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Generations of generative classes *

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Abstract. We study generating sets of diagrams for generative classes. The generative classes appeared solving a series of model-theoretic problems. They are divided into semantic and syntactic ones. The fists ones are witnessed by well-known Fraïssé constructions and Hrushovski constructions. Syntactic generative classes and syntactic generic constructions were introduced by the author. They allow to consider any ω-homogeneous structure as a generic limit of diagrams over finite sets. Therefore any elementary theory is represented by some their generic models. Moreover, an information written by diagrams is realized in these models.

We consider generic constructions both in general case and with some natural restrictions, in particular, with the self-sufficiency property. We study the dominating relation and domination-equivalence for generative classes. These relations allow to characterize the finiteness of generic structure reducing the construction of generic structures to maximal diagrams. We also have that a generic structure is finite if and only if given generative class is finitely generated, i.e., all diagrams of this class are reduced to copying of some finite set of diagrams.

It is shown that a generative class without maximal diagrams is countably generated, i.e., reduced to some at most countable set of diagrams if and only if there is a countable generic structure. And the uncountable generation is equivalent to the absence of generic structures or to the existence only uncountable generative structures.

Keywords: generative class, generic structure, generation of generative class.

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1. Introduction

The notion of generative class was introduced in [9] and used in [10; 11] solving a series of complicated model-theoretic problems. This notion produces syntactic generic constructions which naturally generalize semantic ones including well-known Fraïssé constructions [2–4] and Hrushovski constructions [1; 5–7].

In the present paper we continue to study structural properties of generative classes and related objects [8; 12–14]. In Section 2, we consider a series of notions and results on generative class including the notions of domination and domination-equivalence for generative classes. In Section 3, we introduce the notions of finite, countable, and uncountable generations for generative classes and characterize these syntactic properties in terms of generic structures being semantic objects.

2. Preliminaries

We consider collections of sentence and formulas in first order logic over a language Σ. Thus, as usual, $\vdash$ means proof from no hypotheses deducing $\vdash \varphi$ for a formula φ of language Σ, which may contain function symbols and constants. If deducing φ, hypotheses in a set Φ of formulas can be used, we write $\Phi \vdash \varphi$. Usually Σ will be fixed in context and not mentioned explicitly.

Below we write $X, Y, Z, \ldots$ for finite sets of variables, and denote by $A, B, C, \ldots$ finite sets of elements, as well as finite sets in structures, or else the structures with finite universes themselves.

In diagrams, $A, B, C, \ldots$ denote finite sets of constant symbols disjoint from the constant symbols in Σ and $\Sigma(A)$ is the vocabulary with the constants from A adjoined. $\Phi(A), \Psi(B), \mathrm{X}(C)$ stand for Σ-*diagrams* (of sets A, B, C), that is, *consistent sets of* $\Sigma(A)$-, $\Sigma(B)$-, $\Sigma(C)$-*sentences*, respectively.

Below we assume that for any considered diagram $\Phi(A)$, if a_1, a_2 are distinct elements in A then $\neg(a_1 \approx a_2) \in \Phi(A)$. This means that if c is a constant symbol in Σ, then there is at most one element $a \in A$ such that $(a \approx c) \in \Phi(A)$.

If $\Phi(A)$ is a diagram and B is a set, we denote by $\Phi(A)|_B$ the set $\{\varphi(\bar{a}) \in \Phi(A) \mid \bar{a} \in B\}$. Similarly, for a language Σ, we denote by $\Phi(A)|_\Sigma$ the restriction of $\Phi(A)$ to the set of formulas in the language Σ.

Definition 1. [8–14]. We denote by $[\Phi(A)]^A_B$ the diagram $\Phi(B)$ obtained by replacing a subset $A' \subseteq A$ by a set $B' \subseteq B$ of constants disjoint from Σ and with $|A'| = |B'|$, where $A \setminus A' = B \setminus B'$. Similarly we call the consistent set of formulas denoted by $[\Phi(A)]^A_X$ the type $\Phi(X)$ if it is the result of a bijective substitution into $\Phi(A)$ of variables of X for the constants in A. In this case, we say that $\Phi(B)$ is a *copy* of $\Phi(A)$ and a *representative* of $\Phi(X)$. We also denote the diagram $\Phi(A)$ by $[\Phi(X)]^X_A$.

Remark 1. If the vocabulary contains functional symbols then diagrams $\Phi(A)$ containing equalities and inequalities of terms can generate both finite and infinite structures. The same effect is observed for purely predicate vocabularies if it is written in $\Phi(A)$ that the model for $\Phi(A)$ should be infinite. For instance, diagrams containing axioms for finitely axiomatizable theories have this property.

By the definition, for any diagram $\Phi(A)$, each constant symbol in Σ appears in some formula of $\Phi(A)$. Thus, $\Phi(A)$ can be considered as $\Phi(A \cup K)$, where K is the set of constant symbols in Σ.

We now give conditions on a partial ordering of a collection of diagrams which suffice for it to determine a structure. We modify some of the conditions for structures by d to signify they are conditions on diagrams not structures.

Definition 2. [8–14]. Let Σ be a vocabulary. We say that $(\mathbf{D}_0; \leqslant)$ (or $\mathbf{D}_0$) is *generic*, or *generative*, if $\mathbf{D}_0$ is a class of Σ-diagrams of finite sets so that $\mathbf{D}_0$ is partially ordered by a binary relation $\leqslant$ such that $\leqslant$ is preserved by bijective substitutions, i. e., if $\Phi(A) \leqslant \Psi(B)$, and $A' \subseteq B'$ such that $[\Phi(A)]^A_{A'} = \Phi(A')$ and $[\Psi(B)]^B_{B'} = \Psi(B')$ are defined, then $[\Phi(A)]^A_{A'}$, $[\Psi(B)]^B_{B'}$ are in $\mathbf{D}_0$ and $[\Phi(A)]^A_{A'} \leqslant [\Psi(B)]^B_{B'}$.[1] Furthermore:

(i) if $\Phi(A) \in \mathbf{D}_0$ then for any quantifier free formula $\varphi(\bar{x})$ and any tuple $\bar{a} \in A$ either $\varphi(\bar{a}) \in \Phi(A)$ or $\neg\varphi(\bar{a}) \in \Phi(A)$;

(ii) if $\Phi \leqslant \Psi$ then $\Phi \subseteq \Psi$;[2]

(iii) if $\Phi \leqslant \mathrm{X}$, $\Psi \in \mathbf{D}_0$, and $\Phi \subseteq \Psi \subseteq \mathrm{X}$, then $\Phi \leqslant \Psi$;

(iv) some diagram $\Phi_0(\emptyset)$ is the least element of the system $(\mathbf{D}_0; \leqslant)$, and $\mathbf{D}_0 \setminus \{\Phi_0(\emptyset)\}$ is nonempty;

(v) (the *d-amalgamation property*) for any diagrams $\Phi(A)$, $\Psi(B)$, $\mathrm{X}(C) \in \mathbf{D}_0$, if there exist injections $f_0\colon A \to B$ and $g_0\colon A \to C$ with $[\Phi(A)]^A_{f_0(A)} \leqslant \Psi(B)$ and $[\Phi(A)]^A_{g_0(A)} \leqslant \mathrm{X}(C)$, then there are a diagram $\Theta(D) \in \mathbf{D}_0$ and injections $f_1\colon B \to D$ and $g_1\colon C \to D$ for which $[\Psi(B)]^B_{f_1(B)} \leqslant \Theta(D)$, $[\mathrm{X}(C)]^C_{g_1(C)} \leqslant \Theta(D)$ and $f_0 \circ f_1 = g_0 \circ g_1$; the diagram $\Theta(D)$ is called the *amalgam* of $\Psi(B)$ and $\mathrm{X}(C)$ over the diagram $\Phi(A)$ and witnessed by the four maps (f_0, g_0, f_1, g_1);

[1] Note that $\mathbf{D}_0$ is closed under bijective substitutions since $\leqslant$ is preserved by bijective substitutions and $\leqslant$ is reflexive.

[2] Note that $\Phi(A) \leqslant \Psi(B)$ implies $A \subseteq B$, since if $a \in A$ then $(a \approx a) \in \Phi(A)$, so $\Phi(A) \leqslant \Psi(B)$ implies $\Phi(A) \subseteq \Psi(B)$ and we have $(a \approx a) \in \Psi(B)$, whence $a \in B$.

(vi) (the *local realizability property*) if $\Phi(A) \in \mathbf{D}_0$ and $\Phi(A) \vdash \exists x\, \varphi(x)$, then there are a diagram $\Psi(B) \in \mathbf{D}_0$, $\Phi(A) \leqslant \Psi(B)$, and an element $b \in B$ for which $\Psi(B) \vdash \varphi(b)$;

(vii) (the *d-uniqueness property*) for any diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$ if $A \subseteq B$ and the set $\Phi(A) \cup \Psi(B)$ is consistent then $\Phi(A) = \{\varphi(\bar{b}) \in \Psi(B) \mid \bar{b} \in A\}$.

A diagram Φ is called a *strong subdiagram* of a diagram Ψ if $\Phi \leqslant \Psi$.

A diagram $\Phi(A)$ is said to be (*strongly*) *embeddable* in a diagram $\Psi(B)$ if there is an injection $f\colon A \to B$ such that $[\Phi(A)]^A_{f(A)} \subseteq \Psi(B)$ ($[\Phi(A)]^A_{f(A)} \leqslant \Psi(B)$). The injection f, in this instance, is called a (*strong*) *embedding* of diagram $\Phi(A)$ in diagram $\Psi(B)$ and is denoted by $f\colon \Phi(A) \to \Psi(B)$. A diagram $\Phi(A)$ is said to be (*strongly*) *embeddable* in a structure $\mathcal{M}$ if $\Phi(A)$ is (strongly) embeddable in some diagram $\Psi(B)$, where $\mathcal{M} \models \Psi(B)$. The corresponding embedding $f\colon \Phi(A) \to \Psi(B)$, in this case, is called a (*strong*) *embedding* of diagram $\Phi(A)$ in structure $\mathcal{M}$ and is denoted by $f\colon \Phi(A) \to \mathcal{M}$.

Let $\mathbf{D}_0$ be a class of diagrams, $\mathbf{P}_0$ be a class of structures of some language, and $\mathcal{M}$ be a structure in $\mathbf{P}_0$. The class $\mathbf{D}_0$ is *cofinal* in the structure $\mathcal{M}$ if for each finite set $A \subseteq M$, there are a finite set B, $A \subseteq B \subseteq M$, and a diagram $\Phi(B) \in \mathbf{D}_0$ such that $\mathcal{M} \models \Phi(B)$. The class $\mathbf{D}_0$ is *cofinal* in $\mathbf{P}_0$ if $\mathbf{D}_0$ is cofinal in every structure of $\mathbf{P}_0$. We denote by $\mathbf{K}(\mathbf{D}_0)$ the class of all structures $\mathcal{M}$ with the condition that $\mathbf{D}_0$ is cofinal in $\mathcal{M}$, and by $\mathbf{P}$ a subclass of $\mathbf{K}(\mathbf{D}_0)$ such that each diagram $\Phi \in \mathbf{D}_0$ is true in some structure in $\mathbf{P}$.

Now we extend the relation $\leqslant$ from the generative class $(\mathbf{D}_0; \leqslant)$ to a class of subsets of structures in the class $\mathbf{K}(\mathbf{D}_0)$.

Let $\mathcal{M}$ be a structure in $\mathbf{K}(\mathbf{D}_0)$, A and B be finite sets in $\mathcal{M}$ with $A \subseteq B$. We call A a *strong subset* of the set B (in the structure $\mathcal{M}$), and write $A \leqslant B$, if there exist diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$, for which $\Phi(A) \leqslant \Psi(B)$ and $\mathcal{M} \models \Psi(B)$.

A finite set A is called a *strong subset* of a set $M_0 \subseteq M$ (in the structure $\mathcal{M}$), where $A \subseteq M_0$, if $A \leqslant B$ for any finite set B such that $A \subseteq B \subseteq M_0$ and $\Phi(A) \subseteq \Psi(B)$ for some diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$ with $\mathcal{M} \models \Psi(B)$. If A is a strong subset of M_0 then, as above, we write $A \leqslant M_0$. If $A \leqslant M$ in $\mathcal{M}$ then we refer to A as a *self-sufficient set* (in $\mathcal{M}$).

Notice that, by the d-uniqueness property, the diagrams $\Phi(A)$ and $\Psi(B)$ specified in the definition of strong subsets are defined uniquely. A diagram $\Phi(A) \in \mathbf{D}_0$, corresponding to a self-sufficient set A in $\mathcal{M}$, is said to be a *self-sufficient diagram* (in $\mathcal{M}$).

Definition 3. [8–14]. A class $(\mathbf{D}_0; \leqslant)$ possesses the *joint embedding property* (JEP) if for any diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$, there is a diagram $\mathrm{X}(C) \in \mathbf{D}_0$ such that $\Phi(A)$ and $\Psi(B)$ are strongly embeddable in $\mathrm{X}(C)$.

Clearly, every generative class has JEP since JEP means the d-amalgamation property over the empty set.

Definition 4. [8–14]. A structure $\mathcal{M} \in \mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{D}_0; \leqslant)$, or is *finitely generated over* Σ, if any finite set $A \subseteq M$ is contained in some finite self-sufficient set in $\mathcal{M}$, i. e., there is a finite set B with $A \subseteq B \subseteq M$ and $\Psi(B) \in \mathbf{D}_0$ such that $\mathcal{M} \models \Psi(B)$ and $\Psi(B) \leqslant \mathrm{X}(C)$ for any $\mathrm{X}(C) \in \mathbf{D}_0$ with $\mathcal{M} \models \mathrm{X}(C)$ and $\Psi(B) \subseteq \mathrm{X}(C)$. A class $\mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{D}_0; \leqslant)$, or is *finitely generated over* Σ, if each structure in $\mathbf{P}$ has finite closures (with respect to $(\mathbf{D}_0; \leqslant)$).

Clearly, an at most countable structure $\mathcal{M}$ has finite closures with respect to $(\mathbf{D}_0; \leqslant)$ if and only if $M = \bigcup\limits_{i \in \omega} A_i$ for some self-sufficient sets A_i with $A_i \leqslant A_{i+1}$, $i \in \omega$.

Note that the finite closure property is defined modulo Σ and does not correlate with the cardinalities of algebraic closures. For instance, if Σ contains infinitely many constant symbols then $\mathrm{acl}(A)$ is always infinite whereas a finite set A can or can not be extended to a self-sufficient set.

Besides, for the finite closures of sets A we consider finite self-sufficient extensions B in a given structure $\mathcal{M}$ with respect to $(\mathbf{D}_0; \leqslant)$ only and B can be both a universe of a substructure of $\mathcal{M}$ or not. Moreover, it is permitted that corresponding diagrams $\Psi(B)$ can have only finite, finite and infinite, or only infinite models.

Thus, for instance, a finitely axiomatizable theory without finite models and with a generative class $(\mathbf{D}_0; \subseteq)$, containing diagrams for all finite sets and with axioms in diagrams, has identical finite closures whereas each diagram in $\mathbf{D}_0$ has only infinite models.

Definition 5. [8–14]. A structure $\mathcal{M} \in \mathbf{K}(\mathbf{D}_0)$ is $(\mathbf{D}_0; \leqslant)$-*generic*, or a *generic limit for the class* $(\mathbf{D}_0; \leqslant)$ and denoted by $\mathrm{glim}(\mathbf{D}_0; \leqslant)$, if it satisfies the following conditions:

(a) $\mathcal{M}$ has finite closures with respect to $\mathbf{D}_0$;

(b) if $A \subseteq M$ is a finite set, $\Phi(A), \Psi(B) \in \mathbf{D}_0$, $\mathcal{M} \models \Phi(A)$ and $\Phi(A) \leqslant \Psi(B)$, then there exists a set $B' \leqslant M$ such that $A \subseteq B'$ and $\mathcal{M} \models \Psi(B')$.

Clearly, uncountable $(\mathbf{D}_0; \leqslant)$-generic structures can be non-isomorphic. Indeed, for instance, all infinite structures in the empty language are generic for a given generative class although these structures are non-isomorphic for distinct cardinalities. But, as the following theorem shows, they are isomorphic for at most countable cases.

Theorem 1. [11]. *For any generative class* $(\mathbf{D}_0; \leqslant)$ *with at most countably many diagrams whose copies form* $\mathbf{D}_0$, *there exists at most countable* $(\mathbf{D}_0; \leqslant)$-*generic structure, unique up to isomorphism.*

Theorem 2. [8; 12; 14]. *Every ω-homogeneous structure $\mathcal{M}$ is $(\mathbf{D}_0;\leqslant)$-generic for some generative class $(\mathbf{D}_0;\leqslant)$.*

Thus any first-order theory has a generic model and therefore can be represented by it.

Definition 6. [8–14]. A generative class $(\mathbf{D}_0;\leqslant)$ is *self-sufficient* if the following *axiom of self-sufficiency* holds:

(viii) if $\Phi,\Psi,\mathrm{X}\in\mathbf{D}_0$, $\Phi\leqslant\Psi$, and $\mathrm{X}\subseteq\Psi$, then $\Phi\cap\mathrm{X}\leqslant\mathrm{X}$.

Note that in the proof of Theorem 2 the required generative class $(\mathbf{D}_0;\leqslant)$ is self-sufficient.

Theorem 3. [8–11; 14]. *Let $(\mathbf{D}_0;\leqslant)$ be a self-sufficient class, $\mathcal{M}$ be at most countable $(\mathbf{D}_0;\leqslant)$-generic structure, and $\mathbf{K}$ be the class of all models of $T=\mathrm{Th}(\mathcal{M})$ which has finite closures. Then the generic structure $\mathcal{M}$ is homogeneous.*

Thus, since any ω-homogeneous structure can be considered as generic with respect to a generic class with complete diagrams, a countable structure $\mathcal{M}$ is homogeneous if and only if it is generic for an appropriate self-sufficient generative class $(\mathbf{D}_0;\leqslant)$.

Definition 7. [9–11]. Let $(\mathbf{D}_0;\leqslant)$ and $(\mathbf{D}'_0;\leqslant')$ be generative classes of languages Σ and Σ', respectively, with $\Sigma\subseteq\Sigma'$. We say that the class $(\mathbf{D}'_0;\leqslant')$ *dominates* the class $(\mathbf{D}_0;\leqslant)$, and write $\mathbf{D}_0\trianglelefteq\mathbf{D}'_0$, if for any diagram $\Phi(A)\in\mathbf{D}_0$ there is a diagram $\Phi'(A')\in\mathbf{D}'_0$ such that $\Phi(A)\subseteq\Phi'(A')$, and the condition of there being some systems, which are extensions over A, together with available information on interrelations of elements in these extensions written in the diagram $\Phi(A)$, implies that the same extensions exist over A, and that similar information is available on interrelations of elements in those extensions written in the diagram $\Phi'(A')$.

If $\mathbf{D}_0\trianglelefteq\mathbf{D}'_0$ and $\mathbf{D}'_0\trianglelefteq\mathbf{D}_0$ we say that generative classes $(\mathbf{D}_0;\leqslant)$ and $(\mathbf{D}'_0;\leqslant')$ are *domination-equivalent* and write $\mathbf{D}_0\sim\mathbf{D}'_0$.

Theorem 4. [9–11]. *Let $\mathcal{M}$ and $\mathcal{M}'$ be countable homogeneous structures of languages Σ and Σ', respectively. The following conditions are equivalent:*

(1) *the structure $\mathcal{M}$ is isomorphically embeddable in the structure $\mathcal{M}'\restriction\Sigma$;*

(2) *there are generative classes $(\mathbf{D}_0;\leqslant)$ and $(\mathbf{D}'_0;\leqslant')$ such that $\mathcal{M}$ is $(\mathbf{D}_0;\leqslant)$-generic, $\mathcal{M}'$ is $(\mathbf{D}'_0;\leqslant')$-generic, and $\mathbf{D}_0\trianglelefteq\mathbf{D}'_0$.*

Since mutually embeddable countable homogeneous structures are isomorphic, Theorem 4 implies the following corollary.

Corollary 1. *Let $\mathcal{M}$ and $\mathcal{M}'$ be countable homogeneous structures of a language Σ. The following conditions are equivalent:*

(1) *the structures* $\mathcal{M}$ *and* $\mathcal{M}'$ *are isomorphic;*

(2) *there are domination-equivalent generative classes* $(\mathbf{D}_0;\leqslant)$ *and* $(\mathbf{D}'_0;\leqslant')$ *such that* $\mathcal{M}$ *is* $(\mathbf{D}_0;\leqslant)$*-generic and* $\mathcal{M}'$ *is* $(\mathbf{D}'_0;\leqslant')$*-generic;*

(3) *the structures* $\mathcal{M}$ *and* $\mathcal{M}'$ *are* $(\mathbf{D}_0;\leqslant)$*-generic for some generative class* $(\mathbf{D}_0;\leqslant)$*.*

Proof. (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1). Having the hypothesis we see by Theorem 4 that $\mathcal{M}$ and $\mathcal{M}'$ are mutually embeddable. Since $\mathcal{M}$ and $\mathcal{M}'$ are countable homogeneous, they are isomorphic. $\square$

3. Finite, countable and uncountable generations of generative classes

Definition 8. [13]. Let $(\mathbf{D}_0;\leqslant)$ be a generative class in a language Σ, $\Phi(A)\in\mathbf{D}_0$. The diagram $\Phi(A)$ is called *structural* if it satisfies the following modification of the local realizability property: if $\Phi(A)\vdash\exists x\,\varphi(x)$ then there is a constant term $t(\bar a)$, $\bar a\in A$, such that $\Phi(A)\vdash\varphi(t(\bar a))$.

Note that there exist generative classes containing non-structural diagrams. Indeed, consider a finitely axiomatizable, by an axiom φ_0, complete theory T of relational language and without finite models (for instance, consider the theory of dense linear order without endpoints). Now, take a generative class $(\mathbf{D}_0;\leqslant)$ for T such that all diagrams in $\mathbf{D}_0$ contain φ_0. Obviously, $\mathbf{D}_0$ does not contain structural diagrams since for any $\Phi(A)\in\mathbf{D}_0$, every model $\mathcal{M}\models\Phi(A)$ is infinite, being a model of T, whereas constant terms $t(\bar a)$, $\bar a\in A$, can have values only in the finite set A.

Theorem 5. [13, Theorem 4.1]. *For any diagram* $\Phi(A)\in\mathbf{D}_0$*,* $A\neq\varnothing$*, the following conditions are equivalent:*

(1) $\Phi(A)$ *is structural;*

(2) *there exists a structure* $\mathcal{M}$ *consisting of some constant terms in the language* $\Sigma\cup A$ *and such that* $\mathcal{M}\models\Phi(A)$*.*

Proof. (1) $\Rightarrow$ (2). Let $\Phi(A)$ be structural. Denote by N the set of all constant terms $t(\bar a)$, $\bar a\in A$, in the language $\Sigma\cup A$. For constant terms t_1 and t_2, we put $t_1\sim t_2$ if and only if $\Phi(A)\vdash(t_1\approx t_2)$. Clearly, $\sim$ is an equivalence relation and there is a canonical structure $\mathcal{N}/{\sim}$ having the universe $N/{\sim}$ and satisfying the quantifier-free part of $\Phi(A)$. By (vi$'$) and induction on length of formulas in $\Phi(A)$ we get $\mathcal{N}/\sim\,\models\Phi(A)$. Taking representatives for each $\sim$-class in $N/{\sim}$ we form a required structure $\mathcal{M}$ isomorphic to $\mathcal{N}/{\sim}$.

(2) $\Rightarrow$ (1). If there exists a required structure $\mathcal{M}$ having the universe M consisting of some constant terms (one representative for each $\sim$-class)

in the language $\Sigma \cup A$ and such that $\mathcal{M} \models \Phi(A)$, then, by completeness of the first-order calculus, for any formula $\varphi(x)$ with $\Phi(A) \vdash \exists x\, \varphi(x)$, there is a term $t(\bar{a}) \in M$ such that $\Phi(A) \vdash \varphi(t(\bar{a}))$. Thus, $\Phi(A)$ is structural. $\square$

The structure $\mathcal{N}/\sim$ in the proof of Theorem 5 is called $\Phi(A)$-*canonical*, or simply *canonical* and denoted by $\mathcal{M}_{\Phi(A)}$. The structure $\mathcal{M}$, in the proof, is a *representation* of $\mathcal{M}_{\Phi(A)}$.

By Theorem 5, any structural diagram $\Phi(A)$, for $A \neq \varnothing$, defines the algebra with the universe $N/\sim$ being the restriction of $\mathcal{N}/\sim$ to the functional sublanguage and *finitely* generated by A (relative constant symbols in Σ). At the same time, for quantifier-free diagrams, this condition is sufficient:

Corollary 2. [13, Corollary 4.2]. *Any quantifier-free diagram $\Phi(A) \in \mathbf{D}_0$ is structural.*

Definition 9. [13]. A diagram $\Phi(A) \in \mathbf{D}_0$ is called *self-structural* if $A \neq \varnothing$ and $\Phi(A)$ satisfies the following: if $\Phi(A) \vdash \exists x\, \varphi(x)$ then there is an element $a \in A$ such that $\Phi(A) \vdash \varphi(a)$.

Theorem 6. [13, Theorem 25] *For a generative class $(\mathbf{D}_0; \leqslant)$ with a language having a finite set C of pairwise distinct constants, the following conditions are equivalent:*

(1) *the $(\mathbf{D}_0; \leqslant)$-generic structure is finite;*

(2) *$(\mathbf{D}_0; \leqslant)$ has maximal diagrams;*

(3) *$(\mathbf{D}_0; \leqslant)$ is domination-equivalent to a minimal generative class consisting of a diagram $\Phi_0(\varnothing)$ and of copies of a self-structural diagram $\Phi(A)$;*

(4) *the $(\mathbf{D}_0; \leqslant)$-generic structure is isomorphic, for a quantifier-free diagram $\Phi(A)$, to a representation, with the universe $A \cup C$, of $\Phi(A)$-canonical structure.*

Remark 2. In Theorem 6 the existence of finite C is implied by each of the conditions (1), (2), (3).

Definition 10. A generative class $(\mathbf{D}_0; \leqslant)$ is called λ-*generated*, where λ is a cardinality, if $\mathbf{D}_0$ contains a set Z of diagrams such that each diagram in $\mathbf{D}_0$ is a copy of some diagram in Z. The generative class $(\mathbf{D}_0; \leqslant)$ is called *finitely generated* if it is n-*generated* for some $n \in \omega$. The generative class $(\mathbf{D}_0; \leqslant)$ is called *countably generated* if it is ω-*generated*. If $(\mathbf{D}_0; \leqslant)$ is not countably generated it is called *uncountably generated*.

The following theorem extends the list of criteria for the existence of finite generic structures in Theorem 6:

Theorem 7. *For a generative class $(\mathbf{D}_0; \leqslant)$ the following conditions are equivalent:*

(1) *$(\mathbf{D}_0; \leqslant)$ is finitely generated;*

(2) *each diagram in $(\mathbf{D}_0; \leqslant)$ is extensible till a maximal one;*

(3) *$(\mathbf{D}_0; \leqslant)$ has a maximal diagram.*

Proof. (1) $\Rightarrow$ (2). If $(\mathbf{D}_0;\leqslant)$ is finitely generated by diagrams $\Phi_1,\ldots,\Phi_n$ then their amalgam is a copy of some Φ_i and it is maximal. Thus any diagram Φ being in the list $\Phi_1,\ldots,\Phi_n$ is extensible till a maximal one.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). If $(\mathbf{D}_0;\leqslant)$ has a maximal diagram $\Phi(A)$ then, since $\mathbf{D}_0$ is closed under amalgams, $\Phi(A)$ is a copy of amalgam of $\Phi(A)$ with arbitrary diagram $\Psi(A)\in\mathbf{D}_0$. Therefore, each diagram in $\mathbf{D}_0$ is a copy of a restriction $\Psi(A)|_B$ of $\Phi(A)$ to some $B\subseteq A$. As A is finite there are finitely many these restrictions. Thus, $(\mathbf{D}_0;\leqslant)$ is finitely generated. □

Theorem 8. *For any generative class $(\mathbf{D}_0;\leqslant)$ the following conditions are equivalent:*

(1) *there is a countable $(\mathbf{D}_0;\leqslant)$-generic structure (and the language has at most countable set C of pairwise distinct constants);*

(2) $(\mathbf{D}_0;\leqslant)$ *is countably generated and does not have maximal diagrams.*

Proof. (1) $\Rightarrow$ (2) holds by the definition of countable generic structure and Theorem 6 with Remark 2. In particular, the language has at most countable set of pairwise distinct constants.

(2) $\Rightarrow$ (1). If $(\mathbf{D}_0;\leqslant)$ is countably generated and does not have maximal diagrams then we construct step-by-step a countable generic structure $\mathcal{M}$ as in the proof of Theorem 1 producing at most countably many pairwise distinct constants. □

Theorem 8 immediately implies the following characterization for the uncountably generated generative classes $(\mathbf{D}_0;\leqslant)$.

Corollary 3. *For any generative class $(\mathbf{D}_0;\leqslant)$ the following conditions are equivalent:*

(1) $(\mathbf{D}_0;\leqslant)$ *is uncountably generated;*

(2) *the set C of pairwise distinct constants in the language is uncountable and/or all $(\mathbf{D}_0;\leqslant)$-generic structures are uncountable or they do not exist.*

Corollary 3 with the following theorem allows to divide (in terms of meeting of contradictions for the cardinalities of definable sets) the uncountable generation of $(\mathbf{D}_0;\leqslant)$ into two cases: with or without generic structures.

Theorem 9. [8]. *For any generative class $(\mathbf{D}_0;\leqslant)$ the following conditions are equivalent:*

(1) *there exists a $(\mathbf{D}_0;\leqslant)$-generic structure;*

(2) *there are no type-definable sets X constructed with respect to $(\mathbf{D}_0;\leqslant)$ such that these X meet contradictions for their cardinality;*

(3) *there are no definable sets X constructed with respect to $(\mathbf{D}_0;\leqslant)$ such that these X meet contradictions for their cardinality.*

In conclusion we note that by Theorems above generative classes can, on syntactic level, control existence of finite, countable, or uncountable generic structures, and their absence as well.

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С. В. Судоплатов

Порождения в генерирующих классах

Аннотация. В работе исследуются порождающие множества диаграмм для генерирующих классов. Сами генерирующие классы возникли при решении ряда теоретико-модельных проблем. Они подразделяются на семантические и синтаксические. К первым относятся широко известные конструкции Фраиссе и Хрушовского. Синтаксические генерирующие классы и синтаксические генерические конструкции были введены в работах автора. Они позволяют рассматривать любую ω-однородную структуру в виде генерического предела диаграмм над конечными множествами. Тем самым, любая элементарная теория представляется некоторыми своими генерическими моделями. При этом информация, заданная диаграммами, реализуется в этих моделях.

Мы рассматриваем генерические конструкции как в общем виде, так и при некоторых естественных ограничениях, в частности при выполнении свойства самодостаточности. Исследуется отношение доминирования и эквивалентности по доминированию для генерирующих классов. С помощью этого отношения характеризуется условие конечности генерической структуры, сводящее построение генерической структуры к использованию лишь максимальных диаграмм. Условие конечности генерической структуры также эквивалентно конечной порожденности генерирующего класса, т. е. сведению всех диаграмм данного класса к копированию некоторого конечного множества диаграмм.

Доказано, что счетная порожденность (сведение к некоторому, не более чем счетному множеству диаграмм) генерирующего класса без максимальных диаграмм равносильна существованию счетной генерической структуры, а несчетная порожденность — отсутствию генерических структур или наличию лишь несчетных генерических структур.

Ключевые слова: генерирующий класс, генерическая структура, порождение генерирующего класса.

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