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Generation by Conjugate Elements of Finite Almost Simple Groups With a Sporadic Socle

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Abstract. We study the minimum number of elements in the conjugacy class of an automorphism of a sporadic simple group that generate a subgroup containing all inner automorphisms. These results refine the estimates obtained earlier in the papers by Guralnick and Saxl and by Di Martino, Pellegrini, and Zalesski.

Keywords: sporadic group, Fischer group, conjugacy, generators, Baer–Suzuki theorem

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Научная статья

О порождении конечных почти простых групп со спорадическим цоколем сопряжёнными элементами

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Аннотация. Изучается минимальное число элементов в классе сопряжённости автоморфизма простой спорадической группы, которые порождают подгруппу, содержащую все внутренние автоморфизмы. Результаты уточняют оценки, полученные ранее в работе Гуральника и Саксла и работе Ди Мартино, Пеллегрини и Залесского.

Ключевые слова: спорадическая группа, группа Фишера, сопряжённость, порождающие, теорема Бэра – Судзуки

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Introduction

We only consider finite groups in this paper, so the word "group" will always mean a finite group.

Let S be a nonabelian simple group, which we always identify with the subgroup $\mathrm{Inn}(S)$ of inner automorphisms in the group $\mathrm{Aut}(S)$ of all automorphisms. Then S is the unique minimal normal subgroup of every group G such that $S \leqslant G \leqslant \mathrm{Aut}(S)$. In this case, G is usually called an *almost simple group with socle* S. If $x \in \mathrm{Aut}(S)$ is a nonidentity (possibly, inner) automorphism then the subgroup of $G = \langle x, S\rangle$ that is generated by the conjugacy class of x is normal in G. Consequently, this subgroup includes S and, therefore, coincides with G.

In 2003, R. Guralnick and J. Saxl [10] introduced the notation

$$\alpha(x) = \alpha_S(x)$$

for the minimum number of elements conjugate to $x \neq 1$ in $G = \langle x, S\rangle$ that generate G. In other words, the parameter $\alpha_S(x)$ is defined by the property that, for every natural m, $\alpha_S(x) \leqslant m$ if and only if some elements $x_1, \dots, x_m$ conjugate to x in G generate G. It is not difficult to see that if $y \neq 1$ is a power of x then, for all $g_1, \dots, g_m \in G$, we have the inclusion

$$\langle y^{g_1}, \dots, y^{g_m}\rangle \leqslant \langle x^{g_1}, \dots, x^{g_m}\rangle.$$

Thus, if $S \leqslant \langle y^{g_1}, \dots, y^{g_m}\rangle$ then $S \leqslant \langle x^{g_1}, \dots, x^{g_m}\rangle$ and so $\alpha_S(x) \leqslant \alpha_S(y)$. This means in particular that in order to find upper bounds on $\alpha_S(x)$ for a fixed S, it is sufficient to consider only elements x of prime order.

The main result of [10] constitutes finding explicit, albeit not always best possible, upper bounds on $\alpha_S(x)$ for all nonabelian simple groups S. These bounds and their refinements have been extensively used in applications of the classification of finite simple groups. For example, they are substantially used in proofs of various analogues of the famous Baer-Suzuki theorem, see [3; 5–9; 15–19]. For practical use, the estimates on $\alpha_S(x)$ from [10] are not always sufficient. Refinements of these estimates for certain simple groups were required and obtained, for example, in [2; 15; 16].

In the case where S is a sporadic simple group and x is its nonidentity *inner* automorphism, nearly precise values of $\alpha_S(x)$ were found by L. Di Martino, M. A. Pellegrini, and A. E. Zalesski in [2, Theorem 3.1]. All cases of imprecise estimates in this theorem are in the following list[1]:

1) $(S, x) = (Fi_{22}, 2A)$ and $5 \leqslant \alpha_S(x) \leqslant 6$;

2) $(S, x) = (Fi_{23}, 2A)$ and $5 \leqslant \alpha_S(x) \leqslant 6$;

3) $(S, x) = (Fi_{22}, 3B)$ and $2 \leqslant \alpha_S(x) \leqslant 3$;

4) $(S, x) = (Suz, 3A)$ and $3 \leqslant \alpha_S(x) \leqslant 4$;

5) $S = M$, x is not an involution and $2 \leqslant \alpha_S(x) \leqslant 3$;

6) $S = M$, x is an involution and $3 \leqslant \alpha_S(x) \leqslant 4$.

As a matter of fact, we can indicate the exact value of $\alpha_S(x)$ in cases 1) and 2) above. The Fischer groups Fi_{22}, Fi_{23}, and Fi_{24} are so called 3-*transposition groups*. Each of them is generated[2] by a conjugacy class D that is a class of 3-*transpositions*, i. e. consists of elements of order 2 (*involutions*) such that the product of every two of them has order 1, 2, or 3. The classes of 3-transpositions in Fi_{22}, Fi_{23}, and $Fi_{24} = Fi_{24}'.2$ are $2A$, $2A$, and $2C$, respectively, in the notation of [1]. The groups of 3-transpositions that can be generated by at most five 3-transpositions were classified in 1995 by J. Hall and L. Soicher [11, Theorems (1.1)–(1.3)]. This result and S. Norton's paper [14] imply that F_{24} can be generated by five 3-transpositions, whereas Fi_{22} and Fi_{23} cannot. Also, Fi_{24} cannot be generated by four 3-transpositions, see [11, Theorem (1.1)]. Thus, the following assertion holds which in particular gives precise values for $\alpha_S(x)$ in cases 1) and 2) above.

Proposition 1. *If* $(S, x) \in \{(Fi_{22}, 2A), (Fi_{23}, 2A)\}$ *then* $\alpha_S(x) = 6$. *If* $(S, x) = (Fi_{24}', 2C)$ *then* $\alpha_S(x) = 5$.

[1] Here and later on we use the notation and conventions from the Atlas of finite groups [1]; in particular, we may use the same symbol to denote both a group element and its conjugacy class.

[2] Fi_{24} is not simple, but its subgroup Fi_{24}' of index 2 is. Therefore, the generating class of 3-transpositions for Fi_{24} lies in $Fi_{24} \setminus Fi_{24}'$.

Table 1

Sporadic groups S with $\mathrm{Aut}(S) \neq S$

S	$\lvert S\rvert$	classes of involutions x in $\mathrm{Aut}(S) \setminus S$	$\alpha_S(x) \leqslant$
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	$2C$	4
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$2B$, $2C$	4
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$2C$	4
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	$2B$	4
McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	$2B$	4
$O'N$	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	$2B$	4
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	$2C$, $2D$	5
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	$2C$	5
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$2C$, $2D$	5
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	$2C$	5
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$2D$, $2E$, $2F$	7
$Fi_{24}{}'$	$2^{21} \cdot 3^{16} \cdot 5^5 \cdot 7^3 \cdot 11 \cdot 13 \cdot 23 \cdot 29$	$2C$, $2D$	8

As we have already mentioned, only inner automorphisms of sporadic groups were considered in [2]. For automorphisms in $\mathrm{Aut}(S) \setminus S$, where S is a sporadic group, only the estimates from [10] are known. If $\mathrm{Aut}(S) \neq S$ and $x \in \mathrm{Aut}(S) \setminus S$ is of prime order then S is as given in the first column of Table 1, x is an involution whose conjugacy class is given in the third column, and the estimate on $\alpha_S(x)$ from [10] is given in the fourth column. Observe that since two involutions always generate a solvable group, we have $3 \leqslant \alpha_S(x)$ for all cases included in Table 1.

The main result of this paper is as follows.

Theorem 1. *Let $x \in \mathrm{Aut}(S) \setminus S$ be an automorphism of prime order of a sporadic group S. Then $3 \leqslant \alpha_S(x) \leqslant 4$, except when $(S, x) = (Fi_{24}{}', 2C)$ and $\alpha_S(x) = 5$.*

Combining this result with [2, Theorem 3.1] and Proposition 1 we will also prove the following assertion which includes the cases of inner and outer automorphisms of prime and composite order.

Theorem 2. *Let $x \in \mathrm{Aut}(S)$ be a nonidentity automorphism of a sporadic group S. Then $\alpha_S(x) \leqslant 4$, except in the following cases:*

1) $(S, x) = (Fi_{22}, 2A)$ *and* $\alpha_S(x) = 6$;

2) $(S, x) = (Fi_{23}, 2A)$ *and* $\alpha_S(x) = 6$;

3) $(S, x) = (Fi_{24}', 2C)$ *and* $\alpha_S(x) = 5$.

The following problem still remains open.

Problem 1. *Find* $\alpha_S(x)$ *for every sporadic simple group* S *and its nonidentity automorphism* x.

In order to solve this problem, we have to determine the precise value of $\alpha_S(x)$ in the above-mentioned cases 3) − 6), where [2, Theorem 3.1] does not give such a value, as well as determine whether $\alpha_S(x)$ equals 3 or 4 for S and x from Table 1, except $(S, x) = (Fi_{24}', 2C)$, where we know that $\alpha_S(x) = 5$.

1. Proof of Theorems 1 and 2

Proof of Theorem 1. As we have already mentioned, the inequality $3 \leqslant \alpha_S(x)$ for an involution x follows from the fact that every two involutions generate a solvable group [13, Lemma 2.14].

Taking account of Proposition 1 and the data in Table 1, we will prove Theorem 1 once we establish that $\alpha_S(x) \leqslant 4$ when (S, x) is in the following list:

$$(HS, 2C), (HS, 2D), (He, 2C), (Suz, 2C), (Suz, 2D), (HN, 2C),$$
$$(Fi_{22}, 2D), (Fi_{22}, 2E), (Fi_{22}, 2F), (Fi_{24}', 2D).$$

Since $O_2(G) = 1$ for $G = \langle x, S\rangle$, the Baer–Suzuki theorem [13, Theorem 2.12] implies that x, together with some conjugate x^t, generates a subgroup containing a nonidentity element y of odd order. Also, $y \in S$, because $|G : S| = 2$. If $\alpha_S(y) = 2$ then by definition S contains an element $g \in G$ such that

$$S = \langle y, y^g\rangle \leqslant \langle x, x^t, x^g, x^{tg}\rangle.$$

This will imply the inequality $\alpha_S(x) \leqslant 4$ and the claim will follow.

In view of [2, Theorem 3.1], we have $\alpha_S(y) = 2$ for a nonidentity $y \in S$ of odd order, where (S, x) is from the above list, unless

$$(S, y) \in \{(Suz, 3A), (Fi_{22}, 3A), (Fi_{22}, 3B), (Fi_{24}', 3A), (Fi_{24}', 3B)\}.$$

In particular, $\alpha_S(y) = 2$ if S is one of HS, He, or HN.

In the remaining part of the proof, we are going to establish that even when S is one of the groups Suz, Fi_{22}, or Fi_{24}', the conjugate x^t can be chosen so that the product $y = xx^t$ would be of odd order and belong to neither class $3A$ nor $3B$. Under this choice, we have $\alpha_S(y) = 2$ as required.

We use the known fact from character theory that, given elements a, b and c of a group G, the number $\mathrm{m}(a, b, c)$ of pairs (u, v), where u is conjugate

Table 2

Some class multiplication coefficients

(S,x)	$\mathrm{m}(x,x,y)$
$(Suz,2C)$	$\mathrm{m}(2C,2C,3C)=45$
$(Suz,2D)$	$\mathrm{m}(2D,2D,3C)=45$
$(Fi_{22},2D)$	$\mathrm{m}(2D,2D,3C)=3$
$(Fi_{22},2E)$	$\mathrm{m}(2E,2E,3C)=729$
$(Fi_{22},2F)$	$\mathrm{m}(2F,2F,3C)=1080$
$(Fi_{24}',2D)$	$\mathrm{m}(2D,2D,3C)=1224720$

to a, v is conjugate to b, and $uv=c$, can be found from the character table using the formula

$$\mathrm{m}(a,b,c)=\frac{|G|}{|\,\mathrm{C}_G(a)||\,\mathrm{C}_G(b)|}\sum_{\chi\in\mathrm{Irr}(G)}\frac{\chi(a)\chi(b)\overline{\chi(c)}}{\chi(1)},$$

see [12, Exercise (3.9), p. 45]. To make sure that x and some of its conjugates generate a subgroup containing a conjugate of y, it is sufficient to show that $\mathrm{m}(x,x,y)>0$. We use the character tables of $Suz.2$, $Fi_{22}.2$, and $Fi_{24}'.2$ available in both [1] and the computer algebra system `GAP` [4]. The remaining cases can be treated using the `GAP` function

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> ClassMultiplicationCoefficient()
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to calculate $\mathrm{m}(x,x,y)$ for x and y listed in Table 2. It turns out that in all these cases y can be chosen from the class denoted by $3C$ in [1]. The proof of Theorem 1 is complete. □

Proof of Theorem 2. For inner nonidentity automorphisms of S, the claim holds in view of [2, Theorem 3.1] and Proposition 1. For elements of prime order in $\mathrm{Aut}(S)\setminus S$, the claim holds by Theorem 1. It remains to consider the elements of composite order in the difference $\mathrm{Aut}(S)\setminus S$. Let x be such an element. If y is a power of x, $|y|$ is prime, and $(S,y)\notin\{(Fi_{22},2A),(Fi_{24}',2C)\}$ (for example, when x is not a 2-element), then

$$\alpha_S(x)\leqslant\alpha_S(y)\leqslant 4,$$

and the claim holds for x. Consequently, we may assume that S is either Fi_{22} or Fi_{24}', x is a 2-element, and y is the power of x of order 2 in class $2A$ if $S=Fi_{22}$ and in class $2C$ if $S=Fi_{24}'$. Then either x or some power of x has order 4. The character tables of $Fi_{22}.2$ and $Fi_{24}'.2$ from [4] contain information about which conjugacy classes the prime powers of elements belong to. This information implies that both $Fi_{22}.2$ and $Fi_{24}'.2$ have no elements of order 4 whose squares would belong to classes $2A$ or $2C$. The proof is complete. □

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References

1. Conway J.H., Curtis R.T., Norton S.P., Parker R.A., Wilson R.A. Atlas of finite groups. Oxford, Clarendon Press, 1985.
2. Di Martino L., Pellegrini M.A., Zalesski A.E. On generators and representations of the sporadic simple groups. *Comm. Algebra*, 2014, vol. 42, no. 2, pp. 880–908. https://doi.org/10.1080/00927872.2012.729629
3. Flavell P., Guest S., Guralnick R. Characterizations of the solvable radical. *Proc. Amer. Math. Soc.*, 2010, vol. 138, no. 4, pp. 1161–1170. https://doi.org/10.1090/S0002-9939-09-10066-7
4. The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.12.2; 2022. http://www.gap-system.org
5. Gordeev N., Grunewald F., Kunyavskii B., Plotkin E. A description of Baer–Suzuki type of the solvable radical of a finite group. *J. Pure Appl. Algebra*, 2009, vol. 213, no. 2, pp. 250–258. https://doi.org/10.1016/j.jpaa.2008.06.006
6. Gordeev N., Grunewald F., Kunyavskii B., Plotkin E. Baer–Suzuki theorem for the solvable radical of a finite group. *C. R. Acad. Sci. Paris, Ser. I*, 2009, vol. 347, no. 5–6, pp. 217–222. https://doi.org/10.1016/j.crma.2009.01.004
7. Gordeev N., Grunewald F., Kunyavskii B., Plotkin E. From Thompson to Baer–Suzuki: a sharp characterization of the solvable radical. *J. Algebra*, 2010, vol. 323, no. 10, pp. 2888–2904. https://doi.org/10.1016/j.jalgebra.2010.01.032
8. Guest S. A solvable version of the Baer–Suzuki theorem. *Trans. Amer. Math. Soc.*, 2010, vol. 362, pp. 5909–5946. https://doi.org/10.1090/S0002-9947-2010-04932-3
9. Guest S., Levy D. Criteria for solvable radical membership via p-elements. *J. Algebra*, 2014, vol. 415, pp. 88–111. https://doi.org/10.1016/j.jalgebra.2014.06.003
10. Guralnick R., Saxl J. Generation of finite almost simple groups by conjugates radical. *J. Algebra*, 2003, vol. 268, pp. 519–571. https://doi.org/10.1016/S0021-8693(03)00182-0
11. Hall J., Soicher L. Presentations of some 3-transposition groups. *Comm. Algebra*, 1995, vol. 23, no. 7, pp. 2517–2559. https://doi.org/10.1080/00927879508825358
12. Isaacs I. M. Character theory of finite groups. Providence, RI. AMS Chelsea. 2006. https://doi.org/10.1090/chel/359
13. Isaacs I. M. Finite group theory. Grad. Stud. Math., 92. Providence, RI. AMS. 2008. https://doi.org/10.1090/gsm/092
14. Norton S.P. Free transposition groups *Comm. Algebra*, 1996, vol. 24, no. 2, pp. 425–432. https://doi.org/10.1080/00927879608825578
15. Revin D.O., Zavarnitsine A.V. On generations by conjugate elements in almost simple groups with socle ${}^2F_4(q^2)'$. *J. Group Theory*, 2024, vol. 27, no. 1, pp. 119–140. https://doi.org/10.1515/jgth-2022-0216
16. Wang Zh., Guo W., Revin D.O. Toward a sharp Baer–Suzuki theorem for the π-radical: exeptional groups of small rank. *Algebra and Logic*, 2023, vol. 62, no. 1, pp. 1–21. https://doi.org/10.1007/s10469-023-09720-3

17. Yang N., Revin D.O., Vdovin E.P. Baer–Suzuki theorem for the π-radical. *Israel J. Math.*, 2021, vol. 245, no. 1, pp. 173–207. https://doi.org/10.1007/s11856-021-2209-y
18. Yang N., Wu Zh., Revin D.O. On the sharp Baer–Suzuki theorem for the π-radical: sporadic groups. *Siberian Math. J.*, 2022, vol. 63, no. 2, pp. 387–394. https://doi.org/10.1134/S0037446622020161
19. Yang N., Wu Zh., Revin D.O., Vdovin E.P. On the sharp Baer–Suzuki theorem for the π-radical of a finite group. *Sb. Math.*, 2023, vol. 214, no. 1, pp. 108–147. https://doi.org/10.4213/sm9698e

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