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Exact Solutions to the Oberbeck–Boussinesq Equations for Shear Flows of a Viscous Binary Fluid with Allowance Made for the Soret Effect *

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Abstract. The paper considers an exact solution to the equations of thermal diffusion of a viscous incompressible fluid in the Boussinesq approximation with neglect of the Dufour effect for a steady shear flow. It is shown that the reduced system of constitutive relations is nonlinear and overdetermined. A nontrivial exact solution of this system is sought in the Lin–Sidorov–Aristov class. The resulting family of exact solutions allows one to describe steady-state inhomogeneous shear flows. This class generalizes the classical Couette, Poiseuille, and Ostroumov–Birikh solutions. It is demonstrated that the system of ordinary differential equations reduced within this class retains the properties of nonlinearity and overdetermination. A theorem on solvability conditions for the overdetermined system is proved; it is reported that, when these conditions are met, the solution is unique. The overdetermined system is solvable owing to the algebraic identity relating the horizontal velocity gradients, which are linear functions of the vertical coordinate. The constructive proof of the computation of hydrodynamic fields consists in the successive integration of the polynomials, the polynomial degree being dependent on the values of the boundary parameters.

Keywords: viscous binary fluid, exact solution, Soret effect, shear flow, overdetermined system.

1. Introduction

The Boussinesq approximation is known to be used in studying convective flows of binary fluids [6; 14; 18]. The study of the properties of the solutions to the Oberbeck–Boussinesq equations describing thermal diffusion flows is complicated by the Soret and Dufour thermodynamic cross effects between the temperature field and the field of dissolved substance concentration [13; 22]. This is mathematically expressed in the coupled nature of the equations of heat transfer and concentration. A class of exact solutions of the Navier–Stokes equations for describing the flows of incompressible binary fluids with allowance made for both effects was proposed in [3] The idea that the Dufour parameter, due to its smallness, can be neglected in the description of fluid flows dominates in the current scientific literature [11; 12; 15; 16; 23]. Exact solutions to the Oberbeck—Boussinesq equations are sought based on this physical assumption.

The finding of exact solutions to the thermal diffusion equations has started fairly recently from the study of unidirectional (stratified) flows in infinite plane and infinitely long cylindrical channels [4; 21]. Various modifications of the Ostroumov–Birikh family are used in the construction of exact solutions [5; 19]. In other words, the exact solutions whose velocity fields are described by the exact Couette solution are studied, and the horizontal (longitudinal) gradients of temperature, pressure, and concentration are taken into account.

For different applications, it is necessary to have a collection of exact solutions characterizing the distribution of hydrodynamic fields in shear flows (the vertical component of the velocity vector is zero) [7–11]. In this case, the Oberbeck—Boussinesq equation system is overdetermined. Five unknown functions (two velocity vector components, pressure, temperature, and concentration) are computed from six equations (the Oberbeck–Boussinesq system and the equations of incompressibility, heat conduction, and concentration). In the study of shear flows, the inhomogeneity of the fields of temperature, pressure, and concentration can induce an inhomogeneous velocity field [1; 3; 17; 20]. Note that, according to the Onsager principle, the inverse effect is true [1; 3; 17; 20].

The solvability of the overdetermined systems describing inhomogeneous flows, which resulted from the reduction of the Navier–Stokes equations in various force fields, was studied in [9; 10]. An exact solution was constructed in the Lin–Sidorov–Aristov class [1; 17; 20]. It describes spatially inhomogeneous vortex flows. Investigations for inhomogeneous shear flows of viscous binary fluids have yet to be started. The study of fluid flows with a homogeneous two-dimensional velocity field depending only on the transverse coordinate (two-dimensional Couette flow), was started in [11]. The

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presented mathematical model describes the convection of a binary fluid in view of the Soret effect in an exact statement. It was shown in [11] that, in a steady-state flow of a fluid with a dissolved substance, there was a more pronounced stratification of hydrodynamic fields as compared to Marangoni heat convection first discussed in [2]. Besides, the exact solutions reported in [8] and taking into account the spatial inhomogeneity of velocities also describe hydrodynamic fields with a greater number of stratification points than in a homogeneous velocity field. Thus, a fluid flow structure (first of all, in the velocity field) is affected by the spatial inhomogeneity of hydrodynamic fields, boundary conditions, and flow-inducing forces.

This paper, applying the technique proposed in [1; 11; 17; 20], studies the solvability of the Oberbeck—Boussinesq equation system for describing the spatially inhomogeneous thermal-diffusion shear flows of incompressible binary fluids in the Lin—Sidorov–Aristov class with allowance made for the Soret effect.

2. Problem Statement

Steady-state shear flows of binary viscous incompressible fluids are described by the following relationships [11]:

$$
\begin{gathered}
V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}=-\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}\right),\\
V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}=-\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}+\frac{\partial^2 V_y}{\partial z^2}\right),\\
\frac{\partial P}{\partial z}=g\left(\beta_1 T+\beta_2 C\right),\quad \frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}=0,\\
V_x\frac{\partial T}{\partial x}+V_y\frac{\partial T}{\partial y}=\chi\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right),\\
V_x\frac{\partial C}{\partial x}+V_y\frac{\partial C}{\partial y}=d\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}+\frac{\partial^2 C}{\partial z^2}\right)+\alpha d\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right).
\end{gathered}
\tag{2.1}
$$

Here, $\mathbf{V}(x,y,z)=(V_x,V_y,0)$ is fluid velocity; g is free fall acceleration; β_1,β_2 are the thermal and concentration expansion coefficients, respectively; $T(x,y,z)$ is temperature deviation from the equilibrium state; $C(x,y,z)$ is the deviation of the concentration of the light phase (admixture) in a binary fluid mixture from the equilibrium value; $P(x,y,z)$ is deviation from hydrostatic pressure, normed to the average constant fluid density ρ; ν is kinematic fluid viscosity; χ is the temperature conductivity coefficient; d is the diffusion coefficient; α is the thermal diffusion coefficient (the Soret parameter) [22].

The quadratically nonlinear system (2.1) is overdetermined since four functions of the five equations (velocities V_x, V_y , pressure P, temperature T, and concentration C) need to be determined. A nontrivial solution to system (2.1) will be sought in the Lin–Sidorov–Aristov class, where the required unknowns are represented by complete linear forms in terms of coordinates [3; 9; 10; 17]:

$$V_x = U(z) + u_1(z)\,x + u_2(z)\,y, \quad V_y = V(z) + v_1(z)\,x + v_2(z)\,y,$$

$$P = P_0(z) + P_1(z)\,x + P_2(z)\,y, \quad T = T_0(z) + T_1(z)\,x + T_2(z)\,y,$$

$$C = C_0(z) + C_1(z)\,x + C_2(z)\,y. \qquad (2.2)$$

Having substituted the exact solution (2.2) into system (2.1), applying the method of undetermined coefficients, we arrive at a system of ordinary differential equations, which, for the convenience of analysis, is here presented as follows:

$$u_1^2 + u_2 v_1 = \nu u_1'', \quad u_1 u_2 + u_2 v_2 = \nu u_2'',$$

$$u_1 v_1 + v_1 v_2 = \nu v_1'', \quad u_2 v_1 + v_2^2 = \nu v_2'', \quad u_1 + v_2 = 0; \qquad (2.3)$$

$$u_1 T_1 + v_1 T_2 = \chi T_1'', \quad u_2 T_1 + v_2 T_2 = \chi T_2''; \qquad (2.4)$$

$$u_1 C_1 + v_1 C_2 = d C_1'' + \alpha d T_1'', \quad u_2 C_1 + v_2 C_2 = d C_2'' + \alpha d T_2''; \qquad (2.5)$$

$$P_1' = g(\beta_1 T_1 + \beta_2 C_1), \quad P_2' = g(\beta_1 T_2 + \beta_2 C_2); \qquad (2.6)$$

$$u_1 U + u_2 V = -P_1 + \nu U'', \quad v_1 U + v_2 V = -P_2 + \nu V''; \qquad (2.7)$$

$$U T_1 + V T_2 = \chi T_0'', \quad U C_1 + V C_2 = d C_0'' + \alpha d T_0''; \qquad (2.8)$$

$$P_0' = g(\beta_1 T_0 + \beta_2 C_0). \qquad (2.9)$$

The prime in system (2.3)-(2.9) marks derivation with respect to the variable z. In the transition from system (2.1) to system (2.3)-(2.9) the property of overdetermination is inherited, i.e. we have sixteen equations for the determination of fifteen nonlinear coefficients of the linear forms in Eq. (2.2). Subsystem (2.3) alone has a “redundancy” of equations with respect to the number of unknowns. Thus, if it proves possible to find a nontrivial uultaneous solution, i.e. the one satisfying all the equations of the overdetermined subsystem (2.3), successive integration can yield a solution to any of the subsystems (2.4)-(2.9).

3. Analyzing the Solvability of the Equation System

The existence of a nontrivial exact solution of system (2.3)-(2.9) is ensured by the fulfilment of the following theorem.

Theorem 1. *The overdetermined system* (2.3) *has a nontrivial exact solution in class* (2.2)*, which is an exact solution of system* (2.1)*, if and only if the spatial accelerations* u_1, u_2, v_1 *and* v_2 *are described by the functions*

$$u_1 = u\cos\vartheta\sin\vartheta = -v_2, \quad u_2 = u\cos^2\vartheta, \quad v_1 = -u\sin^2\vartheta,$$

where u *is a function satisfying the equation* $u'' = 0$ *and* ϑ *is some number.*

Proof. Transform subsystem (2.3) by reducing the number of equations in it. To do this, use the relationship $u_1 + v_2 = 0$ and express the spatial acceleration v_2 via the component u_1 as $v_2 = -u_1$. Substitute this expression into all the differential equations of subsystem (2.3). This yields

$$u_1^2 + u_2 v_1 = \nu u_1'', \quad u_1^2 + u_2 v_1 = -\nu u_1'', \quad u_2'' = 0, \quad v_1'' = 0. \tag{3.1}$$

It follows from the last two equations in system (3.1) that the components v_1, u_2 are linear functions,

$$u_2 = b_1 z + b_2, \quad v_1 = b_3 z + b_4. \tag{3.2}$$

Besides, from the first two equations of system (3.1) it follows that the spatial acceleration u_1 linearly depends on the variable z,

$$u_1 = b_5 z + b_6. \tag{3.3}$$

Herewith, solutions (3.2),(3.3) must satisfy the algebraic consistency condition

$$u_1^2 + u_2 v_1 = 0. \tag{3.4}$$

The substitution of Eqs. (3.2) and (3.3) into the condition expressed by Eq. (3.4) results in the following system of conditions imposed on the coefficients b_1, b_2, b_3, b_4, b_5 and b_6 :

$$b_5^2 + b_1 b_3 = 0, \quad 2b_5 b_6 + b_1 b_4 + b_2 b_3 = 0, \quad b_6^2 + b_2 b_4 = 0. \tag{3.5}$$

The solutions of system (3.5) for the spatial accelerations represented by Eqs. (3.2), (3.3) are particular cases of the following representation:

$$u_1 = u\cos\vartheta\sin\vartheta, \quad u_2 = u\cos^2\vartheta, \quad v_1 = -u\sin^2\vartheta, \tag{3.6}$$

where ϑ is some number and the function u satisfies the equation $u'' = 0$, i.e. it is representable as $u = c_1 z + c_2$.

The substitution of the solution for the spatial accelerations represented by Eq. (3.6) into the consistency condition (3.4) gives the true identity

$$u_1^2 + u_2 v_1 = (u\cos\vartheta\sin\vartheta)^2 + \left(u\cos^2\vartheta\right)\left(-u\sin^2\vartheta\right) = 0.$$

Consequently, the overdetermined system (2.3) has a simultaneous solution. Thus, the theorem has been constructively proved. □

4. Constructing an Exact Solution

The above exact solutions for the spatial accelerations $u_1, v_1, andv_2$ enable us to find the unknown functions U, V, T_i, C_i and P_i $(i = 0, 1, 2)$ defining the hydrodynamic fields.

Theorem 2. *If the simultaneous solution of the overdetermined system* (2.3) *has the form of Eq.* (3.6)*, the solution of Eqs.* (2.3)–(2.9) *does exist.*

Proof. Transform Eqs. (2.4) and (2.5) enabling us to determine, with Eq. (3.6) taken into account, the longitudinal (horizontal) gradients T_1, T_2 of the temperature field T and the gradients C_1, C_2 of the concentration field C. The substitution gives

$$\begin{aligned} \chi T_1'' &= u\cos\vartheta\sin\vartheta T_1 - u\sin^2\vartheta T_2, \\ \chi T_2'' &= u\cos^2\vartheta T_1 - u\cos\vartheta\sin\vartheta T_2, \\ dC_1'' + \alpha d T_1'' &= u\cos\vartheta\sin\vartheta C_1 - u\sin^2\vartheta C_2, \\ dC_2'' + \alpha d T_2'' &= u\cos^2\vartheta C_1 - u\cos\vartheta\sin\vartheta C_2. \end{aligned} \tag{4.1}$$

Multiply the first equation in system (4.1) by $\cos\vartheta$ and subtract the second equation multiplied by $\sin\vartheta$ from it. After double integration, we obtain the relationship

$$\cos\vartheta T_1 - \sin\vartheta T_2 = \gamma_1 z + \gamma_2. \tag{4.2}$$

Hereinafter, the coefficients of the form γ_i $(i \in N)$ are integration constants. It follows from the third and fourth equations in system (4.1) that

$$\cos\vartheta C_1 + \alpha\cos\vartheta T_1 - \sin\vartheta C_2 - \alpha\sin\vartheta T_2 = \gamma_3 z + \gamma_4. \tag{4.3}$$

We express the gradients C_2, T_2 from the relations (4.2), (4.3). To do this, we represent system (4.2), (4.3) in the matrix form as

$$\begin{pmatrix} \sin\vartheta & 0 \\ \alpha\sin\vartheta & \sin\vartheta \end{pmatrix} \begin{pmatrix} T_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} \cos\vartheta T_1 - \gamma_1 z - \gamma_2 \\ \alpha\cos\vartheta T_1 + \cos\vartheta C_1 - \gamma_3 z - \gamma_4 \end{pmatrix}. \tag{4.4}$$

The structure of the solution of the matrix equation (4.4) depends on the value of the coefficient $\sin\vartheta$.

Let us first consider the particular case of ϑ being such that $\sin\vartheta = 0$. Then the left parts of Eq. (4.4) are identically equal to zero and system (4.4) becomes

$$T_1 - \gamma_1 z - \gamma_2 = 0, \quad \alpha T_1 + C_1 - \gamma_3 z - \gamma_4 = 0.$$

Hence, the solution is

$$T_1 = \gamma_1 z + \gamma_2, \quad C_1 = -\alpha T_1 + \gamma_3 z + \gamma_4 = (\gamma_3 - \alpha\gamma_1)\, z + (\gamma_4 - \alpha\gamma_2)\,. \tag{4.5}$$

The obtained solutions (4.5) are substituted into the second equation of system (4.1). In view of Eq. (3.6), when $\sin\vartheta = 0$, Eqs. (2.6), (2.7) acquire the form

$$\chi T_2'' = uT_1, \quad C_2'' = \alpha T_2'' + \frac{1}{d}uC_1, \quad P_1' = g\left(\beta_1 T_1 + \beta_2 C_1\right),$$

$$P_2' = g\left(\beta_1 T_2 + \beta_2 C_2\right), \quad \nu U'' = Vu + P_1, \quad \nu V'' = P_2. \qquad (4.6)$$

Expressions (4.5) and (3.6) are substituted into Eq. (4.6). Successive integration of the ordinary differential equations of this tenth-order system results in a polynomial solution where the gradient P_1 of the pressure field is a second-order polynomial, the gradient P_2 is a fifth-order one, the gradient T_2 of the temperature field and the gradient C_2 of the concentration field are fourth-order polynomials, and the velocities U and V are tenth- and seventh-order ones, respectively.

We now return to the general case of ϑ being such that $\sin\vartheta \neq 0$. This means that the determinant $D = \sin^2\vartheta$ of system (4.4) is nonzero; therefore, system (4.4) is uniquely solvable,

$$\begin{pmatrix} T_2 \\ C_2 \end{pmatrix} = \frac{1}{\sin^2\vartheta}\begin{pmatrix} -\sin\vartheta & 0 \\ \alpha\sin\vartheta & -\sin\vartheta \end{pmatrix}\begin{pmatrix} -\cos\vartheta T_1 + \gamma_1 z + \gamma_2 \\ -\alpha\cos\vartheta T_1 - \cos\vartheta C_1 + \gamma_3 z + \gamma_4 \end{pmatrix}.$$

$$T_2 = \mathrm{ctg}\vartheta T_1 + \frac{-\gamma_1 z - \gamma_2}{\sin\vartheta}, \quad C_2 = \mathrm{ctg}\vartheta C_1 + \frac{(\alpha\gamma_1 - \gamma_3)z + (\alpha\gamma_2 - \gamma_4)}{\sin\vartheta}. \qquad (4.7)$$

Substituting Eq. (4.7) first into the first equation of system (4.1), we have

$$T_1'' = \frac{\sin\vartheta}{\chi}\left(\gamma_1 z + \gamma_2\right)u. \qquad (4.8)$$

With Eq. (4.8) taken into account, the transformation of the third equation of system (4.1) gives

$$C_1'' = \frac{\sin\vartheta}{\chi d}\left[\left(-\alpha\gamma_1(\chi + d) + \gamma_3\chi\right)z + \left(-\alpha\gamma_2(\chi + d) + \gamma_4\chi\right)\right]u. \qquad (4.9)$$

Integrating twice Eqs. (4.8) and (4.9), we arrive at the following solution for the longitudinal (horizontal) gradients T_1, C_1:

$$T_1 = \frac{\sin\vartheta}{12\chi}\left(c_1\gamma_1 z^4 + 2\left(c_1\gamma_2 + c_2\gamma_1\right)z^3 + 6c_2\gamma_2 z^2 + \gamma_5 z + \gamma_6\right),$$

$$C_1 = \frac{\sin\vartheta}{12\chi d}\left[c_1\left(-\alpha\gamma_1(\chi + d) + \gamma_3\chi\right)z^4 + \right.$$
$$+2\left[c_1\left(-\alpha\gamma_2(\chi + d) + \gamma_4\chi\right) + c_2\left(-\alpha\gamma_1(\chi + d) + \gamma_3\chi\right)\right]z^3 +$$
$$\left. +6c_2\left(-\alpha\gamma_2(\chi + d) + \gamma_4\chi\right)z^2 + \gamma_7 z + \gamma_8\right]. \qquad (4.10)$$

Substituting Eq. (4.10) into the linear relations (4.7), we find the form of the remaining gradients of the temperature and concentration fields. Note that, as distinct from the above-discussed case $\sin\vartheta = 0$, all the components T_1, T_2, C_1, C_2 of the representation expressed by Eq. (2.2) are fourth-order polynomials. Then, from the obtained polynomials, the form of the exact solution for the gradients P_1 and P_2 of the pressure field is determined by single integration of Eq. (2.6).

Let us now turn to the integration of subsystem (2.7) enabling us to find the form of the homogeneous components U and V of the velocity field. The substitution of Eq. (2.7) for the spatial accelerations u_1, u_2, v_1 and v_2 into Eq. (3.6) gives the following system of equations:

$$\begin{aligned} Uu\cos\vartheta\sin\vartheta + Vu\cos^2\vartheta &= -P_1 + \nu U'', \\ -Uu\sin^2\vartheta - Vu\cos\vartheta\sin\vartheta &= -P_2 + \nu V''. \end{aligned} \tag{4.11}$$

We multiply the first equation in system (4.11) by $\sin\vartheta$ and the second one by $\cos\vartheta$; summing the resulting relations, we obtain the equation

$$U'' = -\mathrm{ctg}\vartheta V'' + \frac{\sin\vartheta P_1 + \cos\vartheta P_2}{\nu\sin\vartheta}. \tag{4.12}$$

By a double integration of Eq. (4.12) taking into account the expressions obtained for the gradients P_1, P_2, we arrive at a relation between the components U and V of the velocity field represented by Eq. (2.2). Note that the resulting dependence has a linear form and that the coefficients at the velocities U and V in it are constant. The obtained exact solution is then substituted into the second equation of system (4.11) thus yielding an equation for the velocity V. Solving this equation in view of the above-mentioned linear relationship between the components U and V, we can find an exact solution for the velocity U.

In what follows, we find solutions to Eq. (2.8), which gives the form of the background components of the temperature field T and the concentration field C. Similarly to solving system (4.2), (4.3), to find a solution to subsystem (2.8),we represent it as an inhomogeneous matrix algebraic equation with respect to the second derivatives of the background components C_0 and T_0 of the linear forms in Eq. (2.2),

$$\begin{pmatrix} 0 & \chi \\ d & \alpha d \end{pmatrix} \begin{pmatrix} C_0'' \\ T_0'' \end{pmatrix} = \begin{pmatrix} UT_1 + VT_2 \\ UC_1 + VC_2 \end{pmatrix}. \tag{4.13}$$

Note that the determinant of the matrix of the coefficients in the left-hand side of system (4.13) is nonzero; therefore, Eq. (4.13) is solvable with respect to the derivatives of background temperature T_0 and background concentration C_0,

$$\begin{pmatrix} C_0'' \\ T_0'' \end{pmatrix} = \frac{1}{\chi d} \begin{pmatrix} -\alpha d & \chi \\ d & 0 \end{pmatrix} \begin{pmatrix} UT_1 + VT_2 \\ UC_1 + VC_2 \end{pmatrix}. \tag{4.14}$$

The right-hand side of Eq. (4.14) contains the known polynomial functions; consequently, Eq. (4.14) can be easily integrated.

A solution to subsystem (2.9) is found thereafter. The background pressure P_0 results from the integration of the already known polynomial function. Hence, the pressure P_0 itself is a polynomial. As before, the result of this integration can be easily found by the known inhomogeneity contained in the right-hand side of Eq. (2.9). These solutions are rather lengthy and omitted here. □

To summarize the foregoing, it is worth noting that the algorithm of solving the overdetermined system (2.3)-(2.9) is reducible to the integration of the polynomial functions. Such integration is known to be always feasible. Consequently, if a solution of the form of Eq. (3.6) is obtained, the solution of the other subsystems of the overdetermined system under study does exist.

5. Result Interpretation

The obtained exact solutions seem to be of great value for theoretical and applied research. Without constraining the generality of the reasoning, we illustrate the novelty of the solution class by studying the velocity field. To be specific, we will consider Marangoni convection in an infinite horizontal fluid layer, which, for a steady-state flow, is determined by the boundary conditions

$$\eta V_x{}'\left(x,y,h\right) = -\sigma_1\frac{\partial T}{\partial x} - \sigma_2\frac{\partial C}{\partial x}, \quad \eta V_y{}'\left(x,y,h\right) = -\sigma_1\frac{\partial T}{\partial y} - \sigma_2\frac{\partial C}{\partial y},$$

$$T\left(x,y,0\right) = Ay, \quad T\left(x,y,h\right) = \vartheta + Bx,$$

$$C\left(x,y,0\right) = 0, \quad C\left(x,y,h\right) = Dx + Ey,$$

$$P\left(x,y,h\right) = 0, \quad V_x\left(x,y,h\right) = W\cos\alpha + \Omega y, \quad V_y\left(x,y,h\right) = W\sin\alpha.$$

Here, η is the dynamic viscosity of the fluid. The quantity h determines the thickness of the extensive horizontal fluid layer under study; A, B, D, E, W, α, ϑ, Ω are the parameters specified at the fluid layer boundaries.

The deduction of these conditions is similar to the relations obtained in [2; 8]. The analysis of the velocity field shows that specific kinetic energy can have up to six zero values (Fig. 1).

This testifies to the presence of countercurrents and a cellular nature of the flow caused by the inhomogeneous velocity distribution, the interaction of the pressure, temperature, and concentration fields and their effect on fluid velocity. The obtained stratification of the velocity field is much more pronounced than that reported in [2; 8]. It was shown in [2; 8] that the number of stagnation points (zeros of kinetic energy) does not exceed four.

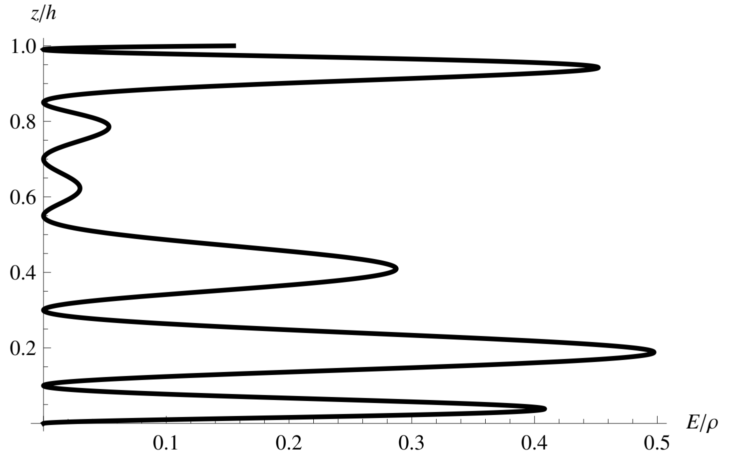

Figure 1. Specific kinetic energy profile $\frac{E}{\rho} = \frac{1}{2}\left(V_x^2 + V_y^2\right)$

6. Conclusion

The paper has studied the overdetermined system of nonlinear partial equations that describes a steady-state shear flow of a binary viscous incompressible fluid. The study has shown the existence of a nontrivial exact solution of the reduced Navier–Stokes equations in the Boussinesq approximation (the Oberbeck–Boussinesq system), supplemented by the incompressibility, heat conduction, and concentration equations. The found polynomial exact solution satisfying the algebraic consistency condition belongs to the Lin—Sidorov—Aristov class. This solution describes convection and advection (horizontal convection) in fluid moving beyond the Coriolis field and in the rotating equatorial zone of the World Ocean in the approximation of the f-plane. The formulas presented in the paper describe the stratification of hydrodynamic fields from the reference values. Of special interest is the study of the velocity field, which illustrates the existence of several stagnation points and countercurrents in fluid flow, particularly in sea water.

References

1. Aristov S.N. *Eddy currents in thin liquid layers. Dr. Sci. Diss.*. Vladivostok, 1990, 303 p. (in Russian)
2. Aristov S.N., Prosviryakov E.Y. On laminar flows of planar free convection. *Rus. J. Nonlin. Dyn.*, 2013, vol. 9, no. 4, pp. 651–657. (in Russian)
3. Aristov S.N., Prosviryakov E.Y. A new class of exact solutions for three-dimensional thermal diffusion equations. *Theoretical Foundations of Chemical Engineering*, 2016, vol. 50, no. 3, pp. 286-293. https://doi.org/10.1134/S0040579516030027

4. Bekezhanova V.B., Goncharova O.N. On approaches to solving the problem of an interface deformation in a two-layer system with evaporation. *Izvestiya of Altai State University*, 2018, no. 1 (99), pp. 69-74. https://doi.org/10.14258/izvasu(2018)1-12 (in Russian)
5. Birikh R.V. Thermocapillary convection in a horizontal layer of liquid. *J. Appl. Mech. Tech. Phys.*, 1966, no. 7, pp. 43-44.
6. Boussinesq J. *Theorie analitique de la chaleur*. Paris, GauthierVillars, 1903, vol. 2, 625 p.
7. Burmasheva N.V., Prosviryakov E.Yu. Convective layered flows of a vertically whirling viscous incompressible fluid. Velocity field investigation. *Journal of Samara State Technical University, Ser. Physical and Mathematical Sciences*, 2019, vol. 23, no. 2, pp. 341–360. https://doi.org/10.14498/vsgtu1670
8. Burmasheva N.V., Prosviryakov E.Yu. Thermocapillary convection of a vertical swirling liquid. *Theoretical Foundations of Chemical Engineering*, 2020, vol. 54, no. 1, pp. 230-239. https://doi.org/10.1134/S0040579519060034
9. Burmasheva N.V., Prosviryakov E.Yu. Exact solution of Navier—Stokes equations describing spatially inhomogeneous flows of a rotating fluid. *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2020, vol. 26, no. 2, pp. 79–87. https://doi.org/10.21538/0134-4889-2020-26-2-79-87 (in Russian)
10. Burmasheva N.V., Prosviryakov E.Yu. A class of exact solutions for two–dimensional equations of geophysical hydrodynamics with two Coriolis parameters. *The Bulletin of Irkutsk State University. Series Mathematics*, 2020 vol. 32, pp. 33-48. https://doi.org/10.26516/1997-7670.2020.32.33 (in Russian)
11. Burmasheva N.V., Prosviryakov E.Yu. On Marangoni shear convective flows of inhomogeneous viscous incompressible fluids in view of the Soret effect. *Journal of King Saud University – Science*, 2020, vol. 32, iss. 8, pp. 3364-3371. https://doi.org/10.1016/j.jksus.2020.09.023
12. Chandra Reddy P., Raju M.C., Raju G.S.S. MHD natural convective heat generation/absorbing and radiating fluid past a vertical plate embedded in porous medium–an exact solution. *Journal of the Serbian Society for Computational Mechanics*, 2018, vol. 12, no. 2, pp.106-127. https://doi.org/10.24874/jsscm.2018.12.02.08
13. Dufour L. Ueber die diffusion der gase durch poröse wände und die sie begleitenden temperaturveränderungen. *Arc. Phys. Nat. Sci. Geneve*, 1872, vol. 45, pp. 490-492. https://doi.org/10.1002/andp.18732240311
14. Gershuni G.Z., Zhukovitskii E.M. *Convective stability of incompressible fluids*. Jerusalem, Keter Publications/Wiley, 1976, 330 p.
15. Goncharova O.N., Rezanova E.V., Lyulin Y.V., Kabov O.A. Modeling of two-layer liquid-gas flow with account for evaporation. *Thermophysics and Aeromechanics*, 2015, vol. 22, no. 5, pp. 631-637. https://doi.org/10.1134/S086986431505011X
16. Lavrenteva O. M., Holenberg Y., Nir A. Marangoni and natural convection in a horizontal layer of viscoplastic fluid with concentration dependent yield stress. Exact analytical solutions. *Microgravity Sci. Technol.,* 2009, vol. 21, pp. 59–65. https://doi.org/10.1007/s12217-009-9127-7
17. Lin C.C. Note on a class of exact solutions in magneto-hydrodynamics. *Archive for Rational Mechanics and Analysis*, 1958, vol. 1, pp. 391-395.
18. Oberbeck A. Uber die warmeleitung der flussigkeiten bei der berucksichtigung der stromungen infolge von temperaturdifferenzen. *Annal. Phys. Chem.*, 1879, bd. 7, no. 6, pp. 271-292.
19. Ostroumov G.A. *Free convection under the condition of the internal problem*. Washington, NACA Technical Memorandum 1407, National Advisory Committee for Aeronautics, 1958.

20. Sidorov A.F. Two classes of solutions of the fluid and gas mechanics equations and their connection to traveling wave theory. *Journal of Applied Mechanics and Technical Physics*, 1989, vol. 30, no. 2, pp. 197-203. https://doi.org/10.1007/BF00852164
21. Shefer I.A. Influence of the transverse temperature drop on the stability of two-layer fluid flows with evaporation. *Fluid Dynamics*, 2019, vol. 54, no. 5, pp. 603-613. https://doi.org/10.1134/S0015462819040098
22. Soret C. Sur l'état d'équilibre que prend au point de vue de sa concentration une dissolution saline primitivement homohéne dont deux parties sont portees a des températures différentes. *Arch. Sci. Phys. Nat.*, 1879, vol. 2, pp. 48-61.
23. Umavathi J.C., Sheremet M.A., Patil S.L. Soret effects on the mixed convection flow using Robin boundary conditions. *Heat Transfer-Asian Research*, 2019. https://doi.org/10.1002/htj.21604

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Точные решения уравнений Обербека – Буссинеска для сдвиговых течений вязкой бинарной жидкости с учетом эффекта Соре

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Аннотация. Рассмотрено точное решение уравнений термодиффузии вязкой несжимаемой жидкости в приближении Буссинеска при пренебрежении эффектом Дюфора для установившегося сдвигового течения. Обозначено, что редуцированная система определяющих соотношений является нелинейной и переопределенной. Нетривиальное точное решение данной системы ищется в классе Линя – Сидорова – Аристова. Полученное семейство точных решений позволяет описывать установившиеся сдвиговые неоднородные течения. Данный класс позволяет обобщить классические решения Куэтта, Пуазейля и Остроумова – Бириха. Показано, что редуцированная в рамках этого класса система обыкновенных дифференциальных уравнений сохраняет свойства нелинейности и переопределенности. Доказана теорема об условиях разрешимости переопределенной системы, и показано, что при их выполнении решение единственно. Решение переопределенной системы возможно благодаря алгебраическому тождеству, связывающее горизонтальные градиенты скоростей, которые являются линейными функциями от вертикальной координаты.

Конструктивное доказательство вычисления гидродинамических полей заключается в последовательном интегрировании полиномов, причем степень полиномов зависит от значений краевых параметров.

Ключевые слова: вязкая бинарная жидкость, точное решение, эффект Соре, сдвиговое течение, переопределенная система.

Список литературы

1. Аристов С. Н. Вихревые течения в тонких слоях жидкости : автореф. дис. ... д-ра физ.-мат. наук : 01.02.05. Владивосток, 1990. 303 с.
2. Аристов С. Н., Просвиряков Е. Ю. О слоистых течениях плоской свободной конвекции // Нелинейная динамика. 2013. Т. 9, № 4. С. 651–657.
3. Аристов С. Н., Просвиряков Е. Ю. Новый класс точных решений трехмерных уравнений термодиффузии // Теоретические основы химической технологии. 2016. Т. 50, № 3. С. 294–301. https://doi.org/10.7868/S0040357116030027
4. Бекежанова В. Б., Гончарова О. Н. О подходах к решению задачи о деформации межфазной границы в двухслойной системе с испарением // Известия Алтайского государственного университета. 2018. № 1 (99). С. 69–74. https://doi.org/10.14258/izvasu(2018)1-12
5. Бирих Р. В. О термокапиллярной конвекции в горизонтальном слое жидкости // Прикладная механика и техническая физика. 1966. № 3. С. 69–72.
6. Boussinesq J. Theorie analitique de la chaleur. Vol. 2. Paris : GauthierVillars, 1903. 625 p.
7. Burmasheva N. V., Prosviryakov E. Yu. Convective layered flows of a vertically whirling viscous incompressible fluid. Velocity field investigation // Вестник Самарского государственного технического университета. Серия: Физико-математические науки. 2019. Т. 23, № 2. С. 341–360. https://doi.org/10.14498/vsgtu1670
8. Бурмашева Н. В., Просвиряков Е. Ю. Термокапиллярная конвекция вертикально завихренной жидкости // Теоретические основы химической технологии. 2020. Т. 54, № 1. С. 114–124. https://doi.org/10.31857/S0040357119060034
9. Бурмашева Н. В., Просвиряков Е. Ю. Точное решение уравнений Навье – Стокса, описывающее пространственно неоднородные течения вращающейся жидкости // Труды Института математики и механики УрО РАН. 2020. Т. 26, № 2. С. 79–87. https://doi.org/10.21538/0134-4889-2020-26-2-79-87
10. Бурмашева Н. В., Просвиряков Е. Ю. Класс точных решений для двумерных уравнений геофизической гидродинамики с двумя параметрами Кориолиса // Известия Иркутского государственного университета. Серия Математика. 2020. Т. 32. С. 33–48. https://doi.org/10.26516/1997-7670.2020.32.33
11. Burmasheva N. V., Prosviryakov E. Yu. On Marangoni shear convective flows of inhomogeneous viscous incompressible fluids in view of the Soret effect // Journal of King Saud University — Science. 2020. Vol. 32, Iss. 8. P. 3364–3371. https://doi.org/10.1016/j.jksus.2020.09.023
12. Chandra Reddy P., Raju M. C., Raju G. S. S. MHD natural convective heat generation/absorbing and radiating fluid past a vertical plate embedded in porous medium–an exact solution // Journal of the Serbian Society for Computational Mechanics. 2018. Vol. 12, N 2. P. 106–127. https://doi.org/10.24874/jsscm.2018.12.02.08

13. Dufour L. Ueber die diffusion der gase durch poröse wände und die sie begleitenden temperaturveränderungen // Arc. Phys. Nat. Sci. Geneve. 1872. Vol. 45. P. 490–492. https://doi.org/10.1002/andp.18732240311
14. Гершуни Г. З., Жуховицкий Е. М. Конвективная устойчивость несжимаемой жидкости. М. : Наука, 1972.
15. Гончарова О. Н., Резанова Е. В., Люлин Ю. В., Кабов О. А. Моделирование двухслойных течений жидкости и газа с учетом испарения // Теплофизика и аэромеханика. 2015. Т. 22, № 5. С. 655–661.
16. Lavrenteva O. M., Holenberg Y., Nir A. Marangoni and natural convection in a horizontal layer of viscoplastic fluid with concentration dependent yield stress. Exact analytical solutions // Microgravity Sci. Technol. 2009. Vol. 21. P. 59–65. https://doi.org/10.1007/s12217-009-9127-7
17. Lin C. C. Note on a class of exact solutions in magneto-hydrodynamics // Archive for Rational Mechanics and Analysis. 1958. Vol. 1. P. 391–395.
18. Oberbeck A. Uber die warmeleitung der flussigkeiten bei der berucksichtigung der stromungen infolge von temperaturdifferenzen// Annal. Phys. Chem. 1879. Bd. 7, N 6. S. 271–292.
19. Остроумов Г. А. Свободная конвекция в условиях внутренней задачи. М. : Гос. изд-во техн.-теорет. лит., 1952. 256 с.
20. Сидоров А. Ф. О двух классах решений уравнений механики жидкости и газа и их связи с теорией бегущих волн// Прикладная механика и техническая физика. 1989. Т. 2. С. 34–40.
21. Шефер И. А. Влияние поперечного перепада температур на устойчивость двухслойных течений жидкости с испарением // Известия Российской академии наук. Механика жидкости и газа. 2019. № 5. С. 15–25. https://doi.org/10.1134/S0568528119040091
22. Soret C. Sur l'état d'èquilibre que prend au point de vue de sa concentration une dissolution saline primitivement homohéne dont deux parties sont portées a des températures différentes // Arch. Sci. Phys. Nat. 1879. Vol. 2. P. 48–61.
23. Umavathi J. C., Sheremet M. A., Patil S. L. Soret effects on the mixed convection flow using Robin boundary conditions // Heat Transfer-Asian Research. 2019. https://doi.org/10.1002/htj.21604

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