Серия «Математика»
2018. Т. 24. С. 82—101

Онлайн-доступ к журналу:
http://mathizv.isu.ru

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УДК 510.67
MSC 03C30, 03C15, 03C50
DOI https://doi.org/10.26516/1997-7670.2018.24.82

Combinations of structures *

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Abstract. We investigate combinations of structures by families of structures relative to families of unary predicates and equivalence relations. Conditions preserving ω-categoricity and Ehrenfeuchtness under these combinations are characterized. The notions of e-spectra are introduced and possibilities for e-spectra are described.

It is shown that ω-categoricity for disjoint P-combinations means that there are finitely many indexes for new unary predicates and each structure in new unary predicate is either finite or ω-categorical. Similarly, the theory of E-combination is ω-categorical if and only if each given structure is either finite or ω-categorical and the set of indexes is either finite, or it is infinite and E_i-classes do not approximate infinitely many n-types for $n \in \omega$. The theory of disjoint P-combination is Ehrenfeucht if and only if the set of indexes is finite, each given structure is either finite, or ω-categorical, or Ehrenfeucht, and some given structure is Ehrenfeucht.

Variations of structures related to combinations and E-representability are considered.

We introduce e-spectra for P-combinations and E-combinations, and show that these e-spectra can have arbitrary cardinalities.

The property of Ehrenfeuchtness for E-combinations is characterized in terms of e-spectra.

Keywords: combination of structures, P-combination, E-combination, e-spectrum.

1. Introduction

The aim of the paper is to introduce operators (similar to [9;10;12;14]) on classes of structures producing structures approximating given structure, as well as to study properties of these operators. These operators are

* The research is partially supported by Russian Foundation for Basic Researches (Grant No. 17-01-00531) and by Committee of Science in Education and Science Ministry of the Republic of Kazakhstan (Grant No. AP05132546).

connected with natural topological properties related to families of theories [2–4; 7; 8].

In Section 2 we define P-operators, E-operators, and corresponding combinations of structures. In Section 3 we characterize the preservation of ω-categoricity for P-combinations and E-combinations as well as Ehrenfeuchtness for P-combinations. In Section 4 we pose and investigate questions on variations of structures under P-operators and E-operators. The notions of e-spectra for P-operators and E-operators are introduced in Section 5. Here values for e-spectra are described. In Section 6 the preservation of Ehrenfeuchtness for E-combinations is characterized.

Throughout the paper we consider structures of relational languages.

2. P-operators, E-operators, combinations

Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \rightleftharpoons \bigcup\limits_{i\in I} \mathcal{A}_i$ expanded by the predicates P_i is the *P-union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the *P-operator*. The structure $\mathcal{A}_P$ is called the *P-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

By the definition, without loss of generality we can assume for

$$\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$$

that all languages $\Sigma(\mathcal{A}_i)$ coincide interpreting new predicate symbols for $\mathcal{A}_i$ by empty relation.

Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}' \upharpoonright P_i) \upharpoonright \Sigma(\mathcal{A}_i)$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write $\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}' \upharpoonright \bigcap\limits_{i\in I} \overline{P_i}$, maybe applying Morleyzation. Moreover, we write $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$ for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

Notice that each structure $\mathcal{A}$ in a predicate language Σ can be represented as a P-combination. Indeed, taking formulas $\varphi_i(x)$, whose sets of solutions cover A, we can take φ_i-restrictions $\mathcal{A}_i$ of $\mathcal{A}$ with $P_i(x) \equiv \varphi_i(x)$. The P-combination of $\mathcal{A}_i$ restricted to Σ forms $\mathcal{A}$.

Clearly, if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$ [14]. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$,

$\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_P = \mathrm{Th}(\mathcal{A}_P)$, which is denoted by $\mathrm{Comb}_P(T_i)_{i\in I}$.

On the opposite side, if all P_i coincide then $P_i(x) \equiv (x \approx x)$ and removing the symbols P_i we get the restriction of $\mathcal{A}_P$ which is the combination of the structures $\mathcal{A}_i$ [10; 12].

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ being the *E-union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the *E-operator*. The structure $\mathcal{A}_E$ is also called the *E-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes.

If $\mathcal{A}_E \prec \mathcal{A}'$, the restriction $\mathcal{A}' \upharpoonright (A' \setminus A_E)$ is denoted by $\mathcal{A}'_\infty$. Clearly, $\mathcal{A}' = \mathcal{A}'_E \coprod \mathcal{A}'_\infty$, where $\mathcal{A}'_E = \mathrm{Comb}_E(\mathcal{A}'_i)_{i\in I}$, $\mathcal{A}'_i$ is a restriction of $\mathcal{A}'$ to its E-class containing the universe A_i, $i \in I$.

Considering an E-combination $\mathcal{A}_E$ we will identify E-classes A_i with structures $\mathcal{A}_i$.

Clearly, the nonempty structure $\mathcal{A}'_\infty$ exists if and only if I is infinite.

Notice that any E-operator can be interpreted as P-operator replacing or naming E-classes for $\mathcal{A}_i$ by unary predicates P_i. For infinite I, the difference between 'replacing' and 'naming' implies that $\mathcal{A}_\infty$ can have unique or unboundedly many E-classes returning to the E-operator.

Thus, for any E-combination $\mathcal{A}_E$, $\mathrm{Th}(\mathcal{A}_E) = \mathrm{Th}(\mathcal{A}'_E)$, where $\mathcal{A}'_E$ is obtained from $\mathcal{A}_E$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. In this case, similar to structures the E-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_E = \mathrm{Th}(\mathcal{A}_E)$, which is denoted by $\mathrm{Comb}_E(T_i)_{i\in I}$, by $\mathcal{T}_E$, or by $\mathrm{Comb}_E\mathcal{T}$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Note that P-combinations and E-unions can be interpreted by randomizations [1] of structures.

Sometimes we admit that combinations $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ and $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$ are expanded by new relations or old relations are extended by new tuples. In these cases the combinations will be denoted by $\mathrm{EComb}_P(\mathcal{A}_i)_{i\in I}$ and $\mathrm{EComb}_E(\mathcal{A}_i)_{i\in I}$, respectively.

3. ω-categoricity and Ehrenfeuchtness for combinations

Proposition 3.1. *If predicates P_i are pairwise disjoint, the languages $\Sigma(\mathcal{A}_i)$ are at most countable, $i \in I$, $|I| \le \omega$, and the structure $\mathcal{A}_P$ is infinite then the theory $\mathrm{Th}(\mathcal{A}_P)$ is ω-categorical if and only if I is finite and each structure $\mathcal{A}_i$ is either finite or ω-categorical.*

Proof. If I is infinite or there is an infinite structure $\mathcal{A}_i$ which is not ω-categorical then $T = \mathrm{Th}(\mathcal{A}_P)$ has infinitely many n-types, where $n = 1$ if $|I| \geq \omega$ and $n = n_0$ for $\mathrm{Th}(\mathcal{A}_i)$ with infinitely many n_0-types. Hence by Ryll-Nardzewski Theorem $\mathrm{Th}(\mathcal{A}_P)$ is not ω-categorical.

If $\mathrm{Th}(\mathcal{A}_P)$ is ω-categorical then by Ryll-Nardzewski Theorem having finitely many n-types for each $n \in \omega$, we have both finitely many predicates P_i and finitely many n-types for each P_i-restriction, i. e., for $\mathrm{Th}(\mathcal{A}_i)$. □

Notice that Proposition 3.1 is not true if a P-combination is not disjoint: taking, for instance, a graph $\mathcal{A}_1$ with a set P_1 of vertices and with infinitely many R_1-edges such that all vertices have degree 1, as well as taking a graph $\mathcal{A}_2$ with the same set P_1 of vertices and with infinitely many R_2-edges such that all vertices have degree 1, we can choose edges such that $R_1 \cap R_2 = \varnothing$, each vertex in P_1 has $(R_1 \cup R_2)$-degree 2, and alternating R_1- and R_2-edges there is an infinite sequence of $(R_1 \cup R_2)$-edges. Thus, $\mathcal{A}_1$ and $\mathcal{A}_2$ are ω-categorical whereas $\mathrm{Comb}(\mathcal{A}_1, \mathcal{A}_2)$ is not.

Note also that Proposition 3.1 does not hold replacing $\mathcal{A}_P$ by $\mathcal{A}_E$. Indeed, taking infinitely many infinite E-classes with structures of the empty languages we get an ω-categorical structure of the equivalence relation E. At the same time, Proposition 3.1 is preserved if there are finitely many E-classes. In general case $\mathcal{A}_E$ does not preserve the ω-categoricity if and only if E_i-classes *approximate* infinitely many n-types for some $n \in \omega$, i. e., there are infinitely many n-types $q_m(\bar{x})$, $m \in \omega$, such that for any $m \in \omega$, $\varphi_j(\bar{x}) \in q_j(\bar{x})$, $j \leq m$, and classes $E_{k_1}, \ldots, E_{k_m}$, all formulas $\varphi_j(\bar{x})$ have realizations in $A_E \setminus \bigcup\limits_{r=1}^{m} E_{k_r}$. Indeed, assuming that all $\mathcal{A}_i$ are ω-categorical we can lose the ω-categoricity for $\mathrm{Th}(\mathcal{A}_E)$ only having infinitely many n-types (for some n) inside $\mathcal{A}_\infty$. Since all n-types in $\mathcal{A}_\infty$ are locally (for any formulas in these types) realized in infinitely many $\mathcal{A}_i$, E_i-classes approximate infinitely many n-types and $\mathrm{Th}(\mathcal{A}_E)$ is not ω-categorical. Thus, we have the following

Proposition 3.2. *If the languages $\Sigma(\mathcal{A}_i)$ are at most countable, $i \in I$, $|I| \leq \omega$, and the structure $\mathcal{A}_E$ is infinite then the theory $\mathrm{Th}(\mathcal{A}_E)$ is ω-categorical if and only if each structure $\mathcal{A}_i$ is either finite or ω-categorical, and I is either finite, or infinite and E_i-classes do not approximate infinitely many n-types for any $n \in \omega$.*

As usual we denote by $I(T, \lambda)$ the number of pairwise non-isomorphic models of T having the cardinality λ.

Recall that a theory T is *Ehrenfeucht* if T has finitely many countable models ($I(T, \omega) < \omega$) but is not ω-categorical ($I(T, \omega) > 1$). A structure with an Ehrenfeucht theory is also *Ehrenfeucht*.

Theorem 3.3. *If predicates P_i are pairwise disjoint, the languages $\Sigma(\mathcal{A}_i)$ are at most countable, $i \in I$, and the structure $\mathcal{A}_P$ is infinite then* *the theory* $\mathrm{Th}(\mathcal{A}_P)$ *is Ehrenfeucht if and only if the following conditions hold:*

(a) I *is finite;*

(b) *each structure* $\mathcal{A}_i$ *is either finite, or* ω*-categorical, or Ehrenfeucht;*

(c) *some* $\mathcal{A}_i$ *is Ehrenfeucht.*

Proof. If I is finite, each structure $\mathcal{A}_i$ is either finite, or ω-categorical, or Ehrenfeucht, and some $\mathcal{A}_i$ is Ehrenfeucht then $T = \mathrm{Th}(\mathcal{A}_P)$ is Ehrenfeucht since each model of T is composed of disjoint models with universes P_i and

$$I(T,\omega) = \prod_{i\in I} I(\mathrm{Th}(\mathcal{A}_i), \min\{|A_i|, \omega\}). \tag{3.1}$$

Now if I is finite and all $\mathcal{A}_i$ are ω-categorical then by (3.1), $I(T,\omega) = 1$, and if some $I(\mathrm{Th}(\mathcal{A}_i), \omega) \geq \omega$ then again by (3.1), $I(T,\omega) \geq \omega$.

Assuming that $|I| \geq \omega$ we have to show that the non-ω-categorical theory T has infinitely many countable models. Assuming on contrary that $I(T,\omega) < \omega$, i. e., T is Ehrenfeucht, we have a nonisolated powerful type $q(\bar{x}) \in S(T)$ [5], i. e., a type such that any model of T realizing $q(\bar{x})$ realizes all types in $S(T)$. By the construction of disjoint union, $q(\bar{x})$ should have a realization of the type $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$. Moreover, if some $\mathrm{Th}(\mathcal{A}_i)$ is not ω-categorical for infinite A_i then $q(\bar{x})$ should contain a powerful type of $\mathrm{Th}(\mathcal{A}_i)$ and the restriction $r(\bar{y})$ of $q(\bar{x})$ to the coordinates realized by $p_\infty(x)$ should be powerful for the theory $\mathrm{Th}(\mathcal{A}_\infty)$, where $\mathcal{A}_\infty$ is infinite and saturated, as well as realizing $r(\bar{y})$ in a model $\mathcal{M} \models T$, all types with coordinates satisfying $p_\infty(x)$ should be realized in $\mathcal{M}$ too. As shown in [11;12], the type $r(\bar{y})$ has the local realizability property and satisfies the following conditions: for each formula $\varphi(\bar{y}) \in r(\bar{y})$, there exists a formula $\psi(\bar{y},\bar{z})$ of T (where $l(\bar{y}) = l(\bar{z})$), satisfying the following conditions:

(i) for each $\bar{a} \in r(M)$, the formula $\psi(\bar{a},\bar{y})$ is equivalent to a disjunction of principal formulas $\psi_i(\bar{a},\bar{y})$, $i \leq m$, such that $\psi_i(\bar{a},\bar{y}) \vdash r(\bar{y})$, and $\models \psi_i(\bar{a},\bar{b})$ implies, that $\bar{b}$ does not semi-isolate $\bar{a}$;

(ii) for every $\bar{a},\bar{b} \in r(M)$, there exists a tuple $\bar{c}$ such that $\models \varphi(\bar{c}) \wedge \psi(\bar{c},\bar{a}) \wedge \psi(\bar{c},\bar{b})$.

Since the type $p_\infty(x)$ is not isolated each formula $\varphi(\bar{y}) \in r(\bar{y})$ has realizations $\bar{d}$ in $\bigcup\limits_{i\in I} A_i$. On the other hand, as we consider the disjoint union of $\mathcal{A}_i$ and there are no non-trivial links between distinct P_i and $P_{i'}$, the sets of solutions for $\psi(\bar{d},\bar{y})$ with $\models \varphi(\bar{d})$ in $\{\neg P_i(x) \mid\models P_i(d_j) \text{ for some } d_j \in \bar{d}\}$ are either equal or empty being composed by definable sets without parameters. If these sets are nonempty the item (i) can not be satisfied: $\psi(\bar{a},\bar{y})$ is not equivalent to a disjunction of principal formulas. Otherwise all ψ-links for realizations of $r(\bar{y})$ are situated inside the set of solutions for $\bar{p}_\infty(\bar{y}) = \bigcup\limits_{y_j\in\bar{y}} p_\infty(y_j)$. In this case for $\bar{a} \models r(\bar{y})$ the formula $\exists\bar{z}(\psi(\bar{z},\bar{a}) \wedge \psi(\bar{z},\bar{y}))$ does not cover the set $r(M)$ since it does not cover each φ-approximation of $r(M)$. Thus, the property (ii) fails.

Hence, (i) and (ii) can not be satisfied, there are no powerful types, and the theory T is not Ehrenfeucht. □

4. Variations of structures related to combinations and E-representability

Clearly, for a disjoint P-combination $\mathcal{A}_P$ with infinite I, there is a structure $\mathcal{A}' \equiv \mathcal{A}_P$ with a structure $\mathcal{A}'_\infty$. Since the type $p_\infty(x)$ is non-isolated (omitted in $\mathcal{A}_P$), the cardinalities for $\mathcal{A}'_\infty$ are unbounded. Infinite structures $\mathcal{A}'_\infty$ are not necessary elementary equivalent and can be both elementary equivalent to some $\mathcal{A}_i$ or not. For instance, if infinitely many structures $\mathcal{A}_i$ contain unary predicates Q_0, say singletons, without unary predicates Q_1 and infinitely many $\mathcal{A}_{i'}$ for $i' \neq i$ contain Q_1, say again singletons, without Q_0 then $\mathcal{A}'_\infty$ can contain Q_0 without Q_1, Q_1 without Q_0, or both Q_0 and Q_1. For the latter case, $\mathcal{A}'_\infty$ is not elementary equivalent neither $\mathcal{A}_i$, nor $\mathcal{A}_{i'}$.

A natural question arises:

Question 1. *What can be the number of pairwise elementary non-equivalent structures $\mathcal{A}'_\infty$?*

Considering an E-combination $\mathcal{A}_E$ with infinite I, and all structures $\mathcal{A}' \equiv \mathcal{A}_E$, there are two possibilities: each non-empty *E-restriction* of $\mathcal{A}'_\infty$, i. e. a restriction to some E-class, is elementary equivalent to some $\mathcal{A}_i$, $i \in I$, or some E-restriction of $\mathcal{A}'_\infty$ is not elementary equivalent to all structures $\mathcal{A}_i$, $i \in I$.

Similarly Question 1 we have:

Question 2. *What can be the number of pairwise elementary non-equivalent E-restrictions of structures $\mathcal{A}'_\infty$?*

Example 4.1. Let $\mathcal{A}_P$ be a disjoint P-combination with infinite I and composed by infinite $\mathcal{A}_i$, $i \in I$, such that I is a disjoint union of infinite I_j, $j \in J$, where $\mathcal{A}_{i_j}$ contains only unary predicates and unique nonempty unary predicate Q_j being a singleton. Then $\mathcal{A}'_\infty$ can contain any singleton Q_j and finitely or infinitely many elements in $\bigcap\limits_{j\in J} \overline{Q}_j$. Thus, there are $2^{|J|} \cdot (\lambda + 1)$ non-isomorphic $\mathcal{A}'_\infty$, where λ is a least upper bound for cardinalities $\left|\bigcap\limits_{j\in J} \overline{Q}_j\right|$.

For $T = \mathrm{Th}(\mathcal{A}_P)$, we denote by $I_\infty(T, \lambda)$ the number of pairwise non-isomorphic structures $\mathcal{A}'_\infty$ having the cardinality λ.

Clearly, $I_\infty(T,\lambda)\le I(T,\lambda)$.

If structures $\mathcal{A}'_\infty$ exist and do not have links with $\mathcal{A}'_P$ (for instance, for a disjoint P-combination) then $I_\infty(T,\lambda)+1\le I(T,\lambda)$, since if models of T are isomorphic then their restrictions to $p_\infty(x)$ are isomorphic too, and $p_\infty(x)$ can be omitted producing $\mathcal{A}'_\infty=\varnothing$. Here $I_\infty(T,\lambda)+1=I(T,\lambda)$ if and only if all $I(\mathrm{Th}(\mathcal{A}_i),\lambda)=1$ and, moreover, for any $\left(\bigcup\limits_{i\in I}P_i\right)$-restrictions $\mathcal{B}_P,\mathcal{B}'_P$ of $\mathcal{B},\mathcal{B}'\models T$ respectively, where $|B|=|B'|=\lambda$, and their P_i-restrictions $\mathcal{B}_i$, $\mathcal{B}'_i$, there are isomorphisms $f_i\colon\mathcal{B}_i\simeq\mathcal{B}'_i$ preserving P_i and with an isomorphism $\bigcup\limits_{i\in I}f_i\colon\mathcal{B}_P\simeq\mathcal{B}'_P$.

The following example illustrates the equality $I_\infty(T,\lambda)+1=I(T,\lambda)$ with some $I(\mathrm{Th}(\mathcal{A}_i),\lambda)>1$.

Example 4.2. Let P_0 be a unary predicate containing a copy of the Ehrenfeucht example [13] with a dense linear order $\le$ and an increasing chain of singletons coding constants c_k, $k\in\omega$; P_n, $n\ge1$, be pairwise disjoint unary predicates disjoint to P_0 such that $P_1=(-\infty,c'_0)$ $P_{n+2}=[c'_n,c'_{n+1})$, $n\in\omega$, and $\bigcup\limits_{n\ge1}P_n$ forms a universe of prime model (over $\varnothing$) for another copy of the Ehrenfeucht example with a dense linear order $\le'$ and an increasing chain of constants c'_k, $k\in\omega$. Now we extend the language

$$\Sigma=\langle\le,\le',P_n,\{c_n\},\{c'_n\}\rangle_{n\in\omega}$$

by a bijection f between $P_0=\{a\mid a\le c_0\text{ or }c_0\le a\}$ and $\{a'\mid a'\le' c'_0\text{ or }c'_0\le' a'\}$ such that $a\le b\Leftrightarrow f(a)\le' f(b)$. The structures $\mathcal{A}'_\infty$ consist of realizations $p_\infty(x)$ which are bijective with realizations of the type $\{c_n<x\mid n\in\omega\}$.

For the theory T of the described structure $\mathrm{EComb}_P(\mathcal{A}_i)_{i\in I}$ we have $I(T,\omega)=3$ (as for the Ehrenfeucht example and the restriction of T to P_0) and $I_\infty(T,\omega)=2$ (witnessed by countable structures with least realizations of $p_\infty(x)$ and by countable structure with realizations of $p_\infty(x)$ all of which are not least).

For Example 4.1 of a theory T with singletons Q_j in $\mathcal{A}_i$ and for a cardinality $\lambda\ge1$, we have

$$I_\infty(T,\lambda)=\begin{cases}\sum\limits_{i=0}^{\min\{|J|,\lambda\}}C^i_{|J|}, & \text{if } J \text{ and } \lambda \text{ are finite;}\\ |J|, & \text{if } J \text{ is infinite and } |J|>\lambda;\\ 2^{|J|}, & \text{if } J \text{ is infinite and } |J|\le\lambda.\end{cases}$$

Clearly, $\mathcal{A}'\equiv\mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i\equiv\mathcal{A}_i$, $i\in I$. At the same time, there are E-combinations such that all $\mathcal{A}'\equiv\mathcal{A}_E$

can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the *E-representability.* If, for instance, all $\mathcal{A}_i$ are infinite structures of the empty language then any $\mathcal{A}' \equiv \mathcal{A}_E$ is an E-combination of some infinite structures $\mathcal{A}'_j$ of the empty language too.

Thus we have:

Question 3. *What is a characterization of E-representability for all $\mathcal{A}' \equiv \mathcal{A}_E$?*

Definition (cf. [6]). For a first-order formula $\varphi(x_1,\ldots,x_n)$, an equivalence relation E and a formula $\sigma(x)$ we define a (E,σ)-*relativized* formula $\varphi^{E,\sigma}$ by induction:

(i) if φ is an atomic formula then $\varphi^{E,\sigma} = \varphi(x_1,\ldots,x_n) \wedge \bigwedge\limits_{i,j=1}^{n} E(x_i,x_j) \wedge \exists y(E(x_1,y) \wedge \sigma(y))$;

(ii) if $\varphi = \psi\tau\chi$, where $\tau \in \{\wedge,\vee,\to\}$, and $\psi^{E,\sigma}$ and $\chi^{E,\sigma}$ are defined then $\varphi^{E,\sigma} = \psi^{E,\sigma}\tau\chi^{E,\sigma}$;

(iii) if $\varphi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$ and $\psi^{E,\sigma}(x_1,\ldots,x_n)$ is defined then $\varphi^{E,\sigma}(x_1,\ldots,x_n) = \neg\psi^{E,\sigma}(x_1,\ldots,x_n) \wedge \bigwedge\limits_{i,j=1}^{n} (E(x_i,x_j) \wedge \exists y(E(x_1,y) \wedge \sigma(y))$;

(iv) if $\varphi(x_1,\ldots,x_n) = \exists x\psi(x,x_1,\ldots,x_n)$ and $\psi^{E,\sigma}(x,x_1,\ldots,x_n)$ is defined then

$$\varphi^{E,\sigma}(x_1,\ldots,x_n) =$$

$$= \exists x\left(\bigwedge_{i=1}^{n}(E(x,x_i) \wedge \exists y(E(x,y) \wedge \sigma(y)) \wedge \psi^{E,\sigma}(x,x_1,\ldots,x_n)\right);$$

(v) if $\varphi(x_1,\ldots,x_n) = \forall x\psi(x,x_1,\ldots,x_n)$ and $\psi^{E,\sigma}(x,x_1,\ldots,x_n)$ is defined then

$$\varphi^{E,\sigma}(x_1,\ldots,x_n) =$$

$$= \forall x\left(\bigwedge_{i=1}^{n} E(x,x_i) \wedge \exists y(E(x,y) \wedge \sigma(y)) \to \psi^{E,\sigma}(x,x_1,\ldots,x_n)\right).$$

We write E instead of (E,σ) if $\sigma = (x \approx x)$.

Note that two E-classes E_i and E_j with structures $\mathcal{A}_i$ and $\mathcal{A}_j$ (of a language Σ), respectively, are not elementary equivalent if and only if there is a Σ-sentence φ such that $\mathcal{A}_E \restriction E_i \models \varphi^E$ (with $\mathcal{A}_i \models \varphi$) and $\mathcal{A}_E \restriction E_j \models (\neg\varphi)^E$ (with $\mathcal{A}_j \models \neg\varphi$). In this case, the formula φ is called (i,j)-*separating.*

The following properties are obvious:

(1) If φ is (i,j)-separating then $\neg\varphi$ is (j,i)-separating.

(2) If φ is (i,j)-separating and ψ is (i,k)-separating then $\varphi \wedge \psi$ is both (i,j)-separating and (i,k)-separating.

(3) There is a set Φ_i of (i,j)-separating sentences, for j in some $J \subseteq I \setminus \{i\}$, which separates $\mathcal{A}_i$ from all structures $\mathcal{A}_j \not\equiv \mathcal{A}_i$.

The set Φ_i is called *e-separating* (for $\mathcal{A}_i$) and $\mathcal{A}_i$ is *e-separable* (witnessed by Φ_i).

Assuming that some $\mathcal{A}' \equiv \mathcal{A}_E$ is not E-representable, we get an E'-class with a structure $\mathcal{B}$ in $\mathcal{A}'$ which is e-separable from all $\mathcal{A}_i$, $i \in I$, by a set Φ. It means that for some sentences φ_i with $\mathcal{A}_E \upharpoonright E_i \models \varphi_i^E$, i. e., $\mathcal{A}_i \models \varphi_i$, the sentences $\left(\bigwedge\limits_{i\in I_0} \neg\varphi_i\right)^E$, where $I_0 \subseteq_{\rm fin} I$, form a consistent set, satisfying the restriction of $\mathcal{A}'$ to the class E'_B with the universe B of $\mathcal{B}$.

Thus, answering Question 3 we have

Proposition 4.3. *For any E-combination $\mathcal{A}_E$ the following conditions are equivalent:*

(1) *there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable;*

(2) *there are sentences φ_i such that $\mathcal{A}_i \models \varphi_i$, $i \in I$, and the set of sentences $\left(\bigwedge\limits_{i\in I_0} \neg\varphi_i\right)^E$, where $I_0 \subseteq_{\rm fin} I$, is consistent with* $\mathrm{Th}(\mathcal{A}_E)$.

Proposition 4.3 implies

Corollary 4.4. *If $\mathcal{A}_E$ has only finitely many pairwise elementary non-equivalent E-classes then each $\mathcal{A}' \equiv \mathcal{A}_E$ is E-representable.*

5. e-spectra

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by e-$\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-}\mathrm{Sp}(\mathcal{A}')) \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$.

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

The following proposition is obvious:

Proposition 5.1. 1. *For a given language* Σ, $0 \leq e\text{-Sp}(\mathrm{Th}(\mathcal{A}_E)) \leq 2^{\max\{|\Sigma|,\omega\}}$.

2. *A structure* $\mathcal{A}_E$ *is e-largest if and only if* $e\text{-Sp}(\mathcal{A}_E) = 0$*. In particular, an e-minimal structure* $\mathcal{A}_E$ *is e-largest is and only if* $e\text{-Sp}(\mathrm{Th}(\mathcal{A}_E)) = 0$.

3. *Any weakly saturated structure* $\mathcal{A}_E$*, i. e., a structure realizing all types of* $\mathrm{Th}(\mathcal{A}_E)$ *is e-largest.*

4. *For any* E*-combination* $\mathcal{A}_E$*, if* $\lambda \leq e\text{-Sp}(\mathrm{Th}(\mathcal{A}_E))$ *then there is a structure* $\mathcal{A}' \equiv \mathcal{A}_E$ *with* $e\text{-Sp}(\mathcal{A}') = \lambda$*; in particular, any theory* $\mathrm{Th}(\mathcal{A}_E)$ *has an e-largest model.*

5. *For any structure* $\mathcal{A}_E$, $e\text{-Sp}(\mathcal{A}_E) = |\mathrm{TH}(\mathcal{A}'_{E'}) \setminus \mathrm{TH}(\mathcal{A}_E)|$*, where* $\mathcal{A}'_{E'}$ *is an e-largest model of* $\mathrm{Th}(\mathcal{A}_E)$.

6. *Any prime structure* $\mathcal{A}_E$ *is e-minimal* (*but not vice versa as the e-minimality is preserved, for instance, extending an infinite* E*-class of given structure to a greater cardinality*)*. Any small theory* $\mathrm{Th}(\mathcal{A}_E)$ *has an e-minimal model* (*being prime*)*, and in this case, the structure* $\mathcal{A}_E$ *is e-minimal if and only if*

$$\mathrm{TH}(\mathcal{A}_E) = \bigcap_{\mathcal{A}' \equiv \mathcal{A}_E} \mathrm{TH}(\mathcal{A}'),$$

i. e., $\mathcal{A}_E$ *is e-least.*

7. *If* $\mathcal{A}_E$ *is e-least then* $e\text{-Sp}(\mathcal{A}_E) = e\text{-Sp}(\mathrm{Th}(\mathcal{A}_E))$.

8. *If* $e\text{-Sp}(\mathrm{Th}(\mathcal{A}_E))$ *finite and* $\mathrm{Th}(\mathcal{A}_E)$ *has e-least model then* $\mathcal{A}_E$ *is e-minimal if and only if* $\mathcal{A}_E$ *is e-least and if and only if* $e\text{-Sp}(\mathcal{A}_E) = e\text{-Sp}(\mathrm{Th}(\mathcal{A}_E))$.

9. *If* $e\text{-Sp}(\mathrm{Th}(\mathcal{A}_E))$ *is infinite then there are* $\mathcal{A}' \equiv \mathcal{A}_E$ *such that* $e\text{-Sp}(\mathcal{A}') = e\text{-Sp}(\mathrm{Th}(\mathcal{A}_E))$ *but* $\mathcal{A}'$ *is not e-minimal.*

10. *A countable e-minimal structure* $\mathcal{A}_E$ *is prime if and only if each* E*-class* $\mathcal{A}_i$ *is a prime structure.*

Reformulating Proposition 3.2 we have

Proposition 5.2. *For* E*-combinations which are not* EComb*, a countable theory* $\mathrm{Th}(\mathcal{A}_E)$ *without finite models is* ω*-categorical if and only if* $e\text{-Sp}(\mathrm{Th}(\mathcal{A}_E)) = 0$ *and each* E*-class* $\mathcal{A}_i$ *is either finite or* ω*-categorical.*

Note that if there are no links between E-classes (i. e., the Comb is considered, not EComb) and there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, then by Compactness the e-completion can vary adding arbitrary (finitely or infinitely) many new E-classes with a fixed structure which is not elementary equivalent to structures in old E-classes.

Proposition 5.3. *For any cardinality λ there is a theory $T = \mathrm{Th}(\mathcal{A}_E)$ of a language Σ such that $|\Sigma| = |\lambda + 1|$ and e-$\mathrm{Sp}(T) = \lambda$.*

Proof. Clearly, for structures $\mathcal{A}_i$ of fixed cardinality and with empty language we have e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E)) = 0$. For $\lambda > 0$ we take a language Σ consisting of unary predicate symbols P_i, $i < \lambda$. Let $\mathcal{A}_{i,n+1}$ be a structure having a universe $A_{i,n}$ with n elements and $P_i = A_{i,n}$, $P_j = \varnothing$, $i, j < \lambda$, $i \neq j$, $n \in \omega \setminus \{0\}$. Clearly, the structure $\mathcal{A}_E$, formed by all $\mathcal{A}_{i,n}$, is e-minimal. It produces structures $\mathcal{A}' \equiv \mathcal{A}_E$ containing E-classes with infinite predicates P_i, and structures of these classes are not elementary equivalent to the structures $\mathcal{A}_{i,n}$. Thus, for the theory $T = \mathrm{Th}(\mathcal{A}_E)$ we have e-$\mathrm{Sp}(T) = \lambda$. □

In Proposition 5.3, we have e-$\mathrm{Sp}(T) = |\Sigma(T)|$. At the same time the following proposition holds.

Proposition 5.4. *For any infinite cardinality λ there is a theory $T = \mathrm{Th}(\mathcal{A}_E)$ of a language Σ such that $|\Sigma| = \lambda$ and e-$\mathrm{Sp}(T) = 2^\lambda$.*

Proof. Let P_j be unary predicate symbols, $j < \lambda$, forming the language Σ, and $\mathcal{A}_i$ be structures consisting of only finitely many nonempty predicates $P_{j_1}, \ldots, P_{j_k}$ and such that these predicates are independent. Taking for the structures $\mathcal{A}_i$ all possibilities for cardinalities of sets of solutions for formulas $P_{j_1}^{\delta_{j_1}}(x) \wedge \ldots \wedge P_{j_k}^{\delta_{j_k}}(x)$, $\delta_{j_l} \in \{0, 1\}$, we get an e-minimal structure $\mathcal{A}_E$ such that for the theory $T = \mathrm{Th}(\mathcal{A}_E)$ we have e-$\mathrm{Sp}(T) = 2^\lambda$.

Another approach for e-$\mathrm{Sp}(T) = 2^\lambda$ was suggested by E.A. Palyutin. Taking infinitely many $\mathcal{A}_i$ with arbitrarily finitely many disjoint singletons $R_{j_1}, \ldots, R_{j_k}$, where Σ consists of R_j, $j < \lambda$, we get $\mathcal{A}' \equiv \mathcal{A}_E$ with arbitrarily many singletons for any subset of λ producing 2^λ E-classes which are pairwise elementary non-equivalent. □

If e-$\mathrm{Sp}(T) = 0$ the theory T is called *e-non-abnormalized* or *$(e, 0)$-abnormalized*. Otherwise, i. e., if e-$\mathrm{Sp}(T) > 0$, T is *e-abnormalized*. An e-abnormalized theory T with e-$\mathrm{Sp}(T) = \lambda$ is called *(e, λ)-abnormalized*. In particular, an $(e, 1)$-abnormalized theory is *e-categorical*, an (e, n)-abnormalized theory with $n \in \omega \setminus \{0, 1\}$ is *e-Ehrenfeucht*, an (e, ω)-abnormalized theory is *e-countable*, and an $(e, 2^\lambda)$-abnormalized theory is *(e, λ)-maximal*.

If e-$\mathrm{Sp}(T) = \lambda$ and T has a model $\mathcal{A}_E$ with e-$\mathrm{Sp}(\mathcal{A}_E) = \mu$ then $\mathcal{A}_E$ is called *$(e, \varkappa)$-abnormalized*, where $\varkappa$ is the least cardinality with $\mu + \varkappa = \lambda$.

By proofs of Propositions 5.3 and 5.4 we have

Corollary 5.5. *For any cardinalities $\mu \leq \lambda$ and the least cardinality $\varkappa$ with $\mu + \varkappa = \lambda$ there is an (e, λ)-abnormalized theory T with an $(e, \varkappa)$-abnormalized model $\mathcal{A}_E$.*

Let $\mathcal{A}_E$ and $\mathcal{B}_{E'}$ be structures and $\mathcal{C}_{E''} = \mathcal{A}_E \coprod \mathcal{B}_{E'}$ be their disjoint union, where $E'' = E \coprod E'$. We denote by $\mathrm{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'})$ the number of elementary pairwise non-equivalent structures $\mathcal{D}$ which are both a restriction of $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and a restriction of $\mathcal{B}' \equiv \mathcal{B}_{E'}$ to some E'-class as well as $\mathcal{D}$ is not elementary equivalent to the structures $\mathcal{A}_i$ and $\mathcal{B}_j$.

We have:

$$\mathrm{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) \leq \min\{e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E)), e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{B}_{E'}))\},$$

$$\max\{e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E)), e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{B}_{E'}))\} \leq e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{C}_{E''})),$$

$$e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E)) + e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{B}_{E'})) = e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{C}_{E''})) + \mathrm{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}).$$

Indeed, all structures witnessing the value $e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{C}_{E''}))$ can be obtained by $\mathrm{Th}(\mathcal{A}_E)$ or $\mathrm{Th}(\mathcal{B}_{E'})$ and common structures are counted for $\mathrm{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'})$.

If $\mathcal{A}_E = \mathcal{B}_{E'}$ then $\mathrm{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$. Assuming that $\mathcal{A}_E$ and $\mathcal{B}_{E'}$ do not have elementary equivalent classes $\mathcal{A}_i$ and $\mathcal{B}_j$, the number $\mathrm{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'})$ can vary from 0 to $2^{|\Sigma|+\omega}$.

Indeed, if $\mathrm{Th}(\mathcal{A}_E)$ or $\mathrm{Th}(\mathcal{B}_{E'})$ does not produce new, elementary non-equivalent classes then $\mathrm{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = 0$. Otherwise we can take structures $\mathcal{A}_i$ and $\mathcal{B}_i$ with one unary predicate symbol P such that P has $2i$ elements for $\mathcal{A}_i$ and $2i + 1$ elements for $\mathcal{B}_i$, $i \in \omega$. In this case we have $\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E)) = 1$, $\mathrm{Sp}(\mathrm{Th}(\mathcal{B}_{E'})) = 1$, $\mathrm{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = 1$, and $\mathcal{C}_{E''}$ witnessed by structures with infinite interpretations for P. Extending the language by unary predicates P_i, $i < \lambda$, and interpreting P_i in disjoint structures as for P above, we get $\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E)) = \lambda$, $\mathrm{Sp}(\mathrm{Th}(\mathcal{B}_{E'})) = \lambda$, $\mathrm{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = \lambda$. Thus we have

Proposition 5.6. *For any cardinality λ there are structures $\mathcal{A}_E$ and $\mathcal{B}_{E'}$ of a language Σ such that $|\Sigma| = |\lambda + 1|$ and $\mathrm{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = \lambda$.*

Applying proof of Proposition 5.4 with even and odd cardinalities for intersections of predicates in $\mathcal{A}_i$ and $\mathcal{B}_j$ respectively, we have $\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E)) = 2^\lambda$, $\mathrm{Sp}(\mathrm{Th}(\mathcal{B}_{E'})) = 2^\lambda$, $\mathrm{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = 2^\lambda$. In particular, we get

Proposition 5.7. *For any infinite cardinality λ are structures $\mathcal{A}_E$ and $\mathcal{B}_{E'}$ of a language Σ such that $|\Sigma| = \lambda$ and $\mathrm{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = 2^\lambda$.*

Replacing E-classes by unary predicates P_i (not necessary disjoint) being universes for structures $\mathcal{A}_i$ and restricting models of $\mathrm{Th}(\mathcal{A}_P)$ to the set of realizations of $p_\infty(x)$ we get the *e-spectrum* $e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_P))$, i. e., the number of pairwise elementary non-equivalent restrictions of $\mathcal{M} \models \mathrm{Th}(\mathcal{A}_P)$ to $p_\infty(x)$. We also get the notions of (e,λ)-abnormalized theory $\mathrm{Th}(\mathcal{A}_P)$, of (e,λ)-abnormalized model of $\mathrm{Th}(\mathcal{A}_P)$, and related notions.

Note that for any countable theory $T = \mathrm{Th}(\mathcal{A}_P)$, $e\text{-Sp}(T) \leq I(T,\omega)$. In particular, if $I(T,\omega)$ is finite then $e\text{-Sp}(T)$ is finite too. Moreover, if T is ω-categorical then $e\text{-Sp}(T) = 0$, and if T is an Ehrenfeucht theory, then $e\text{-Sp}(T) < I(T,\omega)$. Illustrating the finiteness for Ehrenfeucht theories we consider

Example 5.8. Similar to Example 4.2, let T_0 be the Ehrenfeucht theory of a structure $\mathcal{M}_0$, formed from the structure $\langle \mathbb{Q}; < \rangle$ by adding singletons R_k for elements c_k, $c_k < c_{k+1}$, $k \in \omega$, such that $\lim\limits_{k\to\infty} c_k = \infty$. It is well known that the theory T_3 has exactly 3 pairwise non-isomorphic models:

(a) a prime model $\mathcal{M}_0$ ($\lim\limits_{k\to\infty} c_k = \infty$);

(b) a prime model $\mathcal{M}_1$ over a realization of powerful type $p_\infty(x) \in S^1(\varnothing)$, isolated by sets of formulas $\{c_k < x \mid k \in \omega\}$;

(c) a saturated model $\mathcal{M}_2$ (the limit $\lim\limits_{k\to\infty} c_k$ is irrational).

Now we introduce unary predicates $P_i = \{a \in M_0 \mid a < c_i\}$, $i < \omega$, on $\mathcal{M}_0$. The structures $\mathcal{A}_i = \mathcal{M}_0 \upharpoonright P_i$ form the P-combination $\mathcal{A}_P$ with the universe M_0. Realizations of the type $p_\infty(x)$ in $\mathcal{M}_1$ and in $\mathcal{M}_2$ form two elementary non-equivalent structures $\mathcal{A}_\infty$ and $\mathcal{A}'_\infty$ respectively, where $\mathcal{A}_\infty$ has a dense linear order with a least element and $\mathcal{A}'_\infty$ has a dense linear order without endpoints. Thus, $e\text{-Sp}(T_0) = 2$ and T_0 is e-Ehrenfeucht.

As E.A. Palyutin noticed, varying unary predicates P_i in the following way: $P_{2i} = \{a \in M_0 \mid a < c_{2i}\}$, $P_{2i+1} = \{a \in M_0 \mid a \leq c_{2i+1}\}$, we get $e\text{-Sp}(T_3) = 4$ since the structures $\mathcal{A}'_\infty$ have dense linear orders with(out) least elements and with(out) greatest elements.

Modifying Example above, let T_n be the Ehrenfeucht theory of a structure $\mathcal{M}^n$, formed from the structure $\langle \mathbb{Q}; < \rangle$ by adding constants c_k, $c_k < c_{k+1}$, $k \in \omega$, such that $\lim\limits_{k\to\infty} c_k = \infty$, and unary predicates $R_0, \ldots, R_{n-2}$ which form a partition of the set $\mathbb{Q}$ of rationals, with

$$\models \forall x, y\, ((x < y) \to \exists z\, ((x < z) \wedge (z < y) \wedge R_i(z))),\ i = 0, \ldots, n-2.$$

The theory T_n has exactly $n+1$ pairwise non-isomorphic models:

(a) a prime model $\mathcal{M}^n$ ($\lim\limits_{k\to\infty} c_k = \infty$);

(b) prime models $\mathcal{M}^n_i$ over realizations of powerful types $p_i(x) \in S^1(\varnothing)$, isolated by sets of formulas $\{c_k < x \mid k \in \omega\} \cup \{P_i(x)\}$, $i = 0, \ldots, n-2$ ($\lim\limits_{k\to\infty} c_k \in P_i$);

(c) a saturated model $\mathcal{M}_infty^n$ (the limit $\lim\limits_{k\to\infty} c_k$ is irrational).

Now we introduce unary predicates $P_i = \{a \in M^n \mid a < c_i\}$, $i < \omega$, on $\mathcal{M}^n$. The structures $\mathcal{A}_i = \mathcal{M}^n \upharpoonright P_i$ form the P-combination $\mathcal{A}_P$ with the universe M^n. Realizations of the type $p_\infty(x)$ in $\mathcal{M}^n_i$ and in $\mathcal{M}^n_\infty$ form $n-1$

elementary non-equivalent structures $\mathcal{A}^n_j$, $j\leq n-2$, and $\mathcal{A}^n_\infty$, where $\mathcal{A}^n_j$ has a dense linear order with a least element in R_j, and $\mathcal{A}^n_\infty$ has a dense linear order without endpoints. Thus, e-$\mathrm{Sp}(T_n)=n$ and T_n is e-Ehrenfeucht.

Note that in the example above the type $p_\infty(x)$ has $n-1$ completions by formulas $R_0(x),\ldots,R_{n-2}(x)$.

Example 5.9. Taking a disjoint union $\mathcal{M}$ of $m\in\omega\setminus\{0\}$ copies of $\mathcal{M}_0$ in the language $\{<_j,R_k\}_{j<m,k\in\omega}$ and unary predicates $P_i=\{a\mid\mathcal{M}\models\exists x(a<x\wedge R_i(x))\}$ we get the P-combination $\mathcal{A}_P$ with the universe M for the structures $\mathcal{A}_i=\mathcal{M}\restriction P_i$, $i\in\omega$. We have e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_P))=3^m-1$ since each connected component of $\mathcal{M}$ produces at most two possibilities for dense linear orders or can be empty on the set of realizations of $p_\infty(x)$, and at least one connected component has realizations of $p_\infty(x)$.

Marking the relations $<_j$ by the same symbol $<$ we get the theory T with

$$e\text{-}\mathrm{Sp}(T)=\sum_{l=1}^{m}(l+1)=\frac{m(m+1)}{2}+m=\frac{m^2+3m}{2}.$$

Examples 5.8 and 5.9 illustrate that having a powerful type $p_\infty(x)$ we get e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_P))\neq 1$, i. e., there are no e-categorical theories $\mathrm{Th}(\mathcal{A}_P)$ with a powerful type $p_\infty(x)$. Moreover, we have

Theorem 5.10. *For any theory* $\mathrm{Th}(\mathcal{A}_P)$ *with non-symmetric or definable semi-isolation on the complete type* $p_\infty(x)$, e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_P))\neq 1$.

Proof. Assuming the hypothesis we take a realization a of $p_\infty(x)$ and construct step-by-step a $(a,p_\infty(x))$-*thrifty model* $\mathcal{N}$ of $\mathrm{Th}(\mathcal{A}_P)$, i. e., a model satisfying the following condition: if $\varphi(x,y)$ is a formula such that $\varphi(a,y)$ is consistent and there are no consistent formulas $\psi(a,y)$ with $\psi(a,y)\vdash p_\infty(x)$ then $\varphi(a,\mathcal{N})=\varnothing$.

At the same time, since $p_\infty(x)$ is non-isolated, for any realization a of $p_\infty(x)$ the set $p_\infty(x)\cup\{\neg\varphi(a,x)\mid\varphi(a,x)\vdash p_\infty(x)\}$ is consistent. Then there is a model $\mathcal{N}'\models\mathrm{Th}(\mathcal{A}_P)$ realizing $p_\infty(x)$ and which is not $(a',p_\infty(x))$-thrifty for any realization a' of $p_\infty(x)$.

If semi-isolation is non-symmetric, $\mathcal{N}\restriction p_\infty(x)$ and $\mathcal{N}'\restriction p_\infty(x)$ are not elementary equivalent since the formula $\varphi(a,y)$ witnessing the non-symmetry of semi-isolation has solutions in $\mathcal{N}'\restriction p_\infty(x)$ and does not have solutions in $\mathcal{N}\restriction p_\infty(x)$.

If semi-isolation is definable and witnessed by a formula $\psi(a,y)$ then again $\mathcal{N}\restriction p_\infty(x)$ and $\mathcal{N}'\restriction p_\infty(x)$ are not elementary equivalent since $\neg\psi(a,y)$ is realized in $\mathcal{N}'\restriction p_\infty(x)$ and it does not have solutions in $\mathcal{N}\restriction p_\infty(x)$

Thus, e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_P))>1$. $\square$

Since non-definable semi-isolation implies that there are infinitely many 2-types, we have

Corollary 5.11. *For any theory* $\mathrm{Th}(\mathcal{A}_P)$ *with* e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_P)) = 1$ *the structures* $\mathcal{A}'_\infty$ *are not* ω*-categorical.*

Applying modifications of the Ehrenfeucht example as well as constructions in [12], the results for e-spectra of E-combinations are modified for P-combinations:

Proposition 5.12. *For any cardinality* λ *there is a theory* $T = \mathrm{Th}(\mathcal{A}_P)$ *of a language* Σ *such that* $|\Sigma| = \max\{\lambda, \omega\}$ *and* e-$\mathrm{Sp}(T) = \lambda$.

Proof. Clearly, if $p_\infty(x)$ is inconsistent then e-$\mathrm{Sp}(T) = 0$. Thus, the assertion holds for $\lambda = 0$.

If $\lambda = 1$ we take a theory T_1 with disjoint unary predicates P_i, $i \in \omega$, and a symmetric irreflexive binary relation R such that each vertex has R-degree 2, each P_i has infinitely many connected components, and each connected component on P_i has diameter i. Now structures on $p_\infty(x)$ have connected components of infinite diameter, all these structures are elementary equivalent, and e-$\mathrm{Sp}(T_1) = 1$.

If $\lambda = n > 1$ is finite, we take the theory T_n in Example 5.8 with e-$\mathrm{Sp}(T_n) = n$, as well as we can take a generic Ehrenfeucht theory T'_λ with $\mathrm{RK}(T'_\lambda) = 2$ and with $\lambda - 1$ limit model $\mathcal{M}_i$ over the type $p_\infty(x)$, $i < \lambda - 1$, such that each $\mathcal{M}_i$ has a Q_j-chains, $j \leq i$, and does not have Q_k-chains for $k > i$. Restricting the limit models to $p_\infty(x)$ we get λ elementary non-equivalent structures including the prime structure $\mathcal{N}^0$ without Q_i-chains and structures $\mathcal{M}_i \restriction p_\infty(x)$, $i < \lambda - 1$, which are elementary non-equivalent by distinct (non)existence of Q_j-chains.

Similarly, taking $\lambda \geq \omega$ disjoint binary predicates R_j for the Ehrenfeucht example in 5.8 we have λ structures with least elements in R_j which are not elementary equivalent each other. Producing the theory T_λ we have e-$\mathrm{Sp}(T_\lambda) = \lambda$.

Modifying the generic Ehrenfeucht example taking λ binary predicates Q_j with Q_j-chains which do not imply Q_k-chains for $k > i$ we get λ elementary non-equivalent restrictions to $p_\infty(x)$. $\square$

Note that as in Example 5.8 the type $p_\infty(x)$ for the Ehrenfeucht-like example T_λ has λ completions by the formulas $R_j(x)$ whereas the type $p_\infty(x)$ for the generic Ehrenfeucht theory is complete. At the same time having λ completions for the $p_\infty(x)$-restrictions related to T_λ, the $p_\infty(x)$-restrictions the generic Ehrenfeucht examples with complete $p_\infty(x)$ can violet the uniqueness of the complete 1-type like the Ehrenfeucht example T_0, where $\mathcal{A}_\infty$ realizes two complete 1-types: the type of the least element and the type of elements which are not least.

Proposition 5.13. *For any infinite cardinality* λ *there is a theory* $T = \mathrm{Th}(\mathcal{A}_P)$ *of a language* Σ *such that* $|\Sigma| = \lambda$ *and* e-$\mathrm{Sp}(T) = 2^\lambda$.

Proof. Let T be the theory of independent unary predicates R_j, $j < \lambda$, (defined by the set of axioms $\exists x\, (R_{k_1}(x) \wedge \ldots \wedge R_{k_m}(x) \wedge \neg R_{l_1}(x) \wedge \ldots \wedge$

$\neg R_{l_n}(x))$, where $\{k_1,\ldots,k_m\}\cap\{l_1,\ldots,l_n\}=\varnothing$) such that countably many of them form predicates P_i, $i<\omega$, and infinitely many of them are independent with P_i. Thus, T can be considered as $\mathrm{Th}(\mathcal{A}_P)$. Restrictions of models of T to sets of realizations of the type $p_\infty(x)$ witness that predicates R_j distinct with all P_i are independent. Denote indexes of these predicates R_j by J. Since $p_\infty(x)$ is non-isolated, for any family $\Delta=(\delta_j)_{j\in J}$, where $\delta_j\in\{0,1\}$, the types $q_\Delta(x)=\{R_j^{\delta_j}\mid j\in J\}$ can be pairwise independently realized and omitted in structures $\mathcal{M}\upharpoonright p_\infty(x)$ for $\mathcal{M}\models T$. Then any predicate R_j can be independently realized and omitted in these restrictions. Thus there are 2^λ restrictions with distinct theories, i. e., e-$\mathrm{Sp}(T)=2^\lambda$. □

Since for E-combinations $\mathcal{A}_E$ and P-combinations $\mathcal{A}_P$ and their limit structures $\mathcal{A}_\infty$, being respectively structures on E-classes and $p_\infty(x)$, the theories $\mathrm{Th}(\mathcal{A}_\infty)$ are defined by types restricted to $E(x,y)$ and $p_\infty(x)$, and for any countable theory there are either countably many types or continuum many types, Propositions 5.3, 5.4, 5.12, and 5.13 implies the following

Theorem 5.14. *If $T=\mathrm{Th}(\mathcal{A}_E)$ (respectively, $T=\mathrm{Th}(\mathcal{A}_P)$) is a countable theory then e-$\mathrm{Sp}(T)\in\omega\cup\{\omega,2^\omega\}$. All values in $\omega\cup\{\omega,2^\omega\}$ have realizations in the class of countable theories of E-combinations (of P-combinations).*

6. Ehrenfeuchtness for E-combinations

Theorem 6.1. *If the language $\bigcup\limits_{i\in I}\Sigma(\mathcal{A}_i)$ is at most countable and the structure $\mathcal{A}_E$ is infinite then the theory $T=\mathrm{Th}(\mathcal{A}_E)$ is Ehrenfeucht if and only if e-$\mathrm{Sp}(T)<\omega$ (which is equivalent here to e-$\mathrm{Sp}(T)=0$) and for an e-largest model $\mathcal{A}_{E'}\models T$ consisting of E'-classes $\mathcal{A}_j$, $j\in J$, the following conditions hold:*

(a) *for any $j\in J$, $I(\mathrm{Th}(\mathcal{A}_j),\omega)<\omega$;*

(b) *there are positively and finitely many $j\in J$ such that $I(\mathrm{Th}(\mathcal{A}_j),\omega)>1$;*

(c) *if $I(\mathrm{Th}(\mathcal{A}_j),\omega)\leq 1$ then there are always finitely many $\mathcal{A}_{j'}\equiv\mathcal{A}_j$ or always infinitely many $\mathcal{A}_{j'}\equiv\mathcal{A}_j$ independent of $\mathcal{A}_{E'}\models T$.*

Proof. If e-$\mathrm{Sp}(T)<\omega$ and the conditions (a)–(c) hold then the theory T is Ehrenfeucht since each countable model $\mathcal{A}_{E''}\models T$ is composed of disjoint models with universes $E''_k=A_k$, $k\in K$, and $I(T,\omega)$ is a sum $\sum\limits_{l=0}^{e\text{-}\mathrm{Sp}(T)}$ of finitely many possibilities for models with l representatives with respect to the elementary equivalence of E''-classes that are not presented in a prime (i. e., e-minimal) model of T. These possibilities are composed by finitely many possibilities of $I(\mathrm{Th}(\mathcal{A}_k),\omega) > 1$ for $\mathcal{A}_{k'} \equiv \mathcal{A}_k$ and finitely many of $\mathcal{A}_{k''} \not\equiv \mathcal{A}_k$ with $I(\mathrm{Th}(\mathcal{A}_{k''}),\omega) > 1$. Moreover, there are $\hat{C}(I(\mathrm{Th}(\mathcal{A}_k),\omega), m_i)$ possibilities for substructures consisting of $\mathcal{A}_{k'} \equiv \mathcal{A}_k$ where m_i is the number of E-classes having the theory $\mathrm{Th}(\mathcal{A}_k)$, $\hat{C}(n,m) = C^m_{n+m-1}$ is the number of combinations with repetitions for n-element sets with m places. The formula for $I(T,\omega)$ is based on the property that each E''-class with the structure $\mathcal{A}_k$ can be replaced, preserving the elementary equivalence of $\mathcal{A}_{E''}$, by arbitrary $\mathcal{B} \equiv \mathcal{A}_k$.

Now we assume that the theory T is Ehrenfeucht. Since models of T with distinct theories of E-classes are not isomorphic, we have e-$\mathrm{Sp}(T) < \omega$. Applying the formula for $I(T,\omega)$ we have the conditions (a), (b). The condition (c) holds since varying unboundedly many $\mathcal{A}_{j'} \equiv \mathcal{A}_j$ we get $I(T,\omega) \geq \omega$.

The conditions e-$\mathrm{Sp}(T) < \omega$ and e-$\mathrm{Sp}(T) = 0$ are equivalent. Indeed, if e-$\mathrm{Sp}(T) > 0$ then taking an e-minimal model $\mathcal{M}$ we get, by Compactness, unboundedly many E-classes, which are elementary non-equivalent to E-classes in $\mathcal{M}$. It implies that $I(T,\omega) \geq \omega$. □

Since any prime structure is e-minimal (but not vice versa as the e-minimality is preserved, for instance, extending an infinite E-class of given structure to a greater cardinality preserving the elementary equivalence) and any Ehrenfeucht theory T, being small, has a prime model, any Ehrenfeucht theory $\mathrm{Th}(\mathcal{A}_E)$ has an e-minimal model.

We investigate combinations of structures by families of structures relative to families of unary predicates and equivalence relations. Conditions preserving ω-categoricity and Ehrenfeuchtness under these combinations are characterized. The notions of e-spectra are introduced and possibilities for e-spectra are described.

7. Conclusion

We introduced and studied combinations of structures by families of structures relative to families of unary predicates and equivalence relations. Conditions preserving ω-categoricity and Ehrenfeuchtness under these combinations are characterized. The notions of e-spectra are introduced and possibilities for e-spectra are described.

References

1. Andrews U., Keisler H.J. Separable models of randomizations. *J. Symbolic Logic*, 2015, vol. 80, no. 4, pp. 1149-1181.https://doi.org/10.1017/jsl.2015.33

2. Baldwin J.T., Plotkin J.M. A topology for the space of countable models of a first order theory. *Zeitshrift Math. Logik and Grundlagen der Math.*, 1974, vol. 20, no. 8–12, pp. 173-178.https://doi.org/10.1002/malq.19740200806
3. Bankston P. Ulptraproducts in topology. *General Topology and its Applications*, 1977, vol. 7, no. 3, pp. 283–308.
4. Bankston P. A survey of ultraproduct constructions in general topology. *Topology Atlas Invited Contributions*, 2003, vol. 8, no. 2, pp. 1-32.
5. Benda M. Remarks on countable models. *Fund. Math.*, 1974, vol. 81, no. 2, pp. 107-119.https://doi.org/10.4064/fm-81-2-107-119
6. Henkin L. Relativization with respect to formulas and its use in proofs of independence. *Composito Mathematica*, 1968, vol. 20, pp. 88-106.
7. Newelski L. Topological dynamics of definable group actions. *J. Symbolic Logic*, 2009, vol. 74, no. 1, pp. 50-72.https://doi.org/10.2178/jsl/1231082302
8. Pillay A. Topological dynamics and definable groups. *J. Symbolic Logic*, 2013, vol. 78, no. 2, pp. 657-666.https://doi.org/10.2178/jsl.7802170
9. Sudoplatov S.V. Transitive arrangements of algebraic systems. *Siberian Math. J.*, 1999, vol. 40, no. 6, pp. 1142-1145. https://doi.org/10.1007/BF02677538
10. Sudoplatov S.V. Inessential combinations and colorings of models. *Siberian Math. J.*, 2003, vol. 44, no. 5, pp. 883–890.https://doi.org/10.1023/A:1025901223496
11. Sudoplatov S.V. Powerful digraphs. *Siberian Math. J.*, 2007, vol. 48, no. 1, pp. 165–171.https://doi.org/10.1007/s11202-007-0017-1
12. Sudoplatov S.V. *Klassifikatsiya schetnykh modeley polnykh teoriy* [Classification of Countable Models of Complete Theories]. Novosibirsk, NSTU Publ., 2018.(in Russian)
13. Vaught R. Denumerable models of complete theories. *Infinistic Methods*, London, Pergamon, 1961, pp. 303-321.
14. Woodrow R.E. *Theories with a finite number of countable models and a small language*. Ph. D. Thesis. Simon Fraser University, 1976, 99 p.

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Received 19.04.2018

Комбинации структур

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Аннотация. Исследуются комбинации структур, для данных семейств структур, относительно семейств одноместных предикатов и отношений эквивалентности. Охарактеризованы условия сохранения ω-категоричности и эренфойхтовости для этих комбинаций. Введены понятия e-спектров и описаны возможности для e-спектров.

Показано, что ω-категоричность для дизъюнктных P-комбинаций равносильна конечному числу индексов для новых одноместных предикатов с условием конечности или ω-категоричности каждой структуры в новых одноместных предикатах. Аналогично, теория E-комбинации ω-категорична тогда и только тогда, когда каждая данная структура либо конечна, либо ω-категорична, и множество индексов либо конечно, либо бесконечно и при этом E_i-классы не аппроксимируют бесконечное число n-типов для $n \in \omega$. Теория дизъюнктной P-комбинации эренфойхтова тогда и только тогда, когда множество индексов конечно, каждая данная структура либо конечна, либо ω-категорична, либо эренфойхтова, и некоторая структура эренфойхтова.

Рассмотрены вариации структур, относящиеся к комбинациям и E-представимости.

Введены e-спектры для P-комбинаций и E-комбинаций, и показано, что эти e-спектры могут иметь произвольные мощности.

В терминах e-спектров охарактеризовано свойство эренфойхтовости для E-комбинаций.

Ключевые слова: комбинация структур, P-комбинация, e-спектр, E-комбинация.

Список литературы

1. Andrews U. Separable models of randomizations // J. Symbolic Logic. 2015. Vol. 80, N 4. P. 1149–1181. https://doi.org/10.1017/jsl.2015.33
2. Baldwin J. T., Plotkin J. M. A topology for the space of countable models of a first order theory // Zeitshrift Math. Logik and Grundlagen der Math. 1974. Vol. 20, N 8–12. P. 173–178. https://doi.org/10.1002/malq.19740200806
3. Bankston P. Ulptraproducts in topology // General Topology and its Applications. 1977. Vol. 7, N 3. P. 283–308.
4. Bankston P. A survey of ultraproduct constructions in general topology // Topology Atlas Invited Contributions. 2003. Vol. 8, N 2. P. 1–32.
5. Benda M. Remarks on countable models // Fund. Math. 1974. Vol. 81, N 2. P. 107–119. https://doi.org/10.4064/fm-81-2-107-119
6. Henkin L. Relativization with respect to formulas and its use in proofs of independence // Composito Mathematica. 1968. Vol. 20. P. 88–106.
7. Newelski L. Topological dynamics of definable group actions // J. Symbolic Logic. 2009. Vol. 74, N 1. P. 50–72. https://doi.org/10.2178/jsl/1231082302
8. Pillay A. Topological dynamics and definable groups // J. Symbolic Logic. 2013. Vol. 78, N 2. P. 657–666. https://doi.org/10.2178/jsl.7802170
9. Судоплатов С. В. Транзитивные размещения алгебраических систем // Сиб. мат. журн. 1999. Т. 40, № 6. С. 1347–1351. https://doi.org/10.1007/BF02677538
10. Судоплатов С. В. Несущественные совмещения и раскраски моделей // Сиб. мат. журн. 2003. Т. 44, № 5. С. 1132–1141. https://doi.org/10.1023/A:1025901223496

11. Судоплатов С. В. Властные орграфы // Сиб. мат. журн. 2007. Т. 48, № 1. С. 205–213. https://doi.org/10.1007/s11202-007-0017-1
12. Судоплатов С. В. Классификация счетных моделей полных теорий. Новосибирск : НГТУ, 2018.
13. Vaught R. Denumerable models of complete theories // Infinistic Methods. London : Pergamon, 1961. P. 303–321.
14. Woodrow R. E. Theories with a finite number of countable models and a small language. Ph. D. Thesis. Simon Fraser University, 1976. 99 p.

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Поступила в редакцию 19.04.2018