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On Chu Spaces over $SS - Act$ Category

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Abstract. We prove the general properties of morphisms of Chu spaces and functors with a value in the category $Chu(SS - Act)$ of Chu spaces over the category $SS - Act$. As a consequence, for the category $Chu(SS - Act)$ the existence of coproducts and some products is proved, monomorphisms and epimorphisms are characterized; in terms of this category the characteristics of separable and complete separable Chu spaces are given.

Keywords: Cartesian closed category, S-Act, Chu spaces, functors, limits

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Научная статья

О пространствах Чу над категорией $SS - Act$

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Аннотация. Доказываются общие свойства морфизмов пространств Чу и функторов со значением в категории $Chu(SS - Act)$ пространств Чу над категорией $SS - Act$. В качестве следствий доказывается существование копроизведений, некоторых произведений, характеризуются мономорфизмы и эпиморфизмы категории $Chu(SS - Act)$; в терминах этой категории даются характеристики отделимых и полных отделимых пространств Чу.

Ключевые слова: декартова замкнутая категория, S-полигоны, пространства Чу, функторы, пределы

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1. Introduction

A left S–act, or simply S-act, over monoid S is a set A upon which S acts unitarily on the left. A mapping $f : A \longrightarrow B$ is called homomorphism of S-acts if $f(sa) = sf(a)$ for any $a \in A$, $s \in S$, $f(sa) = sf(a)$ [2].

The category whose objects are S-acts, morphisms are homomorphisms of S-acts, and the composition of morphisms is defined as a superposition of the corresponding maps, is denoted by $S - Act$, so that $Ob(S - Act)$ is the class of all S-acts, $Hom_{S-Act}(A, B)$ is the set of all homomorphisms from S-act A to S-act B, the units of the $S - Act$ category are the identical mappings $1_A \in Hom_{S-Act}(A, A)$.

The monoids action on sets occurs in various situations. General properties of S-acts are actively studied, as well as classes of S-acts with specific properties [2; 3; 7; 8]. S-acts are special cases of presheaves of sets on categories and are therefore related to Grothendieck toposes theory. This makes it possible to obtain results about S-acts, as special cases of the results about the presheaves [5].

In [9], the category $Chu(\mathcal{V})$ is introduced. Its objects are Chu spaces $r : A \otimes X \longrightarrow D$, a morphism from r to $r' : A' \otimes X' \to D'$ is an arbitrary triple (f, g, h) of morphisms $f : A \to A'$, $g : X' \to X$, $h : D \to D'$ in the category $\mathcal{V}$ such that $h \circ r \circ (1_A \otimes g) = r' \circ (f \otimes 1_{X'})$. In [9–11], this category was studied for the case when $\mathcal{V}$ is the $S - Act$ category, S is a commutative monoid and the product is a tensor product. In this paper, we study the category $Chu(SS - Act)$ where $SS - Act$ is a category introduced in [6] and is an extension of the category $S - Act$.

2. Preliminary Results

Define the category $SS - Act$ as follows [6]: $Ob(SS - Act) = Ob(S - Act)$, $Hom_{SS-Act}(A, B) = Hom_{S-Act}(S \times A, B)$, the composition of morphisms $u \in Hom_{SS-Act}(A, B)$ and $v \in Hom_{SS-Act}(B, C)$ is defined by equality $(u \cdot v)(s, a) = v(s, u(s, a))$, where $s \in S$, $a \in A$, the identity morphisms in $SS - Act$ are the morphisms $e_A \in Hom_{SS-Act}(A, A)$ where $e_A(s, a) = a$.

The Chu space over the $SS - Act$ category is defined as the set (A, X, D, r), where $A, X, D \in Ob(SS - Act)$, $r \in Hom_{SS-Act}(A \times X, D)$. If this is not confusing, for Chu space (A, X, D, r) we will use the notation $r \in Hom_{SS-Act}(A \times X, D)$.

In accordance with the general definitions [1], we define the category $Chu(SS - Act)$. Let $r \in Hom_{SS-Act}(A \times X, D)$, $r' \in Hom_{SS-Act}(A' \times X', D')$. A morphism or a Chu transform of r into r' is a triple (f, g, h) of $f \in Hom_{SS-Act}(A, A')$, $g \in Hom_{SS-Act}(X', X)$, $h \in Hom_{SS-Act}(D, D')$ such that $h \cdot r \cdot (e_A \times g) = r' \cdot (f \times e_{X'})$. In this case we will write $(f, g, h) : r \to r'$. If $(f', g', h') : r' \to r''$ then the composition of Chu transforms is defined as follows:

$$(f', g', h') \circ (f, g, h) = (f' \cdot f, g \cdot g', h' \cdot h) : r \to r''.$$

3. Main Lemma

If $u : S \times A \to B$ is a homomorphism of S-acts, $t \in S$ then the mapping $tu : S \times A \to B$, given by the equality $(tu)(s, a) = u(st, a)$, is also a homomorphism, so that the set $Hom_{SS-Act}(A, B)$ is endowed with the S-act structure. Similarly to set mappings, we introduce the notation: $Hom_{SS-Act}(A, B) = B^A$.

In [6] it is proved that the category $SS - Act$ is Cartesian closed, i.e., the functors $Hom_{SS-Act}(\bullet \times \bullet, \bullet)$ and $Hom_{SS-Act}(\bullet, \mathcal{H}^{SS}(\bullet, \bullet))$ are isomorphic for some functor $\mathcal{H}^{SS} : (SS - Act)^o \times (SS - Act) \to SS - Act$.

Let us define the functor $\mathcal{H}^{SS}$. If $A, B, A', B' \in Ob(SS - Act)$, $f \in Hom_{SS-Act}(A', A)$, $g \in Hom_{SS-Act}(B, B')$ then

$$\mathcal{H}^{SS}(A, B) = B^A = Hom_{SS-Act}(A, B)$$

and the mapping $g^f = \mathcal{H}^{SS}(f, g) \in Hom_{SS-Act}(B^A, B'^{A'})$ is defined as follows:

$$g^f(s, w) = \mathcal{H}^{SS}(f, g)(s, w) = (sg) \cdot w \cdot (sf),$$

where $(s, w) \in S \times B^A$.

Denote by $p_{A,X,D} : Hom_{SS-Act}(A \times X, D) \to Hom_{SS-Act}(A, D^X)$ a mapping such that $((p_{A,X,D}(r))(s, a))(t, x) = r(ts, (ta, x))$, where $s, t \in S$, $a \in A$, $x \in X$, $r \in Hom_{SS-Act}(A \times X, D)$.

In [6] it is proved that every mapping $p_{A,X,D}$ is bijective and

$$(*) \qquad p_{A',X',D'}(h \cdot r \cdot (v \times g)) = h^g \cdot p_{A,X,D}(r) \cdot v$$

for all $v \in Hom_{SS-Act}(A', A)$, $g \in Hom_{SS-Act}(X', X)$, $h \in Hom_{SS-Act}(D, D')$, $r \in Hom_{SS-Act}(A \times X, D)$. Thus the family of mappings

$$P^{SS} = \{p_{A,X,D} \mid A, X, D \in Ob(SS - Act)\}$$

is an isomorphism of functors

$$P^{SS} : Hom_{SS-Act}(\bullet \times \bullet, \bullet) \to Hom_{SS-Act}(\bullet, \mathcal{H}^{SS}(\bullet, \bullet)).$$

The equality $(*)$ is equivalent to

$(**)$

$p_{A,X',D'}(h \cdot r \cdot (e_A \times g)) = h^g \cdot p_{A,X,D}(r)$ and $p_{A',X,D}(r \cdot (v \times e_X)) = p_{A,X,D}(r) \cdot v$.

For $r \in Hom_{SS-Act}(A \times X, D)$, we introduce the notation:

$$\hat{r} = p_{A,X,D}(r).$$

By $r_{XD} \in Hom_{SS-Act}(D^X \times X, D)$ we also denote the Chu space such that

$$p_{D^X,X,D}(r_{XD}) = \widehat{r_{XD}} = e_{D^X},$$

where $e_{D^X} : S \times D^X \to D^X$ is a unit in the category $SS - Act$. Note that $\hat{r} \in Hom_{SS-Act}(A, D^X)$.

Lemma 1. *(main Lemma)*

1) Let $f \in Hom_{SS-Act}(A, A')$, $g \in Hom_{SS-Act}(X', X)$, $h \in Hom_{SS-Act}(D, D')$, $r \in Hom_{SS-Act}(A \times X, D)$, $r' \in Hom_{SS-Act}(A' \times X', D')$. *Then*

(a) $(f, g, h) \in Hom_{Chu(SS-Act)}(r, r') \Leftrightarrow h^g \cdot p_{A,X,D}(r) = p_{A',X',D'}(r') \cdot f \Leftrightarrow$

$$\Leftrightarrow h^g \cdot \hat{r} = \hat{r'} \cdot f;$$

(*b*) $$(h^g, g, h) \in Hom_{Chu(SS-Act)}(r_{XD}, r_{X'D'});$$

(*c*) $$(\hat{r}, e_X, e_D) \in Hom_{Chu(SS-Act)}(r, r_{XD});$$

(*d*)
$$(f, g, h) \in Hom_{Chu(SS-Act)}(r, r_{X'D'}) \Leftrightarrow f = h^g \cdot p_{A,X,D}(r) \Leftrightarrow f = h^g \cdot \hat{r},$$

in particular,

(*e*) $$(f, g, h) = (h^g, g, h) \circ (\hat{r}, e_X, e_D).$$

2) For all $w \in Hom_{SS-Act}(A, D^X)$, *we have* $p_{A,X,D}(r_{XD} \cdot (w \times e_X)) = w$.

3) There is equality

(*f*) $$r = r_{XD} \cdot (\hat{r} \times e_X)$$

and for $w \in Hom_{SS-Act}(A, D^X)$, $r = r_{XD} \cdot (w \times e_X)$, *we have* $w = \hat{r}$.

4) For all $w \in Hom_{SS-Act}(A, D^X)$ *the following conditions are equivalent:*

(*g*) $$p_{A,X,D}(r) = w;$$

(*h*) $$(w, e_X, e_D) \in Hom_{Chu(SS-Act)}(r, r_{XD});$$

(*i*) $$r = r_{XD} \cdot (w \times e_X).$$

Proof. Let us prove 1). (*a*) By definition of morphisms of Chu spaces, we have

$$(f, g, h) \in Hom_{Chu(SS-Act)}(r, r') \Leftrightarrow h \cdot r \cdot (e_A \times g) = r' \cdot (f \times e_{X'}).$$

Since $p_{A,X',D'}$ is bijective then

$$h \cdot r \cdot (e_A \times g) = r' \cdot (f \times e_{X'}) \Leftrightarrow p_{A,X',D'}(h \cdot r \cdot (e_A \times g)) = p_{A,X',D'}(r' \cdot (f \times e_{X'})).$$

By $(**)$, we have

$p_{A,X',D'}(h \cdot r \cdot (e_A \times g)) = h^g \cdot p_{A,X,D}(r) = h^g \cdot \hat{r}$,

$p_{A,X',D'}(r' \cdot (f \times e_{X'})) = p_{A',X',D'}(r') \cdot f = \hat{r'} \cdot f$.

Hence the desired result is obtained.

(*b*) Since $p_{X^D,X,D}(r_{XD}) = e_{D^X}$ and $p_{D'^{X'},X',D'}(r_{X'D'}) = e_{D'^{X'}}$ then by (*a*) we have

$$(f, g, h) \in Hom_{Chu(SS-Act)}(r_{XD}, r_{X'D'})$$

if and only if $h^g \cdot p_{D^X,X,D}(r_{XD}) = p_{D'^{X'},X',D'}(r_{X'D'}) \cdot f$ if and only if $h^g = f$.

(c) By (a), we have $(f, e_X, e_D) \in Hom_{Chu(SS-Act)}(r, r_{XD})$ if and only if $p_{A,X,D}(r) = p_{D^X,S,D}(r_{XD}) \cdot f$ if and only if $\hat{r} = f$.

(d), (e) Since $p_{D'^{X'},X',D'}(r_{X'D'}) = e_{D'^{X'}}$ then by (a) we have

$$(f, g, h) \in Hom_{Chu(SS-Act)}(r, r_{X'D'})$$

if and only if $h^g \cdot p_{A,X,D}(r) = p_{A',X',D'}(r_{X'D'}) \cdot f$ if and only if $h^g \cdot p_{A,X,D}(r) = f$. By (b) and (c), we have

$(h^g, g, h) \in Hom_{Chu(SS-Act)}(r_{XD}, r_{X',D'})$,

$(\hat{r}, e_X, e_D) \in Hom_{Chu(SS-Act)}(r, r_{XD})$.

Hence $(h^g, g, h) \circ (\hat{r}, e_X, e_D) = (h^g \cdot \hat{r}, g \cdot e_X, e_D \cdot h) = (f, g, h)$.

Let us prove 2). By $(**)$, we have $p_{A',X,D}(r \cdot (v \times e_X)) = p_{A,X,D}(r) \cdot v$ for all $v \in Hom_{SS-Act}(A', A)$. If $v = w \in Hom_{SS-Act}(A, D^X)$ and $r = r_{XD}$ then $p_{A,X,D}(r_{XD} \cdot (w \times e_X)) = p_{D^X,X,D}(r_{XD}) \cdot w = w$.

Let us prove 3). Let $r \in Hom_{SS-Act}(A \times X, D)$. If $w = p_{A,X,D}(r) = \hat{r}$ in 2) then $p_{A,X,D}(r_{XD} \cdot (\hat{r} \times e_X)) = \hat{r} = p_{A,X,D}(r)$. Since $p_{A,X,D}$ is an injective, then $r_{XD} \cdot (\hat{r} \times e_X) = r$.

If $r_{XD} \cdot (w \times e_X) = r_{XD} \cdot (\hat{r} \times e_X)$ for some $w \in Hom_{SS-Act}(A, D^X)$ then by 2) we have $w = p_{A,X,D}(r_{XD} \cdot (w \times e_X)) = p_{A,X,D}(r_{XD} \cdot (\hat{r} \times e_X)) = \hat{r}$.

Let us prove 4). The equivalence of the conditions (g) and (i) is proved in 2), and the equivalence of the conditions (g) and (h) is verified by proving (c). □

4. Monomorphisms and epimorphisms in the category $Chu(SS-Act)$

Let us give conditions characterizing epimorphisms and monomorphisms in the category $Chu(SS-Act)$.

Theorem 1. *Let $r \in Hom_{SS-Act}(A \times X, D)$ and $r' \in Hom_{SS-Act}(A' \times X', D')$. Then a morphism $(f, g, h) \in Hom_{Chu(SS-Act)}(r, r')$ is an epimorphism if and only if $f \in Hom_{SS-Act}(A, A')$ is an epimorphism, $g \in Hom_{SS-Act}(X', X)$ is a monomorphism and $h \in Hom_{SS-Act}(D, D')$ is an epimorphism.*

Proof. **Necessity.** Let $(f, g, h) : r \to r'$ is an epimorphism in the category $Chu(SS-Act)$.

We will show that f is an epimorphism in the category $SS-Act$. Let $f_1, f_2 \in Hom_{SS-Act}(A', E)$ such that $f_1 \cdot f = f_2 \cdot f$. It is necessary to prove the equality $f_1 = f_2$. Define

$$w \in Hom_{SS-Act}((E \times A') \times X', D')$$

and $(f'_1, e_{X'}, e_{D'}), (f'_2, e_{X'}, e_{D'}) \in Hom_{Chu(SS-Act)}(r', w)$ as follows:

$$w(s, ((e, a'), x')) = r'(s, (a', x')),$$
$$f'_1(s, a') = (f_1(s, a'), a'),$$
$$f'_2(s, a') = (f_2(s, a'), a')$$

for all $s \in S$, $a' \in A'$, $x' \in X'$, $e \in E$. It is not difficult to understand that the definition of Chu transforms $(f'_1, e_{X'}, e_{D'})$, $(f'_2, e_{X'}, e_{D'})$ is well defined and equality $(f'_1, e_{X'}, e_{D'}) \cdot (f, g, h) = (f'_2, e_{X'}, e_{D'}) \cdot (f, g, h)$ is true. Since (f, g, h) is an epimorphism in the category $Chu(SS - Act)$ then $f'_1 = f'_2$. Thus, $f_1 = f_2$ and f is an epimorphism in the category $SS - Act$.

Now we will show that g is a monomorphism and h is an epimorphism in the category $SS - Act$. Let

$$g_1, g_2 \in Hom_{SS-Act}(E, X'), h_1, h_2 \in Hom_{SS-Act}(D', F)$$

such that $g \cdot g_1 = g \cdot g_2$ and $h_1 \cdot h = h_2 \cdot h$. It is necessary to prove the equality $g_1 = g_2$ and $h_1 = h_2$. Define $r_2, r_2 \in Hom_{SS-Act}((A' \times E) \times F$ by equalities $\hat{r}_1 = h_1^{g_1} \cdot \hat{r'}$, $\hat{r}_2 = h_2^{g_2} \cdot \hat{r'}$. By Lemma 1, $h^g \cdot \hat{r} = \hat{r'} \cdot f$. Hence

$$\hat{r_1} \cdot f = h_1^{g_1} \cdot \hat{r'} \cdot f = h_1^{g_1} \cdot h^g \cdot \hat{r} = (h_1 \cdot h)^{(g \cdot g_1)}\hat{r} =$$
$$= (h_2 \cdot h)^{(g \cdot g_2)}\hat{r} = h_2^{g_2} \cdot h^g \cdot \hat{r} = h_2^{g_2} \cdot \hat{r'} \cdot f = \hat{r_2} \cdot f,$$

that is $\hat{r_1} \cdot f {=} \hat{r_2} \cdot f$.

Since f is an epimorphism in the category $SS - Act$ then $\hat{r_1} = \hat{r_2}$. Therefore $r_1 = r_2 = r_o : A' \times E \to F$, $\hat{r_o} : A' \to F^E$. By Lemma 1(a) the equality $\hat{r_1} = \hat{r_o}$, or equivalent equality $h_1^{g_1} \cdot \hat{r'} = \hat{r_o} \cdot e_{A'}$, means that $(e_{A'}, g_1, h_1) : r' \to r_o$ is a homomorphism of Chu spaces. Similarly, $(e_{A'}, g_2, h_2) : r' \to r_o$ is a homomorphism of Chu spaces too. By the definition of composition of Chu spaces morphisms, we have

$$(e_{A'}, g_1, h_1) \circ (f, g, h) = (e_{A'} \cdot f, g \cdot g_1, h_1 \cdot h) =$$
$$= (e_{A'} \cdot f, g \cdot g_2, h_2 \cdot h) = (e_{A'}, g_2, h_2) \circ (f, g, h).$$

Since (f, g, h) is an epimorphism in the category $SS - Act$ then

$$(e_{A'}, g_1, h_1) = (e_{A'}, g_2, h_2),$$

so $g_1 = g_2$, $h_1 = h_2$. Thus, g is a monomorphism and h is an epimorphism.

Sufficiency follows directly from the definition of the composition of morphisms of Chu spaces. □

Theorem 2. *Let $r \in Hom_{SS-Act}(A \times X, D)$ and $r' \in Hom_{SS-Act}(A' \times X', D')$. Then a morphism $(f, g, h) \in Hom_{Chu(SS-Act)}(r, r')$ is a monomorphism if and only if $f \in Hom_{SS-Act}(A, A')$ is a monomorphism, $g \in Hom_{SS-Act}(X', X)$ is an epimorphism and $h \in Hom_{SS-Act}(D, D')$ is a monomorphism.*

Proof. **Necessity.** Let $(f,g,h) \in Hom_{Chu(SS-Act)}(r,r')$ is a monomorphism.

We will show that h is a monomorphism in the category $SS-Act$. Let $h_1, h_2 \in Hom_{SS-Act}(E,D)$ such that $h \cdot h_1 = h \cdot h_2$. It is necessary to prove the equality $h_1 = h_2$. Define $w \in Hom_{SS-Act}(A \times X, E \sqcup D)$ and (e_A, e_X, h'_1), $(e_A, e_X, h'_2) \in Hom_{Chu(SS-Act)}(w,r)$ as follows:

$$w(s,(a,x)) = r(s,(a,x)),$$
$$h'_1(s,e) = h_1(s,e), h'_2(s,e) = h_2(s,e),$$
$$h_1(s,d) = h_2(s,d) = d$$

for all $s \in S, a \in A, x \in X, e \in E, d \in D$. It is not difficult to understand that the definition of Chu transforms (e_A, e_X, h'_1), (e_A, e_X, h'_2) is well defined. Since $h \cdot h_1 = h \cdot h_2$ then $(f,g,h) \cdot (e_A, e_X, h'_1) = (f,g,h) \cdot (e_A, e_X, h'_2)$. Since (f,g,h) is a monomorphism in the category $Chu(SS-Act)$ then $h'_1 = h'_2$. Thus, $h_1 = h_2$.

Now we will show that g is an epimorphism in the category $SS-Act$. By Lemma 2 [11], it enough to show that the morphism $\bar{g} : S \times X' \to S \times X$ is epimorphism in the category $S-Act$, where $\bar{g}(s,x') = (s, g(s,x'))$ for $s \in S$, $x' \in X'$. Assume the converse, i.e., $X_1 \neq S \times X$, where $X_1 = \bar{g}(S \times X')$. By X_0 we denote the Rees factor act of S-act $S \times X$ by the Rees congruence ρ_{X_1}. Define

$$w \in Hom_{SS-Act}(A \times (X_0 \times X), D)$$

and

$$(e_A, g_1, e_D), (e_A, g_2, e_D) \in Hom_{Chu(SS-Act)}(w,r)$$

as follows:

$$w(s,(a,(x_0,x))) = r(s,(a,x)),$$
$$g_1(s,x) = (X_1, x),$$
$$g_2(s,x) = ((s,x)/\rho_{X_1}, x),$$

for all $s \in S$, $a \in A$, $x \in X$, $x_0 \in X_0$. Obviously $g_1 \neq g_2$. From the definition of the Chu space w follow the definitions of the Chu transforms (e_A, g_1, e_D), (e_A, g_2, e_D) are well defined. It is not difficult to understand that $g_1 \cdot g = g_2 \cdot g$. Hence $(f,g,h) \cdot (e_A, g_1, e_D) = (f,g,h) \cdot (e_A, g_2, e_D)$. Since (f,g,h) is a monomorphism in the category $Chu(SS-Act)$ then $g_1 = g_2$, contradiction. Thus, g is an epimorphism in the category $SS-Act$.

Finaly we will show that f is a monomorphism in the category $SS-Act$. Let $f_1, f_2 \in Hom_{SS-Act}(E,A)$ such that $f \cdot f_1 = f \cdot f_2$. It is necessary to prove the equality $f_1 = f_2$. Define

$$r_1, r_2 \in Hom_{SS-Act}(E \times X', D)$$

as follows:

$$r_1(s,(e,x')) = r(s,(f_1(s,e),g(s,x'))),$$
$$r_2(s,(e,x')) = r(s,(f_2(s,e),g(s,x')))$$

for all $x' \in X'$, $s \in S$, $e \in E$. Since $(f,g,h) \in Hom_{Chu(SS-Act)}(r,r')$ then

$$(h \cdot r_1)(s,(e,x')) = h(s,r_1(s,(e,x'))) = h(s,r(s,(f_1(s,e),g(s,x')))) = \\ = r'(s,(f(s,f_1(s,e)),x')) = r'(s,((f \cdot f_1)(s,e),x'))$$

for all $x' \in X'$, $s \in S$, $e \in E$. Similarly,

$$(h \cdot r_2)(s,(e,x')) = r'(s,((f \cdot f_2)(s,e)),x')$$

for all $x' \in X'$, $s \in S$, $e \in E$. Since $f \cdot f_1 = f \cdot f_2$ then

$$(h \cdot r_1)(s,(e,x')) = (h \cdot r_2)(s,(e,x')),$$

i.e., $h \cdot r_1 = h \cdot r_2$. A morphism h is a monomorphism in the category $SS-Act$. Hence $r_1 = r_2$, i.e., $r(s,(f_1(s,e),g(s,x'))) = r(s,(f_2(s,e),g(s,x')))$ for all $x' \in X'$, $s \in S$, $e \in E$. Let $x \in X$, $s \in S$, $e \in E$.

We will prove the equality $r(s,(f_1(s,e),x)) = r(s,(f_2(s,e),x))$. Since g is an epimorphism in the category $SS-Act$, then by Lemma 2 [11], g_s is surjective and $x = g(s,x')$ for some $x' \in X'$. Then $r(s,(f_1(s,e),x)) = r(s,(f_1(s,e),g(s,x'))) = r(s,(f_2(s,e),g(s,x'))) = r(s,(f_2(s,e),x))$. Define $w \in Hom_{SS-Act}(E \times X, D)$ as follows: $w(s,(e,x)) = r(s,(f_1(s,e),x))$. Since

$$r(s,(f_2(s,e),x)) = r(s,(f_1(s,e),x)) = w(s,(e,x)) = e_D(s,w(s,(e,e_X(s,x))))$$

then (f_1,e_X,e_D), $(f_2,e_X,e_D) \in Hom_{Chu(SS-Act)}(w,r)$. Obviously,

$$(f,g,h) \cdot (f_1,e_X,e_D) = (f,g,h) \cdot (f_2,e_X,e_D).$$

Since (f,g,h) is a monomorphism in the category $Chu(SS-Act)$ then $f_1 = f_2$.

Sufficiency follows directly from the definition of the composition of morphisms of Chu spaces. □

5. Separable Chu space

The Chu space $r \in Hom_{SS-Act}(A \times X, D)$ is is called *separable* (*complete separable*) if $\hat{r} = p_{A,X,D}(r) \in Hom_{SS-Act}(A, D^X)$ is a monomorphism (isomorphism) in the category $SS-Act$.

Proposition 1. *(on separable and complete separable Chu spaces)*

1) For Chu space $r \in Hom_{SS-Act}(A \times X, D)$, the following conditions are equivalent:

(a) r is separable;

(b) $(\hat{r}, e_X, e_D)$ is a monomorphism in the category $Chu(SS-Act)$, where $(\hat{r}, e_X, e_D) \in Hom_{Chu(SS-Act)}(r, r_{XD})$;

(b') there exists a monomorphism $w \in Hom_{SS-Act}(A, D^X)$ in the category $SS-Act$ such that $(w, e_X, e_D) \in Hom_{Chu(SS-Act)}(r, r_{XD})$;

(c) there exists a morphism $(f, g, h) \in Hom_{Chu(SS-Act)}(r, r_{X,D})$ such that f is a monomorphism in the category $SS-Act$.

2) Let $(f, g, h) \in Hom_{Chu(SS-Act)}(r, r')$. If f is monomorphism in the category $SS-Act$ and r' is a separable Chu space then r is a separable Chu space.

3) For Chu space $r \in Hom_{SS-Act}(A \times X, D)$, the following conditions are equivalent:

(d) r is complete separable;

(e) $(\hat{r}, e_X, e_D)$ is an isomorphism in the category $Chu(SS-Act)$, where $(\hat{r}, e_X, e_D) \in Hom_{Chu(SS-Act)}(r, r_{XD})$;

(f) r is isomorphic to $r_{X',D'}$ for some $X', D' \in Ob(SS-Act)$.

Proof. Let us prove 1). $(a) \Rightarrow (b)$ By Lemma 1(c), $(p_{A,X,D}(r), e_X, e_D) \in Hom_{Chu(SS-Act)}(r, r_{XD})$. Since $p_{A,X,D}(r)$ is a monomorphism and e_X, e_D are isomorphisms in the category $SS-Act$, then $(p_{A,X,D}(r), e_X, e_D)$ is a monomorphism in the category $Chu(SS-Act)$.

$(b) \Rightarrow (b')$ Since $(\hat{r}, e_X, e_D) \in Hom_{Chu(SS-Act)}(r, r_{XD})$ is a monomorphism in the category $Chu(SS-Act)$, then by Theorem 2, $\hat{r}$ is a monomorphism in the category $SS-Act$. Assuming $w = \hat{r}$, we get (b').

$(b') \Rightarrow (c)$ Obviously.

$(c) \Rightarrow (a)$ By Lemma 1(d), we have $h^g \cdot \hat{r} = f$. Since f is a monomorphism, then $\hat{r}$ is a monomorphism too. Thus, r is a separable Chu spaces.

Let us prove 2). Let $r' \in Hom_{SS-Act}(A' \times X', D')$. Since r' is a separable Chu spaces, then by (b'), there exists $w' \in Hom_{SS-Act}(A', D'^{X'})$ such that $(w', e_{X'}, e_{D'}) \in Hom_{Chu(SS-Act)}(r', r_{X'D'})$. Then

$$(w' \cdot f, g, h) \in Hom_{Chu(SS-Act)}(r, r_{X'D'}).$$

By Lemma 1(d), we have $w' \cdot f = h^g \cdot \hat{r}$. Since $w' \cdot f$ is a monomorphism, then $\hat{r}$ is a monomorphism too. Thus, r is a separable Chu spaces.

Let us prove 3). $(d) \Rightarrow (e)$ By Lemma 1(c), we have

$$(\hat{r}, e_X, e_D) \in Hom_{Chu(SS-Act)}(r, r_{XD}).$$

Since r is a complete separable Chu spaces, it follows that $\hat{r}$, e_X, e_D are isomorphisms. Hence $(\hat{r}, e_X, e_D)$ is an isomorphism in the category $Chu(SS-Act)$.

$(e) \Rightarrow (f)$ Obviously.

$(f) \Rightarrow (d)$ Let $(f,g,h) \in Hom_{Chu(SS-Act)}(r, r_{XD})$ is an isomorphism in the category $Chu(SS-Act)$. Then f, g and h are isomorphisms in the category $SS-Act$. Therefore h^g is an isomorphisms. By Lemma 1(d), we have $f = h^g \cdot \hat{r}$. Thus, $\hat{r}$ is an isomorphisms, i.e., r is a complete separable Chu spaces. □

6. Functors with values in the category $Chu(SS-Act)$

Consider the following functors: $P_1 : Chu(SS-Act) \to (SS-Act)$, $P_2 : Chu(SS-Act) \to (SS-Act)^o$, $P_3 : Chu(SS-Act) \to (SS-Act)$, $P_{23} : Chu(SS-Act) \to (SS-Act)^o \times (SS-Act)$ that map each Chu space $r \in Hom_{SS-Act}(A \times X, D)$ to the objects

$$P_1(A,X,D,r) = A,\ P_2(A,X,D,r) = X,$$

$$P_3(A,X,D,r) = D,\ P_{23}(A,X,D,r) = (X,D)$$

and each morphism $(f,g,h) \in Hom_{Chu(SS-Act)}(r,r')$ to the morphisms

$$P_1(f,g,h) = f,\ P_2(f,g,h) = g,\ P_3(f,g,h) = h,\ P_{23}(f,g,h) = (g,h).$$

The fact that these are functors directly follows from the definition of composition of Chu morphisms.

Let Z be a category, $F : Z \to Chu(SS-Act)$ be a functor. By F_1, F_2, F_3, $F_{23} = (F_2, F_3)$ we denote the functors acting as coordinates of F:

$$F_1 = P_1 \circ F : Z \to (SS-Act),\ F_2 = P_2 \circ F : Z \to (SS-Act)^o,$$

$$F_3 = P_3 \circ F : Z \to (SS-Act),\ \ F_{23} = P_{23} \circ F : Z \to (SS-Act)^o \times (SS-Act).$$

Theorem 3. *(on functors in $Chu(SS-Act)$)*

1) Let $F : Z \to Chu(SS-Act)$ be a functor. Then there are uniquely defined functors $F_1 : Z \to SS-Act$, $F_2 : Z \to (SS-Act)^o$, $F_3 : Z \to SS-Act$ such that for any object $z \in Ob(Z)$ and any morphism $a \in Hom_Z(z,z')$, we have

$$F(z) = (F_1(z), F_2(z), F_3(z), r(z)),$$

where $r(z) \in Hom_{SS-Act}(F_1(z) \times F_2(z), F_3(z))$, and

$$F(a) = (F_1(a), F_2(a), F_3(a)) \in Hom_{Chu(SS-Act)}(r(z), r(z')).$$

2) Let $F_1 : Z \to (SS-Act)$, $F_2 : Z \to (SS-Act)^o$, $F_3 : Z \to (SS-Act)$ be the functors and for any $z \in Ob(Z)$, the morphism $r(z) \in Hom_{SS-Act}(F_1(z) \times F_2(z), F_3(z))$ is fixed. Then the following conditions are equivalent:

(a) the mapping set by the equalities

$$F(z) = (F_1(z), F_2(z), F_3(z), r(z)),\ F(a) = (F_1(a), F_2(a), F_3(a)) \quad (1)$$

is a functor $F : Z \to Chu(SS{-}Act)$, where $z \in Ob(Z)$ and $a \in Hom_Z(z, z')$;

(b) for any $a \in Hom_Z(z, z')$ we have

$$(F_1(a), F_2(a), F_3(a)) \in Hom_{Chu(SS-Act)}(r(z), r(z')). \quad (2);$$

(c) for any $a \in Hom_Z(z, z')$ we have

$$F_3(a)^{F_2(a)} \cdot \widehat{r(z)} = \widehat{r(z')} \cdot F_1(a); \quad (3);$$

(d) the family

$$W = \{W(z) = \widehat{r(z)} \in Hom_{SS-Act}(F_1(z), F_3(z)^{F_2(z)}) \mid z \in Ob(Z)\}$$

is a homomorphism of functors $W : F_1 \to F_3^{F_2} = \mathcal{H}^{SS} \circ (F_2, F_3)$.

Proof. Let us prove 1). Since $F(z)$ is a Chu space, it follows that $F(z) = (A, X, D, r(z))$ for some $r(z) \in Hom_{SS-Act}(A \times X, D)$. So, by the notations above, $A = P_1(F(z)) = F_1(z)$, $X = P_2(F(z)) = F_2(z)$, $D = P_3(F(z)) = F_3(z)$. If $a \in Hom_Z(z, z')$ then

$$F(a) = (f, g, h) \in Hom_{Chu(SS-Act)}(r(z), r(z')).$$

Hence $f = P_1(F(a)) = F_1(a)$, $g = P_2(F(a)) = F_2(a)$, $h = P_3(F(a)) = F_3(a)$. Thus, $F(a) = (F_1(a), F_2(a), F_3(a))$.

Let us prove 2). $(a) \Rightarrow (b)$ Obviously.

$(b) \Rightarrow (a)$ Since $r(z) \in Hom_{SS-Act}(F_1(z) \times F_2(z), F_3(z))$, it follows that $F(z) \in Ob(Chu(SS - Act))$. By (2), we have

$$F(a) = (F_1(a), F_2(a), F_3(a)) \in Hom_{Chu(SS-Act)}(r(z), r(z')).$$

Let $b \in Hom_Z(z', z'')$. Since F_1, F_2 and F_3 are functors, then by the definition of composition of Chu transform, we have

$$\begin{aligned} F(b \circ a) &= (F_1(b \circ a), F_2(b \circ a), F_3(b \circ a)) = \\ &= (F_1(b) \cdot F_1(a), F_2(a) \cdot F_2(b), F_3(b) \cdot F_3(a)) = \\ &= (F_1(b), F_2(b), F_3(b)) \circ (F_1(a), F_2(a), F_3(a)) = F(b) \circ F(a), \end{aligned}$$

and $F(e_z) = (F_1(e_z), F_2(e_z), F_3(e_z)) = (1_{F_1(z)}, 1_{F_2(z)}, 1_{F_3(z)}) = 1_{F(z)}$. Thus, F is a functor.

By Lemma 1(a), the conditions (b) and (c) are equivalent.

By definition of a morphism (natural transformation) of functors, the conditions (c) and (d) are equivalent. □

7. Fundamental theorem on the functor H

Consider the following functors:

$$H_1 = \mathcal{H}^{SS} : (SS - Act)^o \times (SS - Act) \to (SS - Act),$$

$$H_2 : (SS - Act)^o \times (SS - Act) \to (SS - Act)^o,$$

$$H_3 : (SS - Act)^o \times (SS - Act) \to (SS - Act),$$

where $H_2(X, D) = X$, $H_2(g, h) = g$, $H_3(X, D) = D$, $H_3(g, h) = h$ for all $X, D \in Ob(SS - Act)$, $g \in Hom_{SS-Act}(X', X)$, $h \in Hom_{SS-Act}(D, D')$.

Then

$$r_{XD} = Hom_{SS-Act}(H_1(X, D) \times H_2(X, D), H_3(X, D)),$$

and by Lemma 1(b), we have

$$(H_1(g, h), H_2(g, h), H_3(g, h)) = (h^g, g, h) \in Hom_{Chu(SS-Act)}(r_{XD}, r_{X'D'}).$$

Therefore the condition (b) of Theorem 3 is true for the category $Z = (SS - Act)^o \times (SS - Act)$ and functors $F_i = H_i$. Hence, by Theorem 3(a), the mapping given by the equalities

$$H(X, D) = (D^X, X, D, r_{XD}), H(g, h) = (h^g, g, h)$$

is a functor $H : (SS - Act)^o \times (SS - Act) \to Chu(SS - Act)$.

Theorem 4. *(on the functor H)*

1) The functor H is full and faithful.

2) Let $F : Z \to Chu(SS - Act)$ be a functor given by the equality (1). There is a canonical functor homomorphism

$$V = \{V(z) \mid z \in Ob(Z)\} : F \to H \circ F_{23},$$

such that $V(z) = (W(z), e_{F_2(z)}, e_{F_3(z)}) \in Hom_{Chu(SS-Act)}(r(z), r_{F_2(z)F_3(z)})$, where $W(z) = \widehat{r(z)} \in Hom_{SS-Act}(F_1(z), F_3(z)^{F_2(z)})$.

3) The functor H is right adjoint for the functor P_{23}.

4) (a) Let $z \in Ob(Z)$. Then
$V(z)$ is a monomorphism $\Leftrightarrow$ $W(z)$ is a monomorphism $\Leftrightarrow$ $F(z)$ is a separable Chu space;
$V(z)$ is an isomorphism $\Leftrightarrow$ $W(z)$ is an isomorphism $\Leftrightarrow$ $F(z)$ is a complete separable Chu space.

(b) The functor homomorphism $V : F \to H \circ F_{23}$ is an isomorphism $\Leftrightarrow$ $F(z)$ is a complete separable Chu space for all $z \in Ob(Z)$.

Proof. 1) Let us show that H is a full and faithful functor, i.e., the mapping

$$Hom_{(SS-Act)^o \times (SS-Act)}((X, D), (X', D')) \to Hom_{Chu(SS-Act)}(r_{XD}, r_{X'D'}),$$

such that $H(g,h) = (h^g, g, h)$, is bijective. The injectivity is obvious. We prove surjectivity. Let $(f,g,h) \in Hom_{Chu(SS-Act)}(r_{XD}, r_{X'D'})$. Since $(h^g, g, h) \in Hom_{Chu(SS-Act)}(r_{XD}, r_{X'D'})$, then by Lemma 1($d$), we have $f = h^g$, i.e., $H(g,h) = (f,g,h)$. Hence, the mapping $H(g,h) \mapsto (h^g, g, h)$ is bijective. Thus, the functor H is full and faithful.

2) There are equalities $(H \circ F_{23})(z) = r_{F_2(z)F_3(z)}$;

$$(H \circ F_{23})(a) = (F_3(a)^{F_2(a)}, F_2(a)F_3(a));$$

$$F(z) = r(z);\ F(a) = (F_1(a), F_2(a), F_3(a)).$$

To prove that V is a functor homomorphism, , it is necessary to prove the equality

$$V(z') \circ F(a) = (H \circ F_{23})(a) \circ V(z) \tag{4}$$

for all $a \in Hom_Z(z, z')$.

Since $V(z') \circ F(a) = (W(z') \cdot F_1(a), e_{F_2(z')} \cdot F_2(a), e_{F_3(z')} \cdot F_3(a))$ and $(H \circ F_{23})(a) \circ V(z) = (F_3(a)^{F_2(a)} \cdot W(z), F_2(a) \cdot e_{F_2(z)}, F_3(a) \cdot e_{F_3(z)})$, then the equality (4) means that the following three equalities are true:

$$W(z') \cdot F_1(a) = F_3(a)^{F_2(a)} \cdot W(z);$$

$$e_{F_2(z')} \cdot F_2(a) = F_2(a) \cdot e_{F_2(z)};\ e_{F_3(z')} \cdot F_3(a) = F_3(a) \cdot e_{F_3(z)}.$$

The first of the equality coincides with equality (3) of Theorem 3 and therefore it is true, the second and the third equalities are obvious.

3) Let us use one of the standard properties of adjoint functors [4], and to do this, we will prove that the anjunction gives an unit and a counit, i.e. there are the functor homomorphisms $\eta : 1_{Chu(SS-Act)} \to H \circ P_{23}$, $\varepsilon : P_{23} \circ H \to 1_{SS-Act^o \times SS-Act}$, such that

$$H\varepsilon \circ \eta H = 1_H,\ \varepsilon P_{23} \circ P_{23}\eta = 1_{P_{23}}, \tag{5}$$

where the functor homomorphisms

$$H\varepsilon : H \circ P_{23} \circ H \to H \text{ and } \eta H : H \to H \circ P_{23} \circ H$$

are defined as follows:

$$(\eta H)(X,D) = \eta(H(X,D)) : H(X,D) \to (H \circ P_{23})(H(X,D)),$$

$$(H\varepsilon)(X,D) = H(\varepsilon(X,D)) : H((P_{23} \circ H)(X,D)) \to H(X,D).$$

Note that $P_{23} \circ H = 1_{(Chu(SS-Act))^o \times Chu(SS-Act)}$.

Define the counit ε of adjunction as the identity homomorphism

$$\varepsilon = \{\varepsilon(X,D) \in Hom_{(SS-Act)^o \times (SS-Act)}((X,D),(X,D)) \mid$$
$$X, D \in Ob(SS-Act)\},$$

$\varepsilon(X, D) = (e_X, e_D)$. Obviously, the functor homomorphisms $H\varepsilon$ and εP_{23} are the identity transformation of functors.

To define the units η of an adjunction we apply the result of 2) to the case $Z = Chu(SS - Act)$ and $F = 1_{Chu(SS-Act)} : Chu(SS - Act) \to Chu(SS - Act)$. Therefore $F_{23} = P_{23}$ and if $r \in Hom_{SS-Act}(A \times X, D)$, then $H \circ P_{23}(r) = H(X, D) = r_{XD}$, $V(r) = (\hat{r}, e_X, e_D) \in Hom_{Chu(SS-Act)}(r, r_{XD})$ and $V : 1_{Chu(SS-Act)} \to H \circ P_{23}$ is a functor homomorphism. Suppose $\eta = V$. Since $\hat{r}_{XD} = e_{D^X}$, it follows that

$$(\eta H)(X, D) = V(r_{XD}) = (e_{D^X}, e_X, e_D) = \\ = 1_H(X, D) \in Hom_{Chu(SS-Act)}(H(X, D), H(X, D))$$

so that $\eta H : H \to H = H \circ P_{23} \circ H$ is the identity transformation of functors. We also have

$$(P_{23}\eta)(r) = P_{23}(\hat{r}, e_X, e_D) = (e_X, e_D) = \\ = 1_{P_{23}(r)} \in Hom_{(SS-Act)^o \times (SS-Act)}(P_{23}(r), P_{23}(r)).$$

Hence $P_{23}\eta : P_{23} \to P_{23} = P_{23} \circ H \circ P_{23}$ is the identity transformation of functors.

Thus, the functor homomorphisms $H\varepsilon$, ηH, εP_{23}, $P_{23}\eta$ are the identity transformation of functors, hence the equalities (5) are hold. Therefore the functor H is right adjoint for the functor P_{23}.

4) directly follows from Theorem 3 and general properties of functor homomorphisms. □

8. Limits, products and coproducts in the category $Chu(SS - Act)$

Theorem 5. *(on limits) Let Z be a category, $F : Z \to Chu(SS-Act)$ be a functor such that $F(z)$ is a complete separable Chu space for all $z \in Ob(Z)$. If every functor $Z \to SS-Act$ has a limit and every functor $Z^o \to SS-Act$ has a colimit, then there exists $limF$ that is complete separable Chu space.*

Proof. By Theorem 3, there are functors $F_1 : Z \to SS - Act$, $F_2 : Z \to (SS-Act)^o$, $F_3 : Z \to SS-Act$ such that $F(z) = (F_1(z), F_2(z), F_3(z), r(z))$, $F(a) = (F_1(a), F_2(a), F_3(a))$, where

$$r(z) \in Hom_{SS-Act}(F_1(z) \times F_2(z), F_3(z)), a \in Hom_Z(z, z').$$

By $F_2^o : Z^o \to SS - Act$ we denote the functor given by the equalities $F_2^o(z) = F_2(z)$ and $F_2^o(a) = F_2(a) \in Hom_{(SS-Act)^o}(F_2(z), F_2(z')) = Hom_{SS-Act}(F_2^o(z'), F_2^o(z))$, where $a \in Hom_{Z^o}(z', z)$. By the conditions of

Theorem, the functor F_2^o has a colimit, and the functor F_3 has a limit, i.e., there are universal cones

$$\varphi_2^o = \{\varphi_2(z) \in Hom_{(SS-Act)^o}(F_2^o(z), X)) \mid z \in Ob(Z)\},$$

$$\varphi_3 = \{\varphi_3(z) \in Hom_{SS-Act}(D, F_3(z)) \mid z \in Ob(Z)\}$$

where the first cone is the colimit, the second cone is the limit. Thus, $X = colimF_2^o$, $D = limF_3$. By properties of dual categories,

$$\varphi_2 = \{\varphi_2(z) \in Hom_{SS-Act}(X, F_2(z)) \mid z \in Ob(Z)\}$$

is the limit cone of the functor F_2 such that $X = limF_2$. Hence, by the properties of the product of categories,

$$\varphi_{23} = \{(\varphi_2(z), \varphi_3(z)) \in Hom_{(SS-Act)^o \times (SS-Act)}((X, D), F_{23}(z)) \mid z \in Ob(Z)\}$$

is the limit cone of the functor $F_{23} = (F_2, F_3)$.

Since the functor H has a left adjoint, it translates the limit cone into the limit cone, i.e., $H(\varphi_{23}) = \{H(\varphi_{23}(z)) \mid z \in Ob(Z)\}$, where $H(\varphi_{23}(z)) = (\varphi_3(z)^{\varphi_2(z)}, \varphi_2(z), \varphi_3(z))$ is the limit cone of the functor $H \circ F_{23}$. In particular, $H(X, D) = r_{XD} = lim(H \circ F_{23})$.

By Theorem 4, there is a canonical functor homomorphism $V : F \to H \circ F_{23}$. Since each $F(z)$ is a complete separable Chu space, then V is an isomorphism of functors. Hence

$$\{V(z)^{-1} \circ H(\varphi_{23}(z)) \in Hom_{Chu(SS-Act)}(r_{XD}, F(z)) \mid z \in Ob(Z)\}$$

is the limit cone of the functor F. Therefore $limF = r_{XD}$. Since $V(z) = (\widehat{r(z)}, 1_{F_2(z)}, 1_{F_3(z)})$ then $V(z)^{-1} = (\widehat{r(z)}^{-1}, 1_{F_2(z)}, 1_{F_3(z)})$, i.e.,

$$V(z)^{-1} \circ H(\varphi_{23})(z) = (\widehat{r(z)}^{-1} \cdot \varphi_3(z)^{\varphi_2(z)}, \varphi_2(z), \varphi_3(z)).$$

□

The proof of Theorem 6 implies the existence of the product in complete separable Chu spaces.

Theorem 6. *Let $r_i \in Hom_{SS-Act}(A_i \times X_i, D_i)$ $(i \in I)$ be the complete separable Chu spaces. The product of Chu spaces r_i, $i \in I$, is the complete separable Chu space $r_{X_0D_0}$ with Chu transforms*

$$((\hat{r}_i)^{-1} \cdot p_i^{q_i}, q_i, p_i) \in Hom_{Chu(SS-Act)}(r_{X_0D_0}, r_i),$$

where $X_0 = \coprod_{i\in I} X_i$, $D_0 = \prod_{i\in I} D_i$, $q_i(s, x_i) = x_i$, $p_i(s, d) = d(i)$ for all $x_i \in X_i$, $d \in \prod_{i\in I} D_i$, $i \in I$.

Proof. Consider a discrete category Z such that objects are elements of the set I. Then the family $\{r_i \in Hom_{SS-Act}(A_i \times X_i, D_i) \mid i \in I\}$ is the same as functor $F : Z \to Chu(SS-Act)$ defined by equality $F(i) = r_i$, and the limit of the functor F is the product of the family. Therefore, the result being proved is a particular case of Theorem 5. □

The following theorem shows that in the category $Chu(SS-Act)$ the coproducts exist for any Chu spaces.

Theorem 7. *Let $r_i \in Hom_{SS-Act}(A_i \times X_i, D_i)$, $i \in I$. The coproduct of the Chu spaces r_i, $i \in I$, is the Chu space*

$$r \in Hom_{SS-Act}(\coprod_{i\in I} A_i \times \prod_{i\in I} X_i, \coprod_{i\in I} D_i)$$

with Chu transforms $(f_i, g_i, h_i) \in Hom_{Chu(SS-Act)}(r_i, r)$, where

$$r(s,(a_i,x)) = r_i(s,(a_i,x(i))),$$
$$f_i(s,a_i) = a_i, g_i(s,x) = x(i),$$
$$h_i(s,d_i) = d_i$$

for all $a_i \in A_i$, $x \in \prod_{i\in I} X_i$, $d_i \in D_i$, $i \in I$.

Proof. Let $i \in I$. The equalities

$$h_i(s,(r_i(s,(a_i,g_i(s,x)))))= r_i(s,(a_i,x(i))) = \\ = r(s,(a_i,x)) = r(s,(f_i(s,a_i),x))$$

for all $a_i \in A_i$, $x \in \prod_{i\in I} X_i$, imply well-definability of the definition of the Chu transform (f_i, g_i, h_i).

Let $t \in Hom_{SS-Act}(B \times Y, D)$, $(f'_i, g'_i, h'_i) \in Hom_{Chu(SS-Act)}(r_i, t)$. By Theorem 4 [6], S-act $\coprod_{i\in I} A_i$ with morphisms f_i, $i \in I$, is a coproduct of S-act A_i $(i \in I)$, S-act $\coprod_{i\in I} D_i$ with morphisms h_i, $i \in I$, is a coproduct of S-acts D_i $(i \in I)$, and S-act $\prod_{i\in I} X_i$ with morphisms g_i, $i \in I$, is a product of S-acts X_i $(i \in I)$ in the category $SS-Act$. Then there are unique morphisms $\tilde{f} \in Hom_{SS-Act}(\coprod_{i\in I} A_i, B)$, $\tilde{g} \in Hom_{SS-Act}(Y, \prod_{i\in I} X_i)$, $\tilde{h} \in Hom_{SS-Act}(\coprod_{i\in I} D_i, D)$ such that $f'_i = \tilde{f} \cdot f_i$, $g'_i = g_i \cdot \tilde{g}$ and $h'_i = \tilde{h} \cdot h_i$ for all $i \in I$, i.e., $\tilde{f}(s,a_i) = f_i(s,a_i)$, $\tilde{g}(s,y)(i) = g'_i(s,y)$ and $\tilde{h}(s,d) = h'_i(s,d)$ for all $a_i \in A_i$, $y \in Y$, $d \in D$, $i \in I$.

Let us prove $(\tilde{f}, \tilde{g}, \tilde{h}) \in Hom_{Chu(SS-Act)}(r,t)$, that is, the equality is hold $\tilde{h}(s, r(s,(a,\tilde{g}(s,y)))) = t(s,(\tilde{f}(s,a),y))$ for all $a \in \coprod_{i\in I} A_i$, $y \in Y$. Since

$(f_i, g_i, h_i) \in Hom_{Chu(SS-Act)}(r_i, r)$, then

$$r(s,(f_i(s,a),\tilde{g}(s,y))) = h_i(s, r_i(s,(a,(g_i \cdot \tilde{g})(s,y))))$$

for all $a \in A_i$, $y \in Y$. Since $h_i' = \tilde{h} \cdot h_i$ and $g_i' = g_i \cdot \tilde{g}$, then

$$\tilde{h}(s, r(s,(a,\tilde{g}(y)))) = \\ = (\tilde{h} \cdot h_i)(s, r_i(s,(a,(g_i \cdot \tilde{g})(s,y)))) = h_i'(s, r_i(s,(a, g_i'(s,y))))$$

for all $a \in A_i$, $y \in Y$. Since $(f_i', g_i', h_i') \in Hom_{Chu(SS-Act)}(r_i, t)$, then $h_i'(s, r_i(s,(a, g_i'(s,y))) = t(s,(f_i'(s,a),y)))$ for all $a \in A_i$, $y \in Y$. Since $\tilde{f}(s,a) = f_i'(s,a)$ then $t(s,(f_i'(s,a),y)) = t(s,(\tilde{f}(s,a),y))$ for all $a \in A_i$, $y \in Y$. Thus,

$$\tilde{h}(s, r(s,(a,\tilde{g}(s,y)))) = t(s,(\tilde{f}(s,a) \times y))$$

for all $a \in \coprod_{i \in I} A_i$, $y \in Y$. □

9. Conclusion

In this paper, we study the category $Chu(SS - Act)$. It is known [6] that the category $SS - Act$ is Cartesian closed and the embedding functor $S - Act \to SS - Act$ has a left adjoint. Using this result, we prove the general properties of morphisms of Chu spaces and functors with a value in the category $Chu(SS - Act)$ of Chu spaces over the category $SS - Act$. As a consequence, for the category $Chu(SS-Act)$ the existence of coproducts and some products is proved, monomorphisms and epimorphisms are characterized; in terms of this category the characteristics of separable and complete separable Chu spaces are given.

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