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On Radon Barycenters of Measures on Spaces of Measures

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Abstract. We study metrizability of compact sets in spaces of Radon measures with the weak topology. It is shown that if all compacta in a given completely regular topological space are metrizable, then every uniformly tight compact set in the space of Radon measures on this space is also metrizable. It is proved that the property that compact sets of measures on a given space are metrizable is preserved for products of this space with spaces that can be embedded into separable metric spaces. In addition, we construct a Radon probability measure on the space of Radon probability measures on a completely regular space such that its barycenter is not a Radon measure.

Keywords: Radon measure, barycenter, metrizable compact set of measures

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Научная статья

О радоновских барицентрах мер на пространствах мер

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Аннотация. Изучается метризуемость компактных множеств в пространствах радоновских мер со слабой топологией. Показано, что если все компакты в данном вполне регулярном топологическом пространстве метризуемы, то всякое равномерно плотное компактное множество в пространстве радоновских мер на этом пространстве также метризуемо. Доказано, что метризуемость компактных множеств мер на данном пространстве сохраняется для произведений этого пространства с пространствами, которые вкладываются в сепарабельные метрические пространства. Кроме того, построен пример радоновской вероятностной меры на пространстве радоновских вероятностных на вполне регулярном пространстве, для которой барицентр не является радоновской мерой.

Ключевые слова: радоновская мера, барицентр, метризуемое компактное множество мер

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1. Introduction

The goal of this paper is two-fold: we study metrizability of compact sets of measures on general spaces and the Radon property of barycenters of measures on spaces of measures. These two questions are connected through the property of uniform tightness of measures, which involves naturally Prohorov spaces in our discussion.

The study of barycenters of measures on spaces of measures is of independent interest, but is also motivated by recent investigations of nonlinear Kantorovich problems of optimal transportation of measures, see [1; 2; 4; 5; 10; 11; 13], and also the recent surveys [8] and [9].

Let us introduce some terminology and notation (see [6]). Throughout X is a completely regular topological space (see [6] or [12]) and $\mathcal{B}(X)$ is its Borel σ-algebra. The space of bounded continuous functions on X is denoted by $C_b(X)$. A nonnegative Borel measure on X (all measures here are bounded) is called Radon if for every $\varepsilon > 0$ there is a compact set K such that $\mu(X\backslash K) < \varepsilon$. A signed measure $\mu = \mu^+ - \mu^-$ is called Radon if its total variation $|\mu| = \mu^+ + \mu^-$ is Radon (equivalently, its positive and negative parts μ^+ and μ^- are Radon). The total variation norm is defined by $\|\mu\| = |\mu|(X)$.

The space of all Radon measures on X is denoted by $\mathcal{M}_r(X)$, the subset of probability measures is denoted by $\mathcal{P}_r(X)$. The space of measures is equipped with the weak topology (see [6] or [7]) generated by all seminorms of the form

$$p_f(\mu) = \left|\int_F f\,d\mu\right|, \quad f \in C_b(X).$$

Throughout compactness is meant in this topology.

A set of measures $M \subset \mathcal{M}_r(X)$ is called uniformly tight if for every $\varepsilon > 0$ there is a compact set such that $|\mu|(X\backslash K) < \varepsilon$ for all $\mu \in M$.

According to the Prohorov theorem, a bounded uniformly tight set of measures has compact closure in the weak topology (see [6, Theorem 8.6.7]).

A space X is called Prohorov if every weakly compact set of Radon probability measures is uniformly tight. For example, all complete metric spaces and all locally compact spaces are Prohorov. On the other hand, there are very simple Souslin spaces that are not Prohorov, for example, the space $\mathbb{Q}$ of rational numbers. The space X is called strongly Prohorov if all compact sets of signed measures are uniformly tight.

The barycenter of a Radon measure μ on a locally convex space E such that all continuous linear functionals on E are μ-integrable is defined as a vector $a \in E$ for which

$$l(a) = \int_E l(x)\,\mu(dx)$$

for every continuous linear functional l (see [6, §7.14(xii)]). We consider a particular case in which E is the space $\mathcal{M}_r(X)$ of Radon measures on a completely regular topological space X and P is a Radon probability measure on the subset $\mathcal{P}_r(X)$ of probability measures. In this case a broader concept of barycenter is used: the barycenter of P is the Borel measure β_P on X defined by the equality

$$\beta_P(B) = \int_{\mathcal{P}_r(X)} p(B)\,P(dp), \quad B \in \mathcal{B}(X).$$

It is known that the function $p \mapsto p(B)$ is Borel measurable on $\mathcal{P}_r(X)$, so the integral is well-defined and β_P is a Borel measure, moreover, the measure β_P is τ-additive (see [6, Proposition 8.9.8 and Corollary 8.9.9]). The measure β_P is Radon if and only if the measure P is concentrated on a countable union of uniformly tight compact sets in $\mathcal{P}_r(X)$ (see [9, Proposition 3.1]). Therefore, such a barycenter need not be an element of the space $E = \mathcal{M}_r(X)$, but may belong to a larger space of Borel measures. However, on many spaces all Borel measures are automatically Radon, for example, this is true for Souslin spaces, but if X is Souslin, then $\mathcal{M}_r(X)$ is also Souslin, hence in this case barycenters are Radon. Another sufficient condition for the existence of a Radon barycenter for all measures in $\mathcal{P}_r(\mathcal{P}_r(X))$ is the Prohorov property of the space X. Note that in [1;4;5] and some other works the term "intensity" is used for barycenters of measures on spaces of measures.

Our first main result gives an example of a Radon measure on the space of Radon probability measures for which the barycenter is not Radon. This result gives a positive answer to the question posed in [9].

Our second main result describes a broad class of spaces X such that all compacta in $\mathcal{P}_r(X)$ are metrizable. In particular, this is true if X is Prohorov and all compacta in X are metrizable. More precisely, we show that if compacta in X are metrizable, then every uniformly tight compact set in $\mathcal{P}_r(X)$ is also metrizable. However, we do not know whether the metrizability of compacta in X implies alone the metrizability of compacta in $\mathcal{P}_r(X)$. Finally, we show that if compacta in $\mathcal{P}_r(X)$ and $\mathcal{P}_r(Y)$ are metrizable, then the same is true for $\mathcal{P}_r(X \times Y)$. A similar result is proved for the whole space of measures $\mathcal{M}_r(X \times Y)$ if Y has a countable family of continuous functions separating points (i.e., can be embedded into a separable metric space).

Of course, a general necessary and sufficient condition for the metrizability of a compact space is the existence of a countable family of continuous functions on this space separating its points. But when we are speaking of metrizability of all compacta in a given space, the assumption that such a sequence exists on the whole space is too strong, so we are interested in other conditions.

2. A non-Radon barycenter

The goal of this section is to construct a Radon measure on the space of Radon probability measures such that its barycenter is not Radon.

Let us consider the product $\mathbb{R}^{[0,1]}$ regarded as the space of functions $\{x\colon [0,1] \to \mathbb{R}\}$ with the standard Tychonoff product topology (see [12]), i.e., the topology of pointwise convergence of functions.

Theorem 1. *There is a Radon probability measure P on the space $\mathcal{P}_r(\mathbb{R}^{[0,1]})$ of Radon probability measures on $\mathbb{R}^{[0,1]}$ such that its barycenter β_P is not a Radon measure.*

Proof. Let δ_a denote Dirac's measure at a. For every function $x\colon [0,1] \to [0,1]$ we take the measure $\mu_x \in \mathcal{P}(\mathbb{R}^{[0,1]})$ defined as follows:

$$\mu_x = \bigotimes_{t\in[0,1]} \nu_{x,t},$$

where $\nu_{x,t} \in \mathcal{P}_r(\mathbb{R})$, $\nu_{x,t} = \begin{cases} (1-x(t))\delta_0 + x(t)\delta_{1/x(t)} \text{ if } x(t) > 0, \\ \delta_0 \text{ if } dx(t) = 0. \end{cases}$

The product-measure μ_x is first defined on the cylindrical σ-algebra of the space $\mathbb{R}^{[0,1]}$, but it extends to a Radon probability measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^{[0,1]})$, because it is tight: the outer measure μ_x^* of the compact set

$$K = \prod_{t\in[0,1]} [0, g(x(t))],$$

where $g(s) = 1/s$ if $s > 0$ and $g(0) = 1$, equals 1; see [6, Section 7.3] about such extensions. The mapping $J\colon x \mapsto \mu_x$ is continuous from $[0,1]^{[0,1]}$ to $\mathcal{P}_r(\mathbb{R}^{[0,1]})$, since the mapping $x \mapsto \nu_{x,t}$ is continuous for each $t \in [0,1]$, so the product-measure is also continuous in x (see [6, Theorem 8.4.10]). Let P be the image of the power $\lambda^{[0,1]}$ of Lebesgue measure λ on $[0,1]$ under the mapping $x \mapsto \mu_x$. Then the measure P belongs to $\mathcal{P}_r(\mathcal{P}_r(\mathbb{R}^{[0,1]}))$ and is concentrated on the compact set $\{\mu_x\colon x \in [0,1]^{[0,1]}\}$, which is the image of the compact set $[0,1]^{[0,1]}$ under J.

The barycenter β_P of P is the product-measure $\beta^{[0,1]}$, where β is the barycenter of the image of Lebesgue measure λ under the mapping

$$s \mapsto (1-s)\delta_0 + s\delta_{1/s}, \quad [0,1] \to \mathcal{P}_r(\mathbb{R}).$$

Thus,

$$\beta = \int_0^1 ((1-s)\delta_0 + s\delta_{1/s})\, ds.$$

We have $\beta(\{0\}) = 1/2$ and

$$\beta([0,t]) = \int_0^1 (1-s)\, ds + \int_{1/t}^1 s\, ds = 1 - \frac{1}{2t^2} \quad \forall\, t \geq 1.$$

It follows that $\beta_P^*(K) = 0$ for every compact set $K \subset \mathbb{R}^{[0,1]}$. Indeed,

$$K \subset \prod_{t\in[0,1]} [-y(t), y(t)]$$

for some function $y\colon [0,1] \to \mathbb{R}^+$. There is N such that $y(t) \leq N$ for infinitely many points $t \in \{t_j, j \in \mathbb{N}\}$. Then

$$\beta_P^*(K) \leq \prod_{j=1}^{\infty} \beta([0,N]) = 0,$$

because $\beta([0,N]) < 1$ for each N. Therefore, β_P is not a Radon measure, moreover, it has no Radon extension from the cylindrical σ-algebra. □

Remark 1. It is worth noting that the measure P has the following property: there is no uniformly tight compact set $\mathcal{K} \subset \mathcal{P}_r(\mathbb{R}^{[0,1]})$ with $P(\mathcal{K}) > 0$. Indeed, otherwise

$$\mathcal{K} \subset \{\mu \in \mathcal{P}_r(\mathbb{R}^{[0,1]})\colon \mu(K) > 1/2\}$$

for some compact set $K \subset \mathbb{R}^{[0,1]}$. We have $K \subset \prod_{t\in[0,1]}[-y(t), y(t)]$ for some function $y\colon [0,1] \to \mathbb{R}_+$. Then

$$P\Bigl(\mu \in \mathcal{P}_r(\mathbb{R}^{[0,1]})\colon \mu(K) > 1/2\Bigr) \le \lambda^{[0,1]}\Bigl(x \in [0,1]^{[0,1]}\colon \mu_x\Bigl(\prod_{t\in[0,1]}[-y(t), y(t)]\Bigr) > 1/2\Bigr).$$

There is N such that the set $\{t \in [0,1]\colon y(t) \le N\}$ is infinite. Let $y(t) \le N$ for $t \in \{t_j, j \in \mathbb{N}\}$. Then

$$\mu_x\Bigl(\prod_{t\in[0,1]}[-y(t), y(t)]\Bigr) \le \prod_{j=1}^{\infty} \nu_{x,t_j}([-N,N]) = \prod_{j=1}^{\infty} h(x(t_j)),$$

where $h(s) = 1 - s$ if $0 \le s < 1/N$ and $h(s) = 1$ if $x \ge 1/N$. We have

$$\lambda^{[0,1]}\Bigl(\{x \in [0,1]^{[0,1]}\colon x(t_j) \in [1/(2N), 1/N) \ \text{ for infinitely many } \ j\Bigr) = 1,$$

hence

$$\lambda^{[0,1]}\Bigl(x \in [0,1]^{[0,1]}\colon h(x(t_j)) \le 1 - 1/(2N) \ \text{ for infinitely many } \ j\Bigr) = 1.$$

Therefore, $\lambda^{[0,1]}\Bigl(x \in [0,1]^{[0,1]}\colon \mu_x\Bigl(\prod_{t\in[0,1]}[-y(t), y(t)]\Bigr) = 0\Bigr) = 1$, which is a contradiction.

3. Metrizability of compacta in spaces of measures

If all compacta in the space of probability measures $\mathcal{P}_r(X)$ are metrizable, then the same is true for the space X itself, because it is homeomorphic to the subset of Dirac measures. Apparently, the converse is not true, so the following partial converse might be of interest.

Theorem 2. *Suppose that all compact sets in X are metrizable. Then uniformly tight compact sets in $\mathcal{M}_r(X)$ are also metrizable.*

Proof. Let $S \subset \mathcal{M}_r(X)$ be a uniformly tight compact set. We can assume that all measures in S have total variation norms at most 1. For each n there is a compact set $K_n \subset X$ such that $|\sigma|(X\backslash K_n) \leq 2^{-n}$ for all $\sigma \in S$. These sets can be taken increasing. We show that there is a countable family of continuous functions on S separating the elements of S. For such functions we pick linear functionals on $\mathcal{M}_r(X)$ of the form

$$\mu \mapsto \int_X f_n(x)\, \mu(dx)$$

with suitable bounded continuous functions f_n on X. Since the compact sets K_n are metrizable, for each fixed n there are functions $g_{n,i} \in C_b(X)$ with $\sup_{x\in X} |g_{n,i}(x)| = \sup_{x\in K_n} |g_{n,i}(x)| = 1$ and the following property:

$$\|\nu\| = \sup_i \int_{K_n} g_{n,i}\, d\nu$$

for every Radon measure ν on K_n. Next, for every $m > n$ we can find open sets $U_{n,m,j}$ in the metrizable compact space K_m such that

$$U_{n,m,j+1} \subset U_{n,m,j}, \quad K_n = \bigcap_{j=1}^{\infty} U_{n,m,j}.$$

In the whole space X there are open sets $W_{n,m,j}$ for which

$$U_{n,m,j} = W_{n,m,j} \cap K_m.$$

Finally, for fixed n, m, j we take a function $\varphi_{n,m,j} \in C_b(X)$ such that

$$0 \leq \varphi_{n,m,j} \leq 1, \quad \varphi_{n,m,j}|_{K_n} = 1, \quad \varphi_{n,m,j}|_{X\backslash W_{n,m,j}} = 0.$$

Such functions exist, since X is completely regular (see, e.g., [6, Lemma 6.1.5]). For the desired functions f_n we take the functions $\varphi_{n,m,j} g_{n,i}$ enumerated by a single index.

Let us show that the integrals of these functions separate measures in S. Suppose that $\sigma_1, \sigma_2 \in S$ are distinct. Then there is n such that the restrictions of these measures to K_n are distinct. Let

$$\delta = \|\sigma_1|_{K_n} - \sigma_2|_{K_n}\| > 0.$$

By our construction, there is a function $g_{n,i}$ such that

$$\left|\int_{K_n} g_{n,i}\, d\sigma_1 - \int_{K_n} g_{n,i}\, d\sigma_2\right| \geq \frac{3}{4}\delta.$$

Next, there is $m > n$ such that

$$|\sigma_1|(X\backslash K_m) + |\sigma_2|(X\backslash K_m) \leq \frac{1}{8}\delta.$$

Then

$$\Bigl|\int_{X\backslash K_m} f_n\,d\sigma_1\Bigr| + \Bigl|\int_{X\backslash K_m} f_n\,d\sigma_2\Bigr| \le \frac{1}{8}\delta \quad \forall\, n\in\mathbb{N}. \tag{3.1}$$

Since the sets $U_{n,m,j}$ decrease to K_n, there is j for which

$$|\sigma_1|(U_{n,m,j}\backslash K_n) + |\sigma_2|(U_{n,m,j}\backslash K_n) \le \frac{1}{8}\delta.$$

Therefore,

$$\Bigl|\int_{U_{n,m,j}\backslash K_n} f_n\,d\sigma_1\Bigr| + \Bigl|\int_{U_{n,m,j}\backslash K_n} f_n\,d\sigma_2\Bigr| \le \frac{1}{8}\delta \quad \forall\, n\in\mathbb{N}. \tag{3.2}$$

Let us now compare the integrals of the function $g_{n,i}\varphi_{n,m,j}$ with respect to σ_1 and σ_2. This function equals $g_{n,i}$ on K_n, so

$$\Bigl|\int_{K_n} g_{n,i}\varphi_{n,m,j}\,d\sigma_1 - \int_{K_n} g_{n,i}\varphi_{n,m,j}\,d\sigma_2\Bigr| \ge \frac{3}{4}\delta.$$

Next, we have (3.1) for this function and

$$\Bigl|\int_{K_m\backslash K_n} g_{n,i}\varphi_{n,m,j}\,d(|\sigma_1|+|\sigma_2|)\Bigr| \le \frac{1}{8}\delta,$$

because $g_{n,i}\varphi_{n,m,j}(x) = 0$ if $x\in K_n\backslash U_{n,m,j}$ and (3.2) holds. Thus, the difference of the integrals of $g_{n,i}\varphi_{n,m,j}$ over the whole space is at least $\delta/2$. □

It is unlikely that the assumption of uniform compactness can be omitted, but we have no confirming examples. Standard examples of non-metrizable spaces with metrizable compacta (say, with countable compacta, see, e.g., [3], [14]) do not work.

We also do not know whether the metrizability of compacta in the space $\mathcal{P}_r(X)$ of probability measures is sufficient for the metrizability of compacta in the whole space of measures. If compacta in $\mathcal{M}_r(X)$ are uniformly tight (i.e., X is strongly Prohorov), then the answer is obviously positive.

If the space X admits a continuous injection j into a completely regular space Y such that compacta in $\mathcal{M}_r(X)$ or $\mathcal{P}_r(X)$ are metrizable, then X also has the respective property, because j generates a continuous injection $\mathcal{M}_r(X)\to\mathcal{M}_r(Y)$. In particular, compacta in $\mathcal{P}_r(X)$ are metrizable if X admits a continuous injection into a metric space Y, since $\mathcal{P}_r(Y)$ is metrizable (see [6, Theorem 8.3.2]).

The next result shows that the metrizability of compacta in the space of measures is preserved by taking products with spaces possessing countable families of continuous functions separating points (the latter is equivalent to the existence of a continuous injection into a separable metric space).

For the class of probability measures, it suffices that compacta in spaces of probability measures on both factors be metrizable.

We need the following simple observation: if S is a metrizable compact set in $\mathcal{M}_r(X)$, then there is a sequence $\{f_n\}$ of bounded continuous functions on X such that the functionals

$$\mu \mapsto \int_X f_n \, d\mu$$

separate measures in S. Indeed, the family of all integrals of functions in $C_b(X)$ separate measures on X, hence on every compact set there is a countable subfamily separating the elements of this subset.

Theorem 3. (i) *Suppose that all compacta in the space $\mathcal{M}_r(X)$ are metrizable. Then the same is true for $\mathcal{M}_r(X \times Y)$ provided that the space Y possesses a countable family of continuous functions separating points.*

(ii) *Suppose that all compacta in the spaces of probability measures $\mathcal{P}_r(X)$ and $\mathcal{P}_r(Y)$ are metrizable. Then the same is true for $\mathcal{P}_r(X \times Y)$.*

Proof. (i) Let $\{g_n\}$ be a sequence of continuous functions on Y separating points. We can assume that these functions are bounded and that their linear combinations with rational coefficients also belong to this sequence. Then we can add to this countable family all finite products of its elements. It is readily seen that the obtained family (again denoted by $\{g_n\}$) separates measures on Y (continuous functions on compact sets in Y are uniformly approximated by functions from this family, which follows from the Stone–Weierstrass theorem).

For a bounded continuous function ϱ on $X \times Y$ and a measure σ on $X \times Y$ we denote by $\varrho \cdot \sigma$ the measure with the Radon–Nikodym density ϱ with respect to σ. Below we use such measures for functions ϱ depending only on one argument.

Let S be a compact set in $\mathcal{M}_r(X \times Y)$. For every function g_n the set of measures $g_n \cdot \mu$, $\mu \in S$, is also compact. Hence its projection S_n on $\mathcal{M}_r(X)$ is compact and then is metrizable by our assumption. As noted above, there is a countable family $\{f_{n,k}\}$ of bounded continuous functions on X such that the integrals of these functions separate measures in S_n.

Let us show that the integrals of the functions $f_{n,k}(x)g_n(y)$ separate measures in S. Suppose that $\sigma_1, \sigma_2 \in S$ assign equal integrals to each function $f_{n,k}(x)g_n(y)$. We verify that for every $f \in C_b(X)$ and every $g \in C_b(Y)$ the integrals of $f(x)g(y)$ with respect to σ_1 and σ_2 coincide. This will imply the equality $\sigma_1 = \sigma_2$. We observe that for every fixed n the projections of $g_n \cdot \sigma_1$ and $g_n \cdot \sigma_2$ on X assign equal integrals to all functions $f_{n,k}$, because the integral of $f_{n,k}(x)$ with respect to the projection of $g_n \cdot \sigma_i$ on X is the integral of $f_{n,k}(x)g_n(y)$ with respect to σ_i. Hence these projections coincide and assign equal integrals to the function f, which means that

$$\int_{X\times Y} f(x)g_n(y)\,\sigma_1(dx\,dy) = \int_{X\times Y} f(x)g_n(y)\,\sigma_2(dx\,dy).$$

We now look at the projections of the measures $f\cdot\sigma_1$ and $f\cdot\sigma_2$ on Y and observe that by the previous identity they assign equal integrals to all functions g_n. Due to our choice of $\{g_n\}$ this implies the coincidence of these projections. Hence they assign equal integrals to g, which completes the proof.

(ii) We need the following criterion of compactness due to Topsøe [15] (see also [7, Theorem 4.5.7]): a bounded subset M of the set $\mathcal{M}_r^+(X)$ of nonnegative measures has compact closure precisely when for every $\varepsilon>0$ and each collection $\mathcal{U}$ of open sets with the property that every compact set in X is contained in a set from $\mathcal{U}$, there exist sets $U_1,\ldots,U_n\in\mathcal{U}$ such that

$$\min\bigl\{\mu(X\backslash U_i):\ 1\le i\le n\bigr\}<\varepsilon \quad \forall\,\mu\in M.$$

Let $S\subset\mathcal{P}_r(X\times Y)$ be compact. By the cited result the set S_0 of measures of the form $f\cdot\sigma$, where $\sigma\in S$, $f\in C_b(X)$ and $1\le f\le 2$, has compact closure. Then the projection of S_0 on Y is contained in a compact set M_0 of nonnegative measures on Y. Such sets are also metrizable under our assumption that compacta in $\mathcal{P}_r(Y)$ are metrizable. Indeed, the image of M_0 under the continuous mapping $\nu\mapsto\nu/\nu(Y)$ is compact in $\mathcal{P}_r(Y)$. Let M_1 be this image. Then M_0 is contained in the image of the metrizable compact set $M_1\times[1,2]$ under the continuous mapping $(\nu,t)\mapsto t\nu$, but this image is also metrizable (see, e.g., [12, Theorem 4.4.15]). Now the same reasoning as in (i) applies once we pick a sequence of functions $g_n\in C_b(Y)$ separating measures on M_0. The only difference is that now we consider functions $f(x)g(y)$ with $1\le f\le 2$ and obtain the equality of the integrals of such functions, but this yields the same for any function $f\in C_b(X)$, because it can be written as $c_1f_1+c_2$, where c_1,c_2 are constants and $1\le f_1\le 2$. □

Remark 2. It is clear from the proof that the assumption about Y can be replaced by the following one: compact sets in $\mathcal{M}_r(Y)$ are metrizable and uniformly tight (i.e., Y has the strong Prohorov property). Indeed, under these assumptions the family M of projections on Y of all measures of the form $f\cdot\sigma$, where $\sigma\in S$ and $f\in C_b(X)$, $|f|\le 1$, is contained in the family of measures $\varphi\cdot\nu$, where ν belongs to the projection S_Y of S on Y and φ is a Borel function with $|\varphi|\le 1$. The projection S_Y is compact, hence in our situation is uniformly tight, which implies the uniform tightness of M. Thus, M is contained in a compact set M_0, which is metrizable by assumption, so the functions g_n used above should be picked with the property to separate measures in M_0 rather than in the whole space $\mathcal{M}_r(Y)$. However, we do not know whether this theorem is true if we only assume that compacta in $\mathcal{M}_r(Y)$ are metrizable.

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