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Notes on Meromorphic Functions with Positive Coefficients Involving Polylogarithm Function

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Abstract. In this paper, we introduce and study a new subclass of meromorphic functions with positive coefficients involving the polylogarithm function and obtain coefficient estimates, growth and distortion theorem, radius of convexity, integral transforms, convex linear combinations and convolution properties for the class $\sigma_{c,p}(\alpha, \lambda)$.

Keywords: Meromorphic, Polylogarithm, Coefficient estimates.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m z^m, \tag{1.1}$$

which are analytic in the punctured open unit disk

$$U^* := \{z : z \in C,\ 0 < |z| < 1\} = U \setminus \{0\}. \tag{1.2}$$

A function f in Σ is said to be meromorphically starlike of order δ if and only if

$$\Re\Big\{-\frac{zf'(z)}{f(z)}\Big\} > \delta;\ (z \in U^*), \tag{1.3}$$

for some δ $(0 \leq \delta < 1)$. We denote by $\Sigma(\delta)$ the class of all meromorphically starlike order δ. Furthermore, a function f in Σ is said to be meromorphically convex of order δ if and only if

$$\Re\Big\{ -\Big(1 + \frac{zf''(z)}{f'(z)}\Big) \Big\} > \delta; \ (z \in U^*), \tag{1.4}$$

for some δ, $(0 \leq \delta < 1)$. We denote by $\Sigma_k(\delta)$ the class of all meromorphically convex order δ. For functions $f \in \Sigma$ given by (1.1) and $g \in \Sigma$

$$g(z) = \frac{1}{z} + \sum_{m=0}^{\infty} b_m z^m, \tag{1.5}$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m b_m z^m. \tag{1.6}$$

Let Σ_p be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m z^m; \ a_m \geq 0 \tag{1.7}$$

which are analytic and univalent in U^*.

For $c \in N$, the set of natural numbers with $c \geq 2$, an absolutely convergent series defined as

$$Li_c(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^c} \tag{1.8}$$

is known as the polylogarithm. This class of functions was invented by Leibnitz [2]. For more works on polylogarithm and meromorphic function (see [4; 6]).

We consider a linear operator $\Omega_c f(z) : \Sigma \to \Sigma$ which is defined by the following Hadamard product (or Convolution):

$$\Omega_c f(z) = \phi_c(z) * f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} \frac{1}{(m+2)^c} a_m z^m, \tag{1.9}$$

where $\phi_c(z) = z^{-2} Li_c(z)$.

Next, we define the linear operator $\mathfrak{D}_c f(z) : \Sigma \to \Sigma$ as follows:

$$\mathfrak{D}_c f(z) = \big\{\Omega_c f(z) - \frac{1}{2^c} a_0\big\} = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{1}{(m+2)^c} a_m z^m. \tag{1.10}$$

For function f in the class Σ_p, we define a linear operator $\mathfrak{D}^n_{c,\ \lambda} f(z)$ as follows

$$\begin{aligned}
\mathfrak{D}^0_{c,\ \lambda} f(z) &= f(z) \\
\mathfrak{D}^1_{c,\ \lambda} f(z) &= (1-\lambda) f(z) + \lambda \frac{(z^2 f(z))'}{z} \ \lambda \geq 0 \\
&= (1-\lambda) f(z) + \lambda z f'(z) = \mathfrak{D}_{c,\ \lambda} f(z) \\
\mathfrak{D}^2_{c,\ \lambda} f(z) &= \mathfrak{D}_{c,\ \lambda} f(z)(\mathfrak{D}^1_{c,\ \lambda} f(z)) \\
&\ \ \vdots
\end{aligned}$$

$$\begin{aligned}
\mathfrak{D}^n_{c,\ \lambda} f(z) &= \mathfrak{D}_{c,\ \lambda} f(z)(\mathfrak{D}^{n-1}_{c,\ \lambda} f(z)) \\
&= \frac{1}{z} + \sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n}{(m+2)^c} a_m z^m \text{ for } n = 1, 2, \cdots
\end{aligned} \tag{1.11}$$

Now, by making use of operator $\mathfrak{D}^n_{c,\ \lambda} f(z)$, we define a new subclass of functions in Σ_p as follows.

Definition 1. *For* $-1 \leq \alpha < 1$ *and* $\lambda \geq 0$ *we let* $\sigma_{c,p}(\alpha, \lambda)$ *be the subclass of* Σ_p *consisting of functions of the form (1.7) and satisfying the analytic criterion*

$$-Re\left\{ \frac{z(\mathfrak{D}^n_{c,\ \lambda} f(z))'}{\mathfrak{D}^n_{c,\ \lambda} f(z)} + \alpha \right\} > \left| \frac{z(\mathfrak{D}^n_{c,\ \lambda} f(z))'}{\mathfrak{D}^n_{c,\ \lambda} f(z)} + 1 \right| \tag{1.12}$$

$\mathfrak{D}^n_{c,\ \lambda} f(z)$ *is given by* (1.11).

The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, growth and distortion properties, radius of convexity, convex linear combination and convolution properties, and integral operators for the class $\sigma_{c,p}(\alpha, \lambda)$.

2. Coefficient inequality

Theorem 1. *A function* f *of the form* (1.7) *is in* $\sigma_{c,p}(\alpha, \lambda)$ *if*

$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n [2m+3-\alpha]}{(m+2)^c} |a_m| \leq 1 - \alpha, \tag{2.1}$$

$-1 \leq \alpha < 1$ *and* $\lambda \geq 0$.

Proof. It is sufficient to show that

$$\left|\frac{z(\mathfrak{D}^n_{c,\ \lambda}f(z))'}{\mathfrak{D}^n_{c,\ \lambda}f(z)}+1\right|+Re\left\{\frac{z(\mathfrak{D}^n_{c,\ \lambda}f(z))'}{\mathfrak{D}^n_{c,\ \lambda}f(z)}+1\right\}\leq 1-\alpha.$$

We have

$$\left|\frac{z(\mathfrak{D}^n_{c,\ \lambda}f(z))'}{\mathfrak{D}^n_{c,\ \lambda}f(z)}+1\right|+Re\left\{\frac{z(\mathfrak{D}^n_{c,\ \lambda}f(z))'}{\mathfrak{D}^n_{c,\ \lambda}f(z)}+1\right\}\leq$$

$$\leq 2\left|\frac{z(\mathfrak{D}^n_{c,\ \lambda}f(z))'}{\mathfrak{D}_c f(z)}+1\right|\leq \frac{2\sum\limits_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n}{(m+2)^c}(m+1)|a_m||z|^m}{\frac{1}{|z|}-\sum\limits_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n}{(m+2)^c}|a_m||z|^m}$$

Letting $z\to 1$ along the real axis, we obtain

$$\leq \frac{2\sum\limits_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n}{(m+2)^c}(m+1)|a_m|}{1-\sum\limits_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n}{(m+2)^c}|a_m|}.$$

The last expression is bounded by $(1-\alpha)$ if

$$\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[2m+3-\alpha]}{(m+2)^c}|a_m|\leq 1-\alpha.$$

Hence the theorem is proved. □

Corollary 1. *Let the function* f *defined by* (1.7) *be in the class* $\sigma_{c,p}(\alpha,\lambda)$. *Then*

$$a_m\leq \sum_{m=1}^{\infty}\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[2m+3-\alpha]},\ (m\geq 1) \tag{2.2}$$

Equality holds for the functions of the form

$$f_m(z)=\frac{1}{z}+\sum_{m=1}^{\infty}\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[2m+3-\alpha]}z^m. \tag{2.3}$$

3. Distortion Theorems

Theorem 2. *Let the function* f *defined by* (1.7) *be in the class* $\sigma_{c,p}(\alpha,\lambda)$. *Then for* $0<|z|=r<1$,

$$\frac{1}{r}-\frac{3^c(1-\alpha)}{[1+2\lambda]^n(5-\alpha)}r\leq|f(z)|\leq\frac{1}{r}+\frac{3^c(1-\alpha)}{[1+2\lambda]^n(5-\alpha)}r \tag{3.1}$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(5-\alpha)} z. \tag{3.2}$$

Proof. Suppose f is in $\sigma_{c,p}(\alpha, \lambda)$. In view of Theorem 1, we have

$$\frac{[1+2\lambda]^n(5-\alpha)}{3^c} \sum_{m=1}^{\infty} a_m \leq \sum_{m=1}^{\infty} \frac{[2m+3-\alpha][1+\lambda(m+1)]^n}{(m+2)^c} \leq (1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty} a_m \leq \frac{3^c(1-\alpha)}{[1+2\lambda]^n(5-\alpha)}.$$

Consequently, we obtain

$$|f(z)| = \left| \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \right| \leq |\frac{1}{z}| + \sum_{m=1}^{\infty} a_m |z|^m \leq$$
$$\leq \frac{1}{r} + r \sum_{m=1}^{\infty} a_m \leq \frac{1}{r} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(5-\alpha)} r.$$

Also

$$|f(z)| = \left| \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \right| \geq |\frac{1}{z}| - \sum_{m=1}^{\infty} a_m |z|^m \geq$$
$$\geq \frac{1}{r} - r \sum_{m=1}^{\infty} a_m \geq \frac{1}{r} - \frac{3^c(1-\alpha)}{[1+2\lambda]^n(5-\alpha)} r.$$

Hence the results (3.1) follow. □

Theorem 3. *Let the function* f *defined by* (1.7) *be in the class* $\sigma_{c,p}(\alpha, \lambda)$. *Then for* $0 < |z| = r < 1$,

$$\frac{1}{r^2} - \frac{3^c(1-\alpha)}{[1+2\lambda]^n(5-\alpha)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(5-\alpha)}.$$

The result is sharp, the extremal function being of the form (2.3).

Proof. From Theorem 1, we have

$$\frac{[1+2\lambda]^n(5-\alpha)}{3^c} \sum_{m=1}^{\infty} m a_m \leq \sum_{m=1}^{\infty} \frac{[2m+3-\alpha][1+\lambda(m+1)]^n}{(m+2)^c} \leq (1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty} ma_m \leq \frac{3^c(1-\alpha)}{[1+2\lambda]^n(5-\alpha)}$$

Consequently, we obtain

$$|f(z)| = \frac{1}{r^2} + \sum_{m=1}^{\infty} ma_m r^{m-1} \leq \frac{1}{r^2} + \sum_{m=1}^{\infty} ma_m \leq$$
$$\leq \frac{1}{r^2} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(5-\alpha)}.$$

Also

$$|f(z)| \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} ma_m r^{m-1} \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} ma_m \geq$$
$$\geq \frac{1}{r^2} - \frac{3^c(1-\alpha)}{[1+2\lambda]^n(5-\alpha)}.$$

This completes the proof. □

4. Class Preserving Integral Operators

In this section we consider the class preserving integral operators of the form (1.7).

Theorem 4. *Let the function f be defined by* (1.7) *be in the class* $\sigma_{c,p}(\alpha,\lambda)$. *Then*

$$F(z) = \mu z^{-\mu-1} \int_0^z t^{\mu} f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{\mu}{\mu+m+1} a_m z^m, \ \mu > 0 \tag{4.1}$$

belongs to the class $\sigma[\delta(\alpha,\lambda,m,\mu)]$, *where*

$$\delta(\alpha,\lambda,m,\mu) = \frac{[1+2\lambda]^n(5-\alpha)(\mu+2) - 3^c\mu(1-\alpha)}{[1+2\lambda]^n(5-\alpha)(\mu+2) + 3^c\mu(1-\alpha)}. \tag{4.2}$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(5-\alpha)} z$.

Proof. Suppose $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ is in $\sigma_{c,p}(\alpha,\lambda)$.

We have

$$F(z) = \mu z^{-\mu-1} \int_0^z t^{\mu} f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{\mu}{\mu+m+1} a_m z^m, \ \mu > 0$$

It is sufficient to show that

$$\sum_{m=1}^{\infty} \frac{m+\delta}{1-\delta} \frac{\mu a_m}{m+\mu+1} \leq 1. \tag{4.3}$$

Since $f(z)$ is in $\sigma_{c,p}(\alpha,\lambda)$, we have

$$\sum_{m=1}^{\infty} \frac{[2m+3-\alpha][1+\lambda(m+1)]^n}{(m+2)^c(1-\alpha)} |a_m| \leq 1. \tag{4.4}$$

Thus (4.3) will be satisfied if

$$\frac{(m+\delta)\mu}{(1-\delta)(m+\mu+1)} \leq \frac{[2m+3-\alpha][1+\lambda(m+1)]^n}{(m+2)^c(1-\alpha)}, \quad \text{for each } m$$

or

$$\delta \leq \frac{[1+\lambda(m+1)]^n[2m+3-\alpha](\mu+m+1) - m\mu(1-\alpha)(m+2)^c}{[1+\lambda(m+1)]^n[2m+3-\alpha](\mu+m+1) + m\mu(1-\alpha)(m+2)^c} \tag{4.5}$$

$$G(m) = \frac{[1+\lambda(m+1)]^n[2m+3-\alpha](\mu+m+1) - m\mu(1-\alpha)(m+2)^c}{[1+\lambda(m+1)]^n[2m+3-\alpha](\mu+m+1) + m\mu(1-\alpha)(m+2)^c}.$$

Then $G(m+1) - G(m) > 0$, for each m. Hence $G(m)$ is increasing function of m. Since

$$G(1) = \frac{[1+2\lambda]^n(5-\alpha)(\mu+2) - 3^c\mu(1-\alpha)}{[1+2\lambda]^n(5-\alpha)(\mu+2) + 3^c\mu(1-\alpha)}.$$

The result follows. □

5. Convex Linear Combinations and Convolution Properties

Theorem 5. *If the function f is in $\sigma_{c,p}(\alpha,\lambda)$ then $f(z)$ is meromorphically convex of order δ $(0 \leq \delta < 1)$ in $|z| < r = r(\alpha,\lambda,\delta)$ where*

$$r(\alpha,\lambda,\delta) = \inf_{n\geq 1} \left\{ \frac{[1+\lambda(m+1)]^n(1-\delta)[2m+3-\alpha]}{(m+2)^c(1-\alpha)m(m+2-\delta)} \right\}^{\frac{1}{m+1}}$$

The result is sharp.

Proof. Let $f(z)$ is in $\sigma_{c,p}(\alpha,\lambda)$. Then by Theorem 1, we have

$$\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[2m+3-\alpha]}{(m+2)^c}|a_m|\leq(1-\alpha). \tag{5.1}$$

It is sufficient to show that $\left|2+\frac{zf''(z)}{f'(z)}\right|\leq 1-\delta$, for $|z|<r=r(\alpha,\lambda,\delta)$, where $r(\alpha,\lambda,\delta)$ is specified in the statement of the theorem. Then

$$\left|2+\frac{zf''(z)}{f'(z)}\right|=\left|\frac{\sum\limits_{m=1}^{\infty}m(m+1)a_mz^{m-1}}{\frac{-1}{z^2}+\sum\limits_{m=1}^{\infty}ma_mz^{m-1}}\right|\leq\sum_{m=1}^{\infty}\frac{m(m+1)a_m|z|^{m+1}}{1-\sum\limits_{m=1}^{\infty}ma_m|z|^{m+1}}.$$

This will be bounded by $(1-\delta)$ if

$$\sum_{m=1}^{\infty}\frac{m(m+2-\delta)}{1-\delta}a_m|z|^{m+1}\leq 1. \tag{5.2}$$

By (5.1), it follow that (5.2) is true if

$$\frac{m(m+2-\delta)}{1-\delta}|z|^{m+1}\leq\frac{[1+\lambda(m+1)]^n[2m+3-\alpha]}{(m+2)^c(1-\alpha)},\ m\geq 1$$

or

$$|z|\leq\left\{\frac{[1+\lambda(m+1)]^n(1-\delta)[2m+3-\alpha]}{(m+2)^c(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}. \tag{5.3}$$

Setting $|z|=r(\alpha,\lambda,\delta)$ in (5.3), the result follows.

The result is sharp for the function

$$f_m(z)=\frac{1}{z}+\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[2m+3-\alpha]}z^m,\ (m\geq 1).$$

□

Theorem 6. *Let* $f_0(z)=\frac{1}{z}$ *and*

$$f_m(z)=\frac{1}{z}+\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[2m+3-\alpha]}z^m,\ (m\geq 1).$$

Then $f(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty}a_mz^m$ *is in the class* $\sigma_{c,p}(\alpha,\lambda)$ *if and only if it can be expressed in the form* $f(z)=\lambda_0f_0(z)+\sum\limits_{m=1}^{\infty}\lambda_mf_m(z)$, *where* $\lambda_0\geq 0,\lambda_m\geq 0\ (m\geq 1)$ *and* $\lambda_0+\sum\limits_{m=1}^{\infty}\lambda_m=1$.

Proof. Let $f(z)=\lambda_0 f_0(z)+\sum\limits_{m=1}^{\infty}\lambda_m f_m(z)$ with $\lambda_0\geq 0, \lambda_m\geq 0\ (m\geq 1)$ and

$$\lambda_0+\sum_{m=1}^{\infty}\lambda_m=1.$$

Then

$$f(z)=\lambda_0 f_0(z)+\sum_{m=1}^{\infty}\lambda_m f_m(z)=$$
$$=\frac{1}{z}+\sum_{m=1}^{\infty}\lambda_m\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[2m+3-\alpha]}z^m.$$

Since

$$\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[2m+3-\alpha]}{(m+2)^c(1-\alpha)}\lambda_m\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[2m+3-\alpha]}=$$
$$=\sum_{m=1}^{\infty}\lambda_m=1-\lambda_0\leq 1.$$

By Theorem 1, f is in the class $\sigma_{c,p}(\alpha,\lambda)$.

Conversely suppose that the function f is in the class $\sigma_{c,p}(\alpha,\lambda)$, since

$$a_m\leq\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[2m+3-\alpha]},\ (m\geq 1)$$
$$\lambda_m=\frac{[1+\lambda(m+1)]^n[2m+3-\alpha]}{(m+2)^c(1-\alpha)}a_m,$$

and $\lambda_0=1-\sum\limits_{m=1}^{\infty}\lambda_m$, it follows that $f(z)=\lambda_0 f_0(z)+\sum\limits_{m=1}^{\infty}\lambda_m f_m(z)$.

This completes the proof of the theorem.

For the functions $f(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty}a_m z^m$ and $g(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty}b_m z^m$ belongs to $\sum\limits_{p}$ we denote by $(f*g)(z)$ the convolution of $f(z)$ and $g(z)$ or

$$(f*g)(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_m b_m z^m.$$

□

Theorem 7. *If the functions* $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ *and* $g(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} b_m z^m$ *are in the class* $\sigma_{c,p}(\alpha,\lambda)$, *then*

$$(f * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m$$

is in the class $\sigma_{c,p}(\alpha,\lambda)$.

Proof. Suppose $f(z)$ and $g(z)$ are in $\sigma_{c,p}(\alpha,\lambda)$. By Theorem 1, we have

$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[2m+3-\alpha]}{(m+2)^c(1-\alpha)} a_m \leq 1$$
$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[2m+3-\alpha]}{(m+2)^c(1-\alpha)} b_m \leq 1.$$

Since $f(z)$ and $g(z)$ are regular are in E, so is $(f*g)(z)$. Furthermore,

$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[2m+3-\alpha]}{(m+2)^c(1-\alpha)} a_m b_m \leq$$
$$\leq \left\{\frac{[1+\lambda(m+1)]^n[2m+3-\alpha]}{(m+2)^c(1-\alpha)}\right\}^2 a_m b_m \leq$$
$$\leq \left(\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[2m+3-\alpha]}{(m+2)^c(1-\alpha)} a_m\right) \cdot$$
$$\cdot \left(\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[2m+3-\alpha]}{(m+2)^c(1-\alpha)} b_m\right) \leq 1.$$

Hence by Theorem 1, $(f*g)(z)$ is in the class $\sigma_{c,p}(\alpha,\lambda)$. □

6. Neighborhoods for the class $\sigma_{c,p}(\alpha,\lambda)$

Neighborhoods for the class $\sigma_{c,p}(\alpha,\lambda)$ which we define as follows:

Definition 2. *A function* $f \in \sum\limits_{p}$ *is said to in the class* $\sigma_{c,p}(\alpha,\lambda,\gamma)$ *if there exists a function* $g \in \sigma_{c,p}(\alpha,\lambda)$ *such that*

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\gamma, \ z \in U, \ (0 \leq \gamma < 1). \tag{6.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [1] and Ruschweyh [5], we define the $\delta-$neighborhhod of a function $f \in \sum_p$ by

$$N_\delta(f) := \Big\{g \in \sum_p : \ g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m : \ \sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta\Big\}. \quad (6.2)$$

Theorem 8. *If $g \in \sigma_{c,p}(\alpha, \lambda)$ and*

$$\gamma = 1 - \frac{\delta(5-\alpha)(1+2\lambda)}{(1+2\lambda)(5-\alpha) - 3^c(1-\alpha)}. \quad (6.3)$$

Then $N_\delta(g) \subset \sigma_{c,p}(\alpha, \lambda, \gamma)$.

Proof. Let $f \in N_\delta(g)$. Then we find from (6.2) that

$$\sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta \quad (6.4)$$

which implies the coefficient inequality

$$\sum_{m=1}^{\infty} |a_m - b_m| \leq \delta, \ (m \in N). \quad (6.5)$$

Since $g \in \sigma_{c,p}(\alpha, \lambda)$, we have $\sum\limits_{m=1}^{\infty} b_m < \frac{3^c(1-\alpha)}{(1+2\lambda)^n(5-\alpha)}$. So that

$$\Big|\frac{f(z)}{g(z)} - 1\Big| \leq \frac{\sum\limits_{m=1}^{\infty} |a_m - b_m|}{1 - \sum\limits_{m=1}^{\infty} b_m} \leq \frac{\delta(1+2\lambda)(5-\alpha)}{(1+2\lambda)(5-\alpha) - 3^c(1-\alpha)} = 1 - \gamma$$

provided γ is given by (6.3). Hence, by definition $f \in \sigma_{c,p}(\alpha, \lambda, \gamma)$ for γ given by (6.3), which completes the proof. □

Remark 1. For $\lambda = 0$ in the results mentioned in all the sections above the class are the same as those of Venkateswarlu et al. [6].

7. Conclusion

This research has introduced a new linear operator related to polylogarithm function and studied some basic properties of geometric function theory. Accordingly, some results related to coefficient estimates, growth and distortion properties, closure theorems and neighborhoods have also been considered, inviting future research for this field of study.

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Заметки о мероморфных функциях с положительными коэффициентами, связанных с полилогарифмической функцией

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Аннотация. В статье вводится и изучается новый подкласс мероморфных функций с положительными коэффициентами, связанных с функцией полилогарифма. Также получены оценки для коэффициентов, теорема роста и искажения, радиус выпуклости, интегральные преобразования, выпуклые линейные комбинации и свойства свертки для класса $\sigma_{c,p}(\alpha, \lambda)$.

Ключевые слова: мероморфный, полилогарифм, оценки коэффициентов.

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