathcal{T}) = -1$, and for nonempty families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) \geq 0$.

For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise $\mathcal{T}$-inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.

If α is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.

We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.

If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

If $\mathcal{T}$ is e-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

As noticed in [4; 5] these notions are valid both for families of complete theories and for arbitrary families of theories including incomplete ones.

The following theorem characterizes the property of e-total transcendency for countable languages.

Theorem 2.1 [9; 13]. *For any family $\mathcal{T}$ with $|\Sigma(\mathcal{T})|\leq\omega$ the following conditions are equivalent:*
(1) $|\mathrm{Cl}_E(\mathcal{T})|=2^\omega$*;*
(2) $e\text{-}\mathrm{Sp}(\mathcal{T})=2^\omega$*;*
(3) $\mathrm{RS}(\mathcal{T})=\infty$*;*
(4) *there exists a* 2*-tree of sentences φ for s-definable sets $\mathcal{T}_\varphi$.*

Let Σ be a language. If Σ is relational we denote by $\mathcal{T}_\Sigma$ the family of all complete theories of the language Σ. If Σ contains functional symbols f then $\mathcal{T}_\Sigma$ is the family of all theories of the language Σ', which is obtained by replacements of all n-ary symbols f by $(n+1)$-ary predicate symbols R_f which interpreted by $R_f=\{(\bar{a},b)\mid f(\bar{a})=b\}$.

Theorem 2.2 [6]. *For any language Σ either $\mathrm{RS}(\mathcal{T}_\Sigma)$ is finite, if Σ consists of finitely many* 0*-ary and unary predicates, and finitely many constant symbols, or $\mathrm{RS}(\mathcal{T}_\Sigma)=\infty$, otherwise.*

For a language Σ we denote by $\mathcal{T}_{\Sigma,n}$ the family of all theories in $\mathcal{T}_\Sigma$ having n-element models, $n\in\omega$, as well as by $\mathcal{T}_{\Sigma,\infty}$ the family of all theories in $\mathcal{T}_\Sigma$ having infinite models.

Theorem 2.3 [6]. *For any language Σ either $\mathrm{RS}(\mathcal{T}_{\Sigma,n})=0$, if Σ is finite or $n=1$ and Σ has finitely many predicate symbols, or $\mathrm{RS}(\mathcal{T}_{\Sigma,n})=\infty$, otherwise.*

Theorem 2.4 [6]. *For any language Σ either $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty})$ is finite, if Σ is finite and without predicate symbols of arities $m\geq 2$ as well as without functional symbols of arities $n\geq 1$, or $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty})=\infty$, otherwise.*

By the definition the families $\mathcal{T}_\Sigma$, $\mathcal{T}_{\Sigma,n}$, $\mathcal{T}_{\Sigma,\infty}$ are E-closed. Thus, combining Theorem 2.1 with Theorems 2.2–2.4 we obtain the following possibilities of cardinalities for the families $\mathcal{T}_\Sigma$, $\mathcal{T}_{\Sigma,n}$, $\mathcal{T}_{\Sigma,\infty}$ depending on Σ and $n\in\omega$:

Proposition 2.5. *For any language Σ either $\mathcal{T}_\Sigma$ is countable, if Σ consists of finitely many* 0*-ary and unary predicates, and finitely many constant symbols, or $|\mathcal{T}_\Sigma|\geq 2^\omega$, otherwise.*

Proposition 2.6. *For any language Σ either $\mathcal{T}_{\Sigma,n}$ is finite, if Σ is finite or $n=1$ and Σ has finitely many predicate symbols, or $|\mathcal{T}_{\Sigma,n}|\geq 2^\omega$, otherwise.*

Proposition 2.7. *For any language Σ either $\mathcal{T}_{\Sigma,\infty}$ is at most countable, if Σ is finite and without predicate symbols of arities $m\geq 2$ as well as without functional symbols of arities $n\geq 1$, or $|\mathcal{T}_{\Sigma,\infty}|\geq 2^\omega$, otherwise.*

Definition [7]. If $\mathcal{T}$ is a family of theories and Φ is a set of sentences, then we put $\mathcal{T}_\Phi=\bigcap\limits_{\varphi\in\Phi}\mathcal{T}_\varphi$ and the set $\mathcal{T}_\Phi$ is called (*type-* or *diagram-*)

definable (by the set Φ) with respect to $\mathcal{T}$, or (*diagram-*)$\mathcal{T}$*-definable*, or simply *d-definable*.

Clearly, finite unions of d-definable sets are again d-definable. Considering infinite unions $\mathcal{T}'$ of d-definable sets $\mathcal{T}_{\Phi_i}$, $i \in I$, one can represent them by sets of sentences with infinite disjunctions $\bigvee\limits_{i\in I} \varphi_i$, $\varphi_i \in \Phi_i$. We call these unions $\mathcal{T}'$ are called d_∞*-definable* sets.

Definition [7]. Let $\mathcal{T}$ be a family of theories, Φ be a set of sentences, α be an ordinal $\leq \mathrm{RS}(\mathcal{T})$ or -1. The set Φ is called α*-ranking* for $\mathcal{T}$ if $\mathrm{RS}(\mathcal{T}_\Phi) = \alpha$. A sentence φ is called α*-ranking* for $\mathcal{T}$ if $\{\varphi\}$ is α*-ranking* for $\mathcal{T}$.

The set Φ (the sentence φ) is called *ranking* for $\mathcal{T}$ if it is α-ranking for $\mathcal{T}$ with some α.

Proposition 2.8 [7]. *For any ordinals $\alpha \leq \beta$, if $\mathrm{RS}(\mathcal{T}) = \beta$ then $\mathrm{RS}(\mathcal{T}_\varphi) = \alpha$ for some (α-ranking) sentence φ. Moreover, there are $\mathrm{ds}(\mathcal{T})$ pairwise $\mathcal{T}$-inconsistent β-ranking sentences for $\mathcal{T}$, and if $\alpha < \beta$ then there are infinitely many pairwise $\mathcal{T}$-inconsistent α-ranking sentences for $\mathcal{T}$.*

Theorem 2.9 [7]. *Let $\mathcal{T}$ be a family of a countable language Σ and with $\mathrm{RS}(\mathcal{T}) = \infty$, α be a countable ordinal, $n \in \omega \setminus \{0\}$. Then there is a d_∞-definable subfamily $\mathcal{T}^* \subset \mathcal{T}$ such that $\mathrm{RS}(\mathcal{T}^*) = \alpha$ and $\mathrm{ds}(\mathcal{T}^*) = n$.*

Definition [16]. Let $F(\Sigma)$ be the set of all formulae in a language Σ, V be an infinite set of variables, V^* be the set of all tuples $\overline{x} \in V^n$, $n \in \omega$. A ternary relation $E \subseteq F(\Sigma) \times F(\Sigma) \times V^*$ is called *special* (for $F(\Sigma)$) if for each $\overline{x} \in V^*$,

$$E_{\overline{x}} = \{(\varphi, \psi) \mid (\varphi, \psi, \overline{x}) \in E\}$$

is an equivalence relation on the set $X \subset F(\Sigma)$ consisting of all formulae φ whose each free variable belongs to $\overline{x}$.

We denote by $\mathrm{SR}(\Sigma)$ the family of all special relations for $F(\Sigma)$.

The special relation $E \in \mathrm{SR}(\Sigma)$ is called *coordinated* if the relation

$$E^* = \{(\varphi, \psi) \mid (\varphi, \psi, \overline{x}) \in E \text{ for some } \overline{x}\}$$

is an equivalence relation on $F(\Sigma)$.

We denote by $\mathrm{id}_F(\Sigma)$ the special relation E for $F(\Sigma)$ whose all $E_{\overline{x}}$-classes are singletons.

A special relation E is called *upward directed* if $E_{\overline{x}} \cup E_{\overline{y}} \subseteq E_{\overline{x}\hat{\ }\overline{y}}$ for any $\overline{x}, \overline{y} \in V^*$.

For each formula $\varphi = \varphi(\overline{x}) \in F(\Sigma)$ and a special relation E we consider the set $\varphi_E = \{\forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})) \mid (\varphi, \psi, \overline{x}) \in E\}$. We denote by ∇_E the set $\bigcup\limits_{\varphi\in F(\Sigma)} \varphi_E$.

Definition. Let E be a special relation for $F(\Sigma)$.

The relation E is called *inessential* if ∇_E consists of identically true formulas. The relation E is called *essential* if it is not inessential.

For a formula $\varphi = \varphi(\overline{x}) \in F(\Sigma)$ the equivalence class $E_{\overline{x}}(\varphi)$ is called *regular* if the set φ_E is consistent.

The relation E is called *weakly regular* if for each formula $\varphi = \varphi(\overline{x}) \in F(\Sigma)$ the equivalence class $E_{\overline{x}}(\varphi)$ is regular.

The relation E is called *locally regular* if each finite part of ∇_E is consistent.

The relation E is called *strongly regular* if ∇_E is consistent.

3. Topologies for families of theories in various languages

The following topological notions are related to known forms of separability both in general case [1] and for families of (in)complete theories [4; 5].

Definition [1]. A topological space $(X, \mathcal{O})$ is a T_0-*space* if for any pair of distinct elements $x_1, x_2 \in X$ there is an open set $U \in \mathcal{O}$ containing exactly one of these elements.

Definition [1]. A topological space $(X, \mathcal{O})$ is a T_1-*space* if for any pair of distinct elements $x_1, x_2 \in X$ there is an open set $U \in \mathcal{O}$ such that $x_1 \in U$ and $x_2 \notin U$.

Definition [1]. A topological space $(X, \mathcal{O})$ is a T_2-*space*, or *Hausdorff* if for any pair of distinct elements $x_1, x_2 \in X$ there are open sets $U_1, U_2 \in \mathcal{O}$ such that $x_1 \in U_1$, $x_2 \in U_2$, and $U_1 \cap U_2 = \varnothing$.

For a family $\mathcal{T}$ of theories in some sublanguages of Σ, for a strongly regular special relation $E \in \mathrm{SR}(\Sigma)$ and a sentence $\varphi \in F(\Sigma)$ we denote by $\mathcal{T}_{\varphi,E}$ the set of all theories $T \in \mathcal{T}$ such that T contains all sentences $\psi \in \mathrm{Sent}(\Sigma(T))$ with $\nabla_E \cup \{\varphi\} \vdash \psi$. In such a case we say that T E-*absorbs* $\nabla_E \cup \{\varphi\}$-consequences in $\Sigma(T)$.

A subfamily $\mathcal{T}_{\varphi,E}$ of a family $\mathcal{T}$ is called a (s, E)-*definable* subfamily of $\mathcal{T}$, or simply a *definable* subfamily of $\mathcal{T}$, or a E-*neighbourhood* for theories $T \in \mathcal{T}_{\varphi,E}$ in $\mathcal{T}$, or simply a *neighbourhood* in $\mathcal{T}$.

The neighbourhoods $\mathcal{T}_{\varphi,E}$ define topologies with the properties T_0, T_1, T_2 above.

We have the following monotone property of the neighbourhoods $\mathcal{T}_{\varphi,E}$.

Proposition 3.1. *If* $\nabla_E \cup \{\varphi\} \vdash \psi$ *then for any family* $\mathcal{T}$, $\mathcal{T}_{\varphi,E} \subseteq \mathcal{T}_{\psi,E}$.

Proof follows by the transitivity of the relation $\vdash$.

Definition. We say that the sentences $\varphi, \psi \in F(\Sigma)$ are ∇_E-*equivalent*, denoted by $\varphi \equiv_{\nabla_E} \psi$, if $\nabla_E \cup \{\varphi\} \vdash \psi$ and $\nabla_E \cup \{\psi\} \vdash \varphi$.

Proposition 3.1 immediately implies:

Corollary 3.2. *If $\varphi\equiv_{\nabla_E}\psi$ then for any family $\mathcal{T}$, $\mathcal{T}_{\varphi,E}=\mathcal{T}_{\psi,E}$.*

Corollary 3.3. *If $\mathcal{T}$ is the set of all consistent (sub)theories of a language Σ then for any sentences $\varphi,\psi\in F(\Sigma)$, $\nabla_E\cup\{\varphi\}\vdash\psi$ if and only if $\mathcal{T}_{\varphi,E}\subseteq\mathcal{T}_{\psi,E}$.*

Proof. In view of Proposition 3.1 it suffices to show that if $\mathcal{T}_{\varphi,E}\subseteq\mathcal{T}_{\psi,E}$ then $\nabla_E\cup\{\varphi\}\vdash\psi$.

Assuming on contrary that $\mathcal{T}_{\varphi,E}\subseteq\mathcal{T}_{\psi,E}$ and $\nabla_E\cup\{\varphi\}\not\vdash\psi$ we have the consistent set $\nabla_E\cup\{\varphi,\neg\psi\}$ which is expansible to a theory $T\in\mathcal{T}$. Since T is closed under deduced sentences we have $T\in\mathcal{T}_{\varphi,E}$, whereas $T\notin\mathcal{T}_{\psi,E}$ since $\neg\psi\in T$. Hence, $\mathcal{T}_{\varphi,E}\not\subseteq\mathcal{T}_{\psi,E}$, contradicting the assumption.

Corollary 3.3 immediately implies:

Corollary 3.4. *If $\mathcal{T}$ is the set of all consistent (sub)theories of a language Σ then for any sentences $\varphi,\psi\in F(\Sigma)$, $\varphi\equiv_{\nabla_E}\psi$ if and only if $\mathcal{T}_{\varphi,E}=\mathcal{T}_{\psi,E}$.*

Remark 3.5. Corollaries 3.3 and 3.4 does not hold for arbitrary family $\mathcal{T}$ since the relation $\equiv_{\nabla_E}$ does not depend on $\mathcal{T}$. We can take, for instance, a singleton $\mathcal{T}=\{T\}$, where T consists of tautologies, and non-∇_E-equivalent sentences $\varphi,\psi\in F(\Sigma)$ producing $\mathcal{T}_{\varphi,E}=\{T\}=\mathcal{T}_{\psi,E}$.

Definition. The relation E is called $(T_0,\mathcal{T})$*-separating* if for any distinct theories $T^1,T^2\in\mathcal{T}$ there is a sentence $\varphi\in F(\Sigma)$ such that $T^1\in\mathcal{T}_{\varphi,E}$ and $T^2\notin\mathcal{T}_{\varphi,E}$, or $T^1\notin\mathcal{T}_{\varphi,E}$ and $T^2\in\mathcal{T}_{\varphi,E}$.

The relation E is called $(T_1,\mathcal{T})$*-separating* if for any distinct theories $T^1,T^2\in\mathcal{T}$ there is a sentence $\varphi\in F(\Sigma)$ such that $T^1\in\mathcal{T}_{\varphi,E}$ and $T^2\notin\mathcal{T}_{\varphi,E}$.

The relation E is called $(T_2,\mathcal{T})$*-separating* or $\mathcal{T}$*-Hausdorff* if for any distinct theories $T^1,T^2\in\mathcal{T}$ there are sentences $\varphi,\psi\in F(\Sigma)$ such that $T^1\in\mathcal{T}_{\varphi,E}$, $T^2\notin\mathcal{T}_{\varphi,E}$, $T^1\notin\mathcal{T}_{\psi,E}$, $T^2\in\mathcal{T}_{\psi,E}$, and $\mathcal{T}_{\varphi,E}\cap\mathcal{T}_{\psi,E}=\emptyset$.

Theories T^1 and T^2 are called (T_i,E)*-separated* if E is $(T_i,\mathcal{T})$-separating for the family $\mathcal{T}=\{T^1,T^2\}$, $i\in\{0,1,2\}$. (T_2,E)-separated theories T^1 and T^2 are called E*-Hausdorff*.

Remark 3.6. There are special relations E which are not separating for given theories at all. Indeed, if $\mathcal{T}$ consists of theories T^i, $i\in I$, and $(\varphi_i\leftrightarrow\varphi_j)\in\nabla_E$ for $\varphi_i\in T^i$, $\varphi_j\in T^j$, $i,j\in I$, then T^i can not be separated each others by neighbourhoods $\mathcal{T}_{\varphi,E}$.

These non-separating special relations E can be both with consistent ∇_E and with inconsistent ∇_E. Inconsistent ∇_E means that for some $\varphi\in T^i$ there is T^j with $T^j\vdash\neg\varphi$. So, for instance, families of positive theories produce non-separating special relations E with consistent ∇_E, whereas families of complete theories in a given language produce non-separating special relations E with inconsistent ∇_E only.

Remark 3.7. By the definition the relation E is $(T_i,\mathcal{T})$-separating if and only if any distinct theories $T^1,T^2\in\mathcal{T}$ are (T_i,E)-separated, $i\in\{0,1\}$.

For the $(T_2,\mathcal{T})$-separability of $T^1,T^2\in\mathcal{T}$ we assume that $T^1,T^2\in\mathcal{T}$ are (T_2,E)-separated by $\mathcal{T}$-disjoint neighbourhoods.

Theorem 3.8. *For a special relation $E\in\mathrm{SR}(\Sigma)$ and distinct theories $T^1,T^2\subset F(\Sigma)$, T^1 and T^2 are (T_0,E)-separated if and only if for some $i\in\{1,2\}$ there is a sentence $\varphi\in F(\Sigma)$ such that $\nabla_E\cup\{\varphi\}\subset T^i$ and $\nabla_E\cup\{\varphi\}\not\subset T^{3-i}$.*

Proof. If T^1 and T^2 are (T_0,E)-separated then we have $T^i\in\{T^1,T^2\}_{\varphi,E}$ and $T^{3-i}\notin\{T^1,T^2\}_{\varphi,E}$ for some $i\in\{1,2\}$. The first one implies that $\nabla_E\cup\{\varphi\}\subset T^i$. Since T^i are closed, the condition $\nabla_E\cup\{\varphi\}\subset T^{3-i}$ implies $T^{3-i}\in\{T^1,T^2\}_{\varphi,E}$ contradicting the assumption. Thus, $\nabla_E\cup\{\varphi\}\not\subset T^{3-i}$.

Conversely, let $\nabla_E\cup\{\varphi\}\subset T^i$ and $\nabla_E\cup\{\varphi\}\not\subset T^{3-i}$. Since the deducibility is monotone (if $\Phi\subseteq\Psi$ for some sets Φ, Ψ of formulae then $\{\varphi\mid\Phi\vdash\varphi\}\subseteq\{\psi\mid\Psi\vdash\psi\}$) then $T^i\in\{T^1,T^2\}_{\varphi,E}$. At the same time by the definition we have $T^{3-i}\notin\{T^1,T^2\}_{\varphi,E}$ implying that T^1 and T^2 are (T_0,E)-separated.

Similarly Theorem 3.8 we have the following:

Theorem 3.9. *For a special relation $E\in\mathrm{SR}(\Sigma)$ and distinct theories $T^1,T^2\subset F(\Sigma)$, T^1 and T^2 are (T_1,E)-separated if and only if there is a sentence $\varphi\in F(\Sigma)$ such that $\nabla_E\cup\{\varphi\}\subset T^1$ and $\nabla_E\cup\{\varphi\}\not\subset T^2$.*

Remark 3.10. For any family $\mathcal{T}$ of complete theories in a given language Σ any special relations E with identically true ∇_E is $\mathcal{T}$-Hausdorff since for each distinct $T^1,T^2\in\mathcal{T}$ there is a sentence $\varphi\in F(\Sigma)$ such that $\varphi\in T^1$ and $\neg\varphi\in T^2$. Moreover, it holds for theories T^1,T^2 with distinct complete restrictions in a common language.

At the same time there are complete theories T^1,T^2 in distinct languages such that for any family $\mathcal{T}$ with $T^1,T^2\in\mathcal{T}$ there are no $\mathcal{T}$-Hausdorff special relations. Indeed, if T^2 is a proper expansion of a theory T^1 then any neighbourhood $\mathcal{T}_{\varphi,E}$ for T^1 contains T^2 contradicting the T_2-Hausdorffness.

Similarly [4] any family $\mathcal{T}$ of theories with a special relation E admits a Hausdorffization, i.e., an expansion $\mathcal{T}'$ of $\mathcal{T}$ such that each theory $T\in\mathcal{T}$ is transformed into an expansion $T'\in\mathcal{T}'$ with the property of T_2-separability for $\mathcal{T}'$ with respect to a special relation $E'\supseteq E$. For this aim it suffices to introduce new 0-ary predicate symbols Q_T for $T\in\mathcal{T}$ such that $Q_T\in T'$ and $\neg Q_T\in T''$ for all $T''\in\mathcal{T}'\setminus\{T'\}$, and extend E till E' by the relation of equality of formulae.

4. Ranks for families of theories in various languages

Starting with the rank $\mathrm{RS}(\cdot)$ and the degree $\mathrm{ds}(\cdot)$ for families of theories in a given language, we define an extended notions of rank $\mathrm{RS}_E(\cdot)$ and degree $\mathrm{ds}_E(\cdot)$ for families $\mathcal{T}$ of theories in various languages with respect to a special relation E.

Definition. For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}_E(\mathcal{T})=-1$, and for nonempty families $\mathcal{T}$ we put $\mathrm{RS}_E(\mathcal{T})\geq 0$.

For a family $\mathcal{T}$ and an ordinal $\alpha=\beta+1$ we put $\mathrm{RS}_E(\mathcal{T})\geq\alpha$ if there are pairwise $\mathcal{T}$-inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n\in\omega$, such that $\mathcal{T}_{\varphi_n,E}$ are pairwise disjoint and $\mathrm{RS}_E(\mathcal{T}_{\varphi_n,E})\geq\beta$, $n\in\omega$.

For a limit ordinal α we put $\mathrm{RS}_E(\mathcal{T})\geq\alpha$ if $\mathrm{RS}_E(\mathcal{T})\geq\beta$ for any $\beta<\alpha$.

We set $\mathrm{RS}_E(\mathcal{T})=\alpha$ if $\mathrm{RS}_E(\mathcal{T})\geq\alpha$ and $\mathrm{RS}_E(\mathcal{T})\not\geq\alpha+1$.

If $\mathrm{RS}_E(\mathcal{T})\geq\alpha$ for any α, we put $\mathrm{RS}_E(\mathcal{T})=\infty$.

A family $\mathcal{T}$ is called *E-totally transcendental* if $\mathrm{RS}_E(\mathcal{T})$ is an ordinal.

By the definition, since there are $\max\{|\Sigma(\mathcal{T})|,\omega\}$ $\Sigma(\mathcal{T})$-sentences, so if $\mathrm{RS}_E(\mathcal{T})<\infty$ then $|\mathrm{RS}_E(\mathcal{T})|\leq\max\{|\Sigma(\mathcal{T})|,\omega\}$.

In particular, the following proposition holds.

Proposition 4.1. *If $|\Sigma(\mathcal{T})|\leq\omega$ then for a special relation E either $|\mathrm{RS}_E(\mathcal{T})|\leq\omega$ or $\mathcal{T}$ is not E-totally transcendental.*

If $\mathcal{T}$ is E-totally transcendental, with $\mathrm{RS}_E(\mathcal{T})=\alpha\geq 0$, we define the *degree* $\mathrm{ds}_E(\mathcal{T})$ of $\mathcal{T}$ with respect to the special relation E as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}_E(\mathcal{T}_{\varphi_i})=\alpha$.

Remark 4.2. Notice that

$$\mathrm{RS}_E(\mathcal{T})=\mathrm{RS}(\mathcal{T}), \tag{4.1}$$

and

$$\mathrm{ds}_E(\mathcal{T})=\mathrm{ds}(\mathcal{T}) \tag{4.2}$$

for $\mathrm{RS}(\mathcal{T})\in\mathrm{Ord}$, if ∇_E consists of tautologies and $\mathcal{T}$ is a family of complete theories in the language Σ. Moreover, the equalities (4.1) and (4.2) hold if E is $\mathcal{T}$-Hausdorff.

In general case we have the inequalities

$$\mathrm{RS}_E(\mathcal{T})\leq\mathrm{RS}(\mathcal{T}), \tag{4.3}$$

and

$$\mathrm{ds}_E(\mathcal{T})\leq\mathrm{ds}(\mathcal{T}) \text{ for } \mathrm{RS}_E(\mathcal{T})=\mathrm{RS}(\mathcal{T})\in\mathrm{Ord} \tag{4.4}$$

for any family $\mathcal{T}$ and a special relation E. This inequalities can have arbitrary difference, with, for instance, $\mathrm{RS}_E(\mathcal{T})=0$ and $\mathrm{RS}(\mathcal{T})$ big enough.

Remark 4.3. Clearly, there are many E-totally transcendental families for various special relations. At the same time, there are families which are not E-totally transcendental. For this aim it is sufficient to take a special relation E with identically true ∇_E and a family $\mathcal{T}$ with $\mathrm{RS}(\mathcal{T})=\infty$ [13].

We remind [13] that all values $\xi\in\mathrm{Ord}\times(\omega\setminus\{0\})\cup\{-1,\infty\}$ are realized as $\xi=(\mathrm{RS}(\mathcal{T}),\mathrm{ds}(\mathcal{T}))$ and $\xi=\mathrm{RS}(\mathcal{T}')$ for appropriate families $\mathcal{T}$, $\mathcal{T}'$ of theories. For this aim we notice that the values -1 and ∞ are realized by the empty family and a family which infinite branching 2-tree, respectively. For the realization $\xi = (\alpha, n) = (\mathrm{RS}(\mathcal{T}), \mathrm{ds}(\mathcal{T}))$ we take 0-ary predicates Q and step-by-step expand a theory T_0 of the empty language by these predicates producing Q-neighbourhoods with RS $\leq \alpha$, and ds $\leq n$ for RS $= \alpha$. This construction is based on the given value $\xi = (\alpha, n)$. If $\alpha = 0$ we just take 0-ary predicates $Q_1, \ldots, Q_n$ and construct theories $T_1, \ldots, T_n$ with $Q_i \in T_i$ and $\neg Q_j \in T_i$ for $j \neq i$. For $\alpha = 1$ and $n = 1$, that corresponds to an e-minimal family [13; 17], we continue the process with countably many predicates Q_k, $k \in \omega$, and with unique accumulation point T_∞ containing formulae $\neg Q_k$ for all k. Having $\alpha = 1$ and $n > 1$ we copy the e-minimal family n times, substituting new 0-ary predicates instead of Q_k and obtaining n accumulation points. For $\alpha = 2$ and $n = 1$ we copy the e-minimal family ω times and mark the obtained family by new predicate Q^2 expanding all obtained theories and producing ω accumulation points. For $\alpha = 2$ and $n > 1$ we copy the obtained family n times, then continue the process for lager ordinal ranks and all natural degrees.

Using the equalities (4.1) and (4.2) we obtain all possibilities for the values of RS_E and ds_E.

Remark 4.4. The values $\mathrm{RS}_E(\mathcal{T})$ satisfy the following monotone properties:

1. If $\mathcal{T} \subseteq \mathcal{T}'$ with a fixed language then $\mathrm{RS}_E(\mathcal{T}) \leq \mathrm{RS}_E(\mathcal{T}')$.

2. If $\mathcal{T} \subseteq \mathcal{T}'$ with an extended language Σ' for $\mathcal{T}'$ and an identical extension $E' \in \mathrm{SR}(\Sigma')$ with respect to a special relation E for $\mathcal{T}'$ then $\mathrm{RS}_E(\mathcal{T}) \leq \mathrm{RS}_{E'}(\mathcal{T}')$.

At the same time the monotonicity fails if E' connects additional formulae. Indeed, taking the example $\mathcal{T}'$ of Hausdorffization of a family $\mathcal{T}$ by a special relation E' in Remark 3.10 we observe that extending E' till a special relation E'' with $(Q_T \leftrightarrow Q_{T'}) \in \nabla_{E''}$ for all new 0-ary predicates $Q_T, Q_{T'}$ the value of RS-rank returns to the initial one: $\mathrm{RS}_{E''}(\mathcal{T}') = \mathrm{RS}_E(\mathcal{T})$. It means that the Hausdorffization can unboundedly increase the value of RS-rank and an appropriate extension of a special relation produces a reduction of RS-rank till a value $\geq \mathrm{RS}_E(\mathcal{T})$.

Using Remark 4.3 we have the following:

Theorem 4.5. *For any ordinals α, β and $m, n \in \omega \setminus \{0\}$ there are families $\mathcal{T}$, $\mathcal{T}'$ with $\mathcal{T} \subseteq \mathcal{T}$ and special relations E, E' with $E \subseteq E'$ such that $\mathrm{RS}_E(\mathcal{T}) = \alpha$, $\mathrm{ds}_E(\mathcal{T}) = m$, $\mathrm{RS}_{E'}(\mathcal{T}') = \beta$, $\mathrm{ds}_{E'}(\mathcal{T}') = n$.*

Proof. If $\alpha < \beta$, or $\alpha = \beta$ and $m < n$ we just use the construction described in Remark 4.3 to obtain lager values of rank and degree. If $\alpha = \beta$ and $m > n$ we take a family $\mathcal{T}$ described in Remark 4.3 with identical relation E and $\mathrm{RS}_E(\mathcal{T}) = \alpha$, $\mathrm{ds}_E(\mathcal{T}) = m$. Now we extend E till a special relation E' identifying correspondent predicates in some $m - n$

copies of s-definable subfamilies $\mathcal{T}^*$ of $\mathcal{T}$ with $\mathrm{RS}_E(\mathcal{T}*) = \alpha$, $\mathrm{ds}_E(\mathcal{T}*) = 1$. After that identification we have the same family $\mathcal{T}$ with $\mathrm{RS}_{E'}(\mathcal{T}) = \beta$, $\mathrm{ds}_{E'}(\mathcal{T}) = n$.

If $\alpha > \beta$ we extend E till a special relation E' identifying copies of families that produce a jump of the rank from β to α following the construction in Remark 4.3. In such a case we do not identify n disjoint s-definable copies witnessing the value n of the degree. After that identification we again preserve the family $\mathcal{T}$ and obtain $\mathrm{RS}_{E'}(\mathcal{T}) = \beta$, $\mathrm{ds}_{E'}(\mathcal{T}) = n$.

Now we characterize families with given RS_E-ranks.

Proposition 4.6. *For any nonempty family $\mathcal{T}$ of a language Σ and a special relation $E \in \mathrm{SR}(\Sigma)$ the following conditions are equivalent:*

(1) $\mathrm{RS}_E(\mathcal{T}) = 0$ *and* $\mathrm{ds}_E(\mathcal{T}) = 1$*;*

(2) *$\mathcal{T}$ does not contain (T_2, E)-separated theories;*

(3) *for any $\varphi, \psi \in \mathrm{Sent}(\Sigma)$ either $\mathcal{T}_{\varphi,E} = \emptyset$ or $\mathcal{T}_{\psi,E} = \emptyset$ or $\mathcal{T}_{\varphi,E} \cap \mathcal{T}_{\psi,E} \neq \emptyset$.*

Proof. (1) $\Rightarrow$ (2). If $\mathcal{T}$ contains (T_2, E)-separated theories T and T' witnessed by neighbourhoods $\mathcal{T}_{\varphi,E}$ and $\mathcal{T}_{\psi,E}$ then these neighbourhoods are disjoint witnessing that $\mathrm{RS}_E(\mathcal{T}) > 0$ or $\mathrm{ds}_E(\mathcal{T}) > 1$.

(2) $\Rightarrow$ (3). Assuming that $\mathcal{T}_{\varphi,E} \neq \emptyset$, $\mathcal{T}_{\psi,E} \neq \emptyset$, $\mathcal{T}_{\varphi,E} \cap \mathcal{T}_{\psi,E}\emptyset$ we obtain the (T_2, E)-separability for theories $T \in \mathcal{T}_{\varphi,E}$ and $T' \in \mathcal{T}_{\psi,E}$.

(3) $\Rightarrow$ (1) follows by the definition of RS_E-rank.

Corollary 4.7. *If $E \in \mathrm{SR}(\Sigma)$ and $\mathcal{T}$ is a nonempty family of a language Σ of theories which is linearly ordered by the inclusion then* $\mathrm{RS}_E(\mathcal{T}) = 0$ *and* $\mathrm{ds}_E(\mathcal{T}) = 1$.

Proof. By Proposition 4.6 it suffices to show that $\mathcal{T}$ does not contain (T_2, E)-separated theories T and T'. Indeed, if these theories exist then there are $\varphi \in T \setminus T'$ and $\phi \in T' \setminus T$ contradicting that $\mathcal{T}$ is linearly ordered.

Remark 4.8. The converse assertion for Corollary 4.7 holds if E is identical [4] or similar to it. At the same time it can fail in general case. For instance, one can form a family $\mathcal{T}$ of theories in disjoint nonempty languages such that a special relation E identifies all theories in $\mathcal{T}$. This identification implies that $\mathrm{RS}_E(\mathcal{T}) = 0$ and $\mathrm{ds}_E(\mathcal{T}) = 1$ whereas all theories in $\mathcal{T}$ are $\subseteq$-incomparable.

Besides, along with the fact that for a family $\mathcal{T}$ of complete theories with the identical relation E the values $\mathrm{RS}_E(\mathcal{T}) = 0$ and $\mathrm{ds}_E(\mathcal{T}) = 1$ imply $|\mathcal{T}| = 1$, a general case with $\mathrm{RS}_E(\mathcal{T}) = 0$ and $\mathrm{ds}_E(\mathcal{T}) = 1$ admits a possibility for a family $\mathcal{T}$ to have arbitrary large cardinality.

The following theorem is a generalization of [4, Theorem 3.4].

Theorem 4.9. *For any family $\mathcal{T}$, a special relation E and $n \in \omega \setminus \{0\}$ the following conditions are equivalent:*

(1) $\mathrm{RS}_E(\mathcal{T})=0$ *and* $\mathrm{ds}_E(\mathcal{T})=n$;

(2) $\mathcal{T}$ *has an n-element E-Hausdorff subfamily* $\{T^1,\ldots,T^n\}$ *such that each element* $T\in\mathcal{T}\setminus\{T^1,\ldots,T^n\}$ *is not (T_2,E)-separated with* $\{T^1,\ldots,T^n\}$, *in* $\mathcal{T}$;

(3) $\mathcal{T}$ *is divided into n disjoint nonempty parts* $\mathcal{T}_{\varphi_1,E},\ldots,\mathcal{T}_{\varphi_n,E}$ *such that each part* $\mathcal{T}_{\varphi_i,E}$ *does not contain (T_2,E)-separated theories.*

Proof. (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3). If $\mathrm{RS}_E(\mathcal{T})=0$ and $\mathrm{ds}_E(\mathcal{T})=n$ then $\mathcal{T}$ is divided into n disjoint nonempty parts $\mathcal{T}_{\varphi_1,E},\ldots,\mathcal{T}_{\varphi_n,E}$ such that each part $\mathcal{T}_{\varphi_i,E}$ has $\mathrm{RS}_E=0$ and $\mathrm{ds}_E=1$. Now by Proposition 4.6 each $\mathcal{T}_{\varphi_i,E}$ does not contain (T_2,E)-separated theories. Moreover, taking theories $T^i\in\mathcal{T}_{\varphi_i,E}$ we observe that $\{T^1,\ldots,T^n\}$ is an E-Hausdorff subfamily of $\mathcal{T}$ such that by $\mathrm{ds}_E(\mathcal{T}_{\varphi_1,E})=\ldots=\mathrm{ds}_E(\mathcal{T}_{\varphi_n,E})=1$ each element $T\in\mathcal{T}\setminus\{T^1,\ldots,T^n\}$ is not (T_2,E)-separated with $\{T^1,\ldots,T^n\}$.

(3) $\Rightarrow$ (2). By the conjecture we take n disjoint nonempty parts $\mathcal{T}_{\varphi_1,E},\ldots,\mathcal{T}_{\varphi_n,E}$ such that each part $\mathcal{T}_{\varphi_i,E}$ does not contain (T_2,E)-separated theories. Now we choose theories $T^i\in\mathcal{T}_{\varphi_i,E}$, $1\leq i\leq n$. By that choice $\{T^1,\ldots,T^n\}$ is E-Hausdorff. At the same time each $T\in\mathcal{T}\setminus\{T^1,\ldots,T^n\}$ belongs to some $\mathcal{T}_{\varphi_i,E}$ and not (T_2,E)-separated with T^i implying that T is not (T_2,E)-separated with $\{T^1,\ldots,T^n\}$, in $\mathcal{T}$.

(2) $\Rightarrow$ (1). Let $\mathcal{T}$ have an n-element E-Hausdorff subfamily $\{T^1,\ldots,T^n\}$ such that each element $T\in\mathcal{T}\setminus\{T^1,\ldots,T^n\}$ is not (T_2,E)-separated with $\{T^1,\ldots,T^n\}$. The E-Hausdorffness implies that there are disjoint neighbourhoods $\mathcal{T}_{\varphi_1,E},\ldots,\mathcal{T}_{\varphi_n,E}$ with $T^i\in\mathcal{T}_{\varphi_i,E}$, $1\leq i\leq n$. Since all $T\in\mathcal{T}\setminus\{T^1,\ldots,T^n\}$ are not (T_2,E)-separated with $\{T^1,\ldots,T^n\}$ there are no more possibilities to divide $\mathcal{T}$ into disjoint definable parts with respect to ∇_E. Therefore $\mathrm{RS}_E(\mathcal{T})=0$ and $\mathrm{ds}_E(\mathcal{T})=n$.

Remark 4.10. Criteria in Theorem 4.9 can be naturally spread for the equalities $\mathrm{RS}_E(\mathcal{T})=\alpha\in\mathrm{Ord}$ and $\mathrm{ds}_E(\mathcal{T})=n\in\omega\setminus\{0\}$ obtaining topological characterizations for values $(\mathrm{RS}_E,\mathrm{ds}_E)=(\alpha,n)$ in terms of topological characterizations for values $(\mathrm{RS}_E,\mathrm{ds}_E)=(\alpha,1)$ and for values $\mathrm{RS}_E<\alpha$.

For instance, the pair $(\mathrm{RS}_E(\mathcal{T}),\mathrm{ds}_E(\mathcal{T}))=(1,1)$, i.e. the *e-minimality* [13; 17] of $\mathcal{T}$, is characterized as follows: $\mathcal{T}$ is divided into infinitely many disjoint (s,E)-definable nonempty subfamilies $\mathcal{T}_i$ of $\mathcal{T}$ each of which does not have E-Hausdorff non-singleton subparts, and $\mathcal{T}$ does not have two disjoint (s,E)-definable infinite subfamilies $\mathcal{T}'$, $\mathcal{T}''$ with infinitely many disjoint (s,E)-definable nonempty subfamilies $\mathcal{T}_j\subseteq\mathcal{T}'$, $\mathcal{T}_k\subseteq\mathcal{T}''$ each of which does not have E-Hausdorff non-singleton subparts.

Remark 4.11. Theorem 2.1 can fail with respect to RS_E-rank. Indeed, a special relation E can produce the non-separability of continuum many theories in a family $\mathcal{T}$ in a countable language implying small ordinal values for $\mathrm{RS}_E(\mathcal{T})$ whereas the cardinality for the closure of $\mathcal{T}$ and the e-spectrum for $\mathcal{T}$ equal 2^ω. At the same time, the value $\mathrm{RS}_E(\mathcal{T})=\infty$ is still characterized by the existence of 2-tree of sentences φ for s-definable sets $\mathcal{T}_{\varphi,E}$.

Similarly, the assertions 2.2–2.7 can fail with respect to a special relation E, since large cardinalities of languages Σ and of families $\mathcal{T}_\Sigma$ of theories do not guarantee these cardinalities with respect to E. In particular, there are large families $\mathcal{T}_\Sigma$ with small ordinal values $\mathrm{RS}_E(\mathcal{T})$.

Remark 4.12. Similarly [4] the RS_E-rank can be naturally modified till $\overline{\mathrm{RS}}_E$-rank based on Boolean combinations of s-definable sets $\mathcal{T}_{\varphi,E}$. It allows to transform a series of results in [4; 5] with respect to special relations E.

5. Conclusion

We studied properties and characteristics for topologies and ranks for families of theories in various languages which is based on special relations connecting formulae in a given language. These relations are used to define and describe kinds of separations with respect to T_0-topologies, T_1-topologies and Hausdorff topologies. Special relations are used to define and study ranks for families of theories in various languages. Possibilities of values for the rank are described, and these possibilities are characterized in topological terms. It would be interesting to apply the general approach based on special relations for a series of natural families of theories.

References

1. Engelking R. *General topology.* Berlin, Heldermann Verlag Publ., 1989, 529 p.
2. Kulpeshov B.Sh., Sudoplatov S.V. Properties of ranks for families of strongly minimal theories. *Siberian Electronic Mathematical Reports*, 2022, vol. 19, no. 1, pp. 120–124. https://doi.org/10.33048/semi.2022.19.011
3. Markhabatov N.D. Ranks for families of permutation theories. *The Bulletin of Irkutsk State University. Series Mathematics*, 2019, vol. 28, pp. 86–95. https://doi.org/10.26516/1997-7670.2019.28.85
4. Markhabatov N.D., Sudoplatov S.V. Topologies, ranks, and closures for families of theories. I. *Algebra and Logic*, 2021, vol. 59, no. 6, pp. 437–455. https://doi.org/10.1007/s10469-021-09620-4
5. Markhabatov N.D., Sudoplatov S.V. Topologies, ranks, and closures for families of theories. II. *Algebra and Logic*, 2021, vol. 60, no. 1, pp. 38–52. https://doi.org/10.1007/s10469-021-09626-y
6. Markhabatov N.D., Sudoplatov S.V. Ranks for families of all theories of given languages. *Eurasian Mathematical Journal*, 2021, vol. 12, no. 2, pp. 52–58. https://doi.org/10.32523/2077-9879-2021-12-2-52-58
7. Markhabatov N.D., Sudoplatov S.V. Definable subfamilies of theories, related calculi and ranks. *Siberian Electronic Mathematical Reports*, 2020, vol. 17, pp. 700–714. https://doi.org/10.33048/semi.2020.17.048
8. Morley M. Categoricity in power. *Trans. Amer. Math. Soc.*, 1965, vol. 114, no. 2, pp. 514–538.

9. Pavlyuk In.I., Sudoplatov S.V. Formulas and properties for families of theories of Abelian groups. *The Bulletin of Irkutsk State University. Series Mathematics*, 2021, vol. 36, pp. 95–109. https://doi.org/10.26516/1997-7670.2021.36.95
10. Pavlyuk In.I., Sudoplatov S.V. Ranks for families of theories of abelian groups. *The Bulletin of Irkutsk State University. Series Mathematics*, 2019, vol. 28, pp. 95–112. https://doi.org/10.26516/1997-7670.2019.28.95
11. Sacks G.E. *Saturated model theory*, New Jersey, London, Singapore, Beijing, Shanghai, Hong Kong, Taipei, Chennai, World Scientific, 2009, 220 p.
12. Sudoplatov S.V. Closures and generating sets related to combinations of structures. *The Bulletin of Irkutsk State University. Series Mathematics*, 2016, vol. 16, pp. 131–144.
13. Sudoplatov S.V. Ranks for families of theories and their spectra. *Lobachevskii Journal of Mathematics*, 2021, vol. 42, no. 12, pp. 2959–2968. https://doi.org/10.1134/S1995080221120313
14. Sudoplatov S.V. Hierarchy of families of theories and their rank characteristics. *The Bulletin of Irkutsk State University. Series Mathematics*, 2020, vol. 33, pp. 80–95. https://doi.org/10.26516/1997-7670.2020.33.80
15. Sudoplatov S.V. Formulas and properties, their links and characteristics. *Mathematics*, 2021, vol. 9, iss. 12, 1391. https://doi.org/10.3390/math9121391
16. Sudoplatov S. V. Special relations for formulas, their equivalence relations and theories. *Siberian Electronic Mathematical Reports*, 2022, vol. 19, no. 1, pp. 259–272. https://doi.org/10.33048/semi.2022.19.020
17. Sudoplatov S.V. Approximations of theories. *Siberian Electronic Mathematical Reports*, 2020, vol. 17, pp. 715–725. https://doi.org/10.33048/semi.2020.17.049

Список источников

1. Энгелькинг Р. Общая топология. М. : Мир, 1986. 752 с.
2. Kulpeshov B. Sh., Sudoplatov S. V. Properties of ranks for families of strongly minimal theories // Siberian Electronic Mathematical Reports. 2022. Vol. 19, N 1. P. 120–124. https://doi.org/10.33048/semi.2022.19.011
3. Markhabatov N. D. Ranks for families of permutation theories // The Bulletin of Irkutsk State University. Series Mathematics. 2019. Vol. 28. P. 86–95. https://doi.org/10.26516/1997-7670.2019.28.85
4. Markhabatov N. D., Sudoplatov S. V. Topologies, ranks, and closures for families of theories. I // Algebra and Logic. 2021. Vol. 59, N 6. P. 437–455. https://doi.org/10.1007/s10469-021-09620-4
5. Markhabatov N. D., Sudoplatov S. V. Topologies, ranks, and closures for families of theories. II // Algebra and Logic. 2021. Vol. 60, No. 1. P. 38–52. https://doi.org/10.1007/s10469-021-09626-y
6. Markhabatov N. D., Sudoplatov S. V. Ranks for families of all theories of given languages // Eurasian Mathematical Journal. 2021. Vol. 12, N 2. P. 52–58. https://doi.org/10.32523/2077-9879-2021-12-2-52-58
7. Markhabatov N. D., Sudoplatov S. V. Definable subfamilies of theories, related calculi and ranks // Siberian Electronic Mathematical Reports. 2020. Vol. 17. P. 700–714. https://doi.org/10.33048/semi.2020.17.048
8. Morley M. Categoricity in power // Trans. Amer. Math. Soc. 1965. Vol. 114, N 2. P. 514–538.

9. Pavlyuk In. I., Sudoplatov S. V. Formulas and properties for families of theories of Abelian groups // The Bulletin of Irkutsk State University. Series Mathematics. 2021. Vol. 36. P. 95–109. https://doi.org/10.26516/1997-7670.2021.36.95
10. Pavlyuk In. I., Sudoplatov S. V. Ranks for families of theories of abelian groups // The Bulletin of Irkutsk State University. Series Mathematics. 2019. Vol. 28. P. 95–112. https://doi.org/10.26516/1997-7670.2019.28.95
11. Сакс Дж. Теория насыщенных моделей. М. : Мир, 1976. 192 с.
12. Sudoplatov S. V. Closures and generating sets related to combinations of structures // The Bulletin of Irkutsk State University. Series Mathematics. 2016. Vol. 16. P. 131–144.
13. Sudoplatov S. V. Ranks for families of theories and their spectra // Lobachevskii Journal of Mathematics. 2021. Vol. 42, N 12. P. 2959–2968. https://doi.org/10.1134/S1995080221120313
14. Sudoplatov S. V. Hierarchy of families of theories and their rank characteristics // The Bulletin of Irkutsk State University. Series Mathematics. 2020. Vol. 33. P. 80–95. https://doi.org/10.26516/1997-7670.2020.33.80
15. Sudoplatov S. V. Formulas and properties, their links and characteristics // Mathematics. 2021. Vol. 9, Issue 12. 1391. 16 pp. https://doi.org/10.3390/math9121391
16. Sudoplatov S. V. Special relations for formulas, their equivalence relations and theories. *Siberian Electronic Mathematical Reports*, 2022, vol. 19, no. 1, pp. 259–272. https://doi.org/10.33048/semi.2022.19.020
17. Sudoplatov S. V. Approximations of theories // Siberian Electronic Mathematical Reports. 2020. Vol. 17. P. 715–725. https://doi.org/10.33048/semi.2020.17.049

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