Серия «Математика»
2022. Т. 39. С. 62—79

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

Research article

УДК 519.642.5
MSC 45H05, 65R20
DOI https://doi.org/10.26516/1997-7670.2022.39.62

Polynomial Spline Collocation Method for Solving Weakly Regular Volterra Integral Equations of the First Kind

Aleksandr N. Tynda[1], **Samad Noeiaghdam**[2,3], **Denis N. Sidorov**[4,2,5]✉

[1] Penza State University, Penza, Russian Federation
[2] Irkutsk National Research Technical University, Irkutsk, Russian Federation
[3] South Ural State University, Chelyabinsk, Russian Federation
[4] Energy Systems Institute SB RAS, Irkutsk, Russian Federation
[5] Irkutsk State University, Irkutsk, Russian Federation
✉ contact.dns@gmail.com

Abstract. The polynomial spline collocation method is proposed for solution of Volterra integral equations of the first kind with special piecewise continuous kernels. The Gauss-type quadrature formula is used to approximate integrals during the discretization of the proposed projection method. The estimate of accuracy of approximate solution is obtained. Stochastic arithmetics is also used based on the Contrôle et Estimation Stochastique des Arrondis de Calculs (CESTAC) method and the Control of Accuracy and Debugging for Numerical Applications (CADNA) library. Applying this approach it is possible to find optimal parameters of the projective method. The numerical examples are included to illustrate the efficiency of proposed novel collocation method.

Keywords: integral equation, discontinuous kernel, spline collocation method, convergence, CESTAC method, CADNA library

Acknowledgements: This work is supported by the Russian Science Foundation (project no. 22-29-01619).

For citation: Tynda A. N., Noeiaghdam S., Sidorov D. N. Polynomial Spline Collocation Method for Solving Weakly Regular Volterra Integral Equations of the First Kind. *The Bulletin of Irkutsk State University. Series Mathematics*, 2022, vol. 39, pp. 62–79. https://doi.org/10.26516/1997-7670.2022.39.62

Научная статья

Метод полиномиальной сплайн-коллокации для решения слабо регулярных интегральных уравнений Вольтерра I рода

А. Н. Тында[1], **С. Нойягдам**[2,3], **Д. Н. Сидоров**[4,2,5]✉

[1] Пензенский государственный университет, Пенза, Российская Федерация

[2] Иркутский национальный исследовательский университет, Иркутск, Российская Федерация

[3] Южно-Уральский государственный университет, Челябинск, Российская Федерация

[4] Институт систем энергетики им. Л. А. Мелентьева СО РАН, Иркутск, Российская Федерация

[5] Иркутский государственный университет, Иркутск, Российская Федерация

✉ contact.dns@gmail.com

Аннотация. Предложен метод полиномиальной сплайн-коллокации для решения интегральных уравнений Вольтерра первого рода с кусочно-непрерывными ядрами. Для аппроксимации интегралов при дискретизации в предлагаемом проекционном методе используется квадратурная формула типа Гаусса. Получена оценка точности приближенного решения. Также используется стохастическая арифметика (CA) на основе метода Controle et Estimation Stochastique des Arrondis de Calculs (CESTAC) и библиотеки Control of Accuracy and Debugging for Numerical Applications (CADNA). Применяя этот подход, можно найти оптимальные параметры проекционного метода. Приведены численные примеры, иллюстрирующие эффективность предложенного нового метода коллокации.

Ключевые слова: интегральное уравнение, разрывное ядро, метод сплайн-коллокации, сходимость, метод CESTAC, библиотека CADNA

Благодарности: Работа выполнена при поддержке Российского научного фонда (проект № 22-29-01619).

Ссылка для цитирования: Tynda A. N., Noeiaghdam S., Sidorov D. N. Polynomial Spline Collocation Method for Solving Weakly Regular Volterra Integral Equations of the First Kind // Известия Иркутского государственного университета. Серия Математика. 2022. Т. 39. С. 62–79.
https://doi.org/10.26516/1997-7670.2022.39.62

1. Introduction

This article focuses on the following weakly regular Volterra equations of the first kind

$$\int_0^t K(t,s)x(s)ds = g(t), \;\; 0 \le s \le t \le T, \;\; g(0) = 0, \tag{1.1}$$

where jump discontinuous kernels are defined as follows

$$K(t,s) = \begin{cases} K_1(t,s), \ t,s \in m_1, \\ \dots \ \dots\dots\dots \\ K_n(t,s), \ t,s \in m_n, \end{cases} \tag{1.2}$$

where $m_i = \{t,s \mid \alpha_{i-1}(t) < s < \alpha_i(t)\}$, $\alpha_0(t) = 0$, $\alpha_n(t) = t$, $i = \overline{1,n}$, $\alpha_i(t)$, $g(t) \in \mathcal{C}^1_{[0,T]}$, functions $K_i(t,s)$ have continuous derivatives with respect to t for $(t,s) \in cl(m_i)$, $K_n(t,t) \neq 0$, $\alpha_i(0) = 0$, $0 < \alpha_1(t) < \alpha_2(t) < \ldots < \alpha_{n-1}(t) < t$, functions $\alpha_1(t), \ldots, \alpha_{n-1}(t)$ increase in a small neighborhood $0 \leq t \leq \tau$, $0 < \alpha_1'(0) \leq \ldots \leq \alpha_{n-1}'(0) < 1$, $cl(m_i)$ denotes closure of set m_i.

Such weakly regular Volterra equations of the first kind with piecewise continuous kernels were first classified and generalised by Sidorov [1] and Lorenzi [2] and extensively studied by many authors during the last decade. Here readers may refer to monograph [7] and references therein. Volterra operator equations of the first kind were studied by Sidorov and Sidorov [8], sufficient conditions for existence of unique solution are obtained. Tynda et al [9] employed direct quadrature methods for solution of equations (1.1) both in linear and nonlinear cases. Muftahov and Sidorov [10] considered the numerical solution of nonlinear systems of such equations. Aghaei et al [11] applied Legendre polynomials approximation method for solution of solution of linear Volterra integral equations with piecewise continuous kernels. The numerical solution of the first kind Volterra convolution integral equations of the first kind with broad class of piecewise smooth kernels was considered by Davies and Duncan [12]. They employed the cubic convolution spline method and proved a stability bound. Such Volterra models enjoys applications in modeling various dynamical processes including storage systems [13; 14]. Generalized quadratures were employed by Sizikov and Sidorov [15] for solving singular Volterra integral equations of Abel type in application to infrared tomography. The numerical solution of the second-kind Volterra integral equation with weakly singular kernel is considered in the piecewise polynomial collocation space by Linag and Brunner [3]. For conventional review of Volterra integral equations theory readers may refer e.g. to monographs by Brunner [4] and by Apartsyn [5]. Some studies of the Volterra integral equations of the first kind have led to the paradoxes as noted by Tynda in zbMATH [6].

In this paper, we also implement the CESTAC method and apply the CADNA library to find the numerical validation of the spline-collocation method to solve the problem (1.1). The priority of this strategy is to find the optimal step, accuracy and error of the numerical method. The paper is organised as following. In Section 2, the spline-collocation method is presented. Also the convergence of the method and smoothness of the solutions are studied. The use of Floating Point Arithmetic (FPA) is discussed in Section 3. The CESTAC method and its principle theorem can be found in Section 4. Using the this theorem we will show, how can we replace the conditions (3.1) with (3.2). The numerical results are illustrated in Section 5. Also in this section the comparative study between the results of the stochastic arithmetic (SA) and the FPA can be found.

2. Polynomial spline-collocation method

In this section, we construct a numerical method for solving problem (1.1) – (1.2), based on the approximation of the exact solution by continuous local splines. First of all, within the framework of conditions (1.2), we replace the original equation (1.1) of the first kind with an equivalent equation of the second kind. To do this, we apply the standard technique of differentiating the equation:

$$x(t) - \int_0^t h(t,s)x(s)ds = f(t),\ h(t,s) = -\frac{K_t'(t,s)}{K(t,t)},\ f(t) = \frac{g'(t)}{K(t,t)}. \quad (2.1)$$

Let us rewrite this equation in operator form

$$(I-H)x = f,\ \text{where } (I-H)x \equiv x(t) - \int_0^t h(t,s)x(s)ds. \quad (2.2)$$

2.1. Numerical scheme

Let us introduce the partition of the interval $[0,T]$ with grid points $t_k,\ k=0,1,\ldots,N$. The introduced grid of nodes is not necessarily uniform and depends on the smoothness properties of the exact solution. Denote by Δ_k the segments $\Delta_k = [t_k, t_{k+1}],\ k=0,1,\ldots,N-1$. Let then

$$\xi_k^j \in \Delta_k,\ j=0,1,\ldots,r-1;\ \xi_k^0 = t_k,\ \xi_k^{r-1} = t_{k+1};\ k=0,1,\ldots,N-1, \quad (2.3)$$

be additional nodes distributed in a certain way over the segment Δ_k. We denote by $P_r(x,\Delta_k)$ an operator, putting to the function $x(t),\ t\in\Delta_k$, in correspondence an interpolation polynomial of degree $r-1$ for $k=\overline{0,N-1}$ constructed on knots ξ_k^j. Let then $x_N(t)$ be a local spline, defined on $[0,T]$ and composed of polynomials $P_r(x,\Delta_k),\ k=0,1,\ldots,N-1$.

We look for an approximate solution of (2.2) as a spline $x_N(t)$ with unknown values $x_N(\xi_k^j),\ k=0,1,\ldots,N-1,\ j=0,1,\ldots,r-1$, at the knots of the grid.

The grid (2.3) depends on the considered class of functions to which the exact solutions belong and will be specified below.

The values $x_N(\xi_k^i)$ in each segment $\Delta_k,\ k=0,1,\ldots,N-1$, are determined step-by-step by the spline-collocation technique from the systems of linear equations

$$(I-H)P_N[x(t),\Delta_k] \equiv P_N[x(t),\Delta_k]-$$
$$-P_N\left[\int\limits_{\Delta_k} P_N^{\tau}[h(t,s)]P_N[x(s),\Delta_k]ds,\ \Delta_k\right] = P_N[f_k(t),\Delta_k]. \tag{2.4}$$

Here P_N is an operator of projection on the set of the local splines of the form $x_N(t)$; $f_k(t)$ is a new right part of equation (2.1) including the integrals over segments $\Delta_j,\ j=0,1,\ldots,k-1$, processed at the previous steps (in these domains, the spline parameters are already known):

$$f_k(t) = f(t) + \sum_{l=0}^{k-1}\int\limits_{t_l}^{t_{l+1}} h(t,s)x_N(s)ds.$$

2.2. Convergence substantiation

Let us rewrite equation (2.1) and projective method (2.4) in the operator form:

$$x - Hx = f,\ H : X \to X,\ X \subset C(\Omega),\ \Omega = [0,T], \tag{2.5}$$
$$x_N - P_N H x_N = P_N f,\ P_N : X \to X_N,\ X_N \subset C(\Omega), \tag{2.6}$$

where X is a dense set in $C(\Omega)$ and X_N are the sets of corresponding local splines.

Since the homogenous Volterra integral equation $x-Hx=0$ has only the trivial solution, the operator $I-H$ is injective. Hence, the operator $I-H$ has the bounded inverse operator $(I-H)^{-1} : X \to X$. For all sufficiently large N we have the estimates

$$\|(I-P_N H)^{-1}\|_{C(\Omega)} = \|\Big((I-H)+(H-P_N H)\Big)^{-1}\|_{C(\Omega)} \leqslant$$
$$\leqslant \frac{\|(I-H)^{-1}\|_{C(\Omega)}}{1-\|(I-H)^{-1}\|_{C(\Omega)}\|H-P_N H\|_{C(\Omega)}} \leqslant 2\|(I-H)^{-1}\|_{C(\Omega)} = A\ (const)$$

if

$$\|H-P_N H\|_{C(\Omega)} \leqslant \frac{1}{2\|(I-H)^{-1}\|_{C(\Omega)}}.$$

Let us show that the last estimate holds for all sufficiently large N. Since $y(t) \equiv \big(Hx\big)(t) \in X$ and X is a dense set in $C(\Omega)$, we have

$$\|H-P_N H\|_{C(\Omega)} = \sup_{x\in X,\|x\|\leqslant 1}\max_{t\in\Omega}|x(t)-P_N x(t)| \leqslant e_N,$$

where $e_N \to 0$ as $N \to \infty$. Therefore, $\|H-P_N H\|_{C(\Omega)} \leqslant \frac{1}{2\|(I-H)^{-1}\|}$ starting with sufficiently large N.

Thus, the operators $(I - P_N H)^{-1}$ are exist and uniformly bounded and equation (2.6) has a unique solution for all sufficiently large N [16]. Taking into account that $P_N x \to x$ as $N \to \infty$ for all $x \in X$, we apply the projection operator P_N both to the left and the right parts of equation (2.5): $x - P_N Hx = P_N f + x - P_N x$. Subtracting this equation from (2.6), we obtain $(I - P_N H)(x_N - x) = P_N x - x$, and $(x_N - x) = (I - P_N H)^{-1}(P_N x - x)$.

This implies

$$\|x_N - x\|_C \leqslant A\|P_N x - x\|_C \leqslant e_N(X). \tag{2.7}$$

Thus, the accuracy of the approximate solution obtained via projective method (2.6) is determined by the accuracy $e_N(X)$ of the approximation of functions from X by the local splines.

2.3. Smooth solutions

In this chapter, we describe in more detail the projection method (2.4) for the case of smooth input functions. Namely, let the functions $K_i(t, s)$, $i = 1, 2, \ldots, n$ and $g(t)$ in (1.1)-(1.2) satisfy additional smoothness conditions. Let $g(t) \in C^{r+1}[0, T]$, $K_i(t, s) \in C^{r+1,r}[0, T]^2$ (continuously differentiable on each variable). The exact solution $x(t)$ of the equation (1.1) in this case belongs to $X = C^r[0, T]$ [7].

We introduce the uniform partition of the interval $[0, T]$ with grid points $t_k = \frac{kT}{N}$, $k = 0, 1, \ldots, N$. Denote by Δ_k the segments $\Delta_k = [t_k, t_{k+1}]$, $k = 0, 1, \ldots, N - 1$. Let $\xi_k^j = \frac{t_{k+1}+t_k}{2} + \frac{t_{k+1}-t_k}{2} y_j$, and $j = 1, 2, \ldots, r - 2$; $\xi_0^k = t_k$, $\xi_{r-1}^k = t_{k+1}$; $k = 0, 2, \ldots, N-1$. where y_j are the roots of the Legendre polynomials of degree $r - 2$.

We denote by $P_r(x, \Delta_k)$ an operator, putting to the function $x(t)$, $t \in \Delta_k$ in correspondence an interpolation polynomial of degree $r - 1$ for $k = \overline{0, N-1}$ constructed on knots ξ_k^j. Let then $x_N(t)$ be a local spline, defined on $[0, T]$ and composed of polynomials $P_r(x, \Delta_k)$, $k = 0, 1, \ldots, N - 1$. We look for an approximate solution of the equation (1.1) as a spline $x_N(t)$, $0 \leqslant t \leqslant T$, with unknown coefficients x_k^j, $k = \overline{0, N-1}$. Let us describe the process of definition x_k^j. At first we find the coefficients x_0^j, $j = 0, 1, \ldots, r-1$, from the system of equations

$$P_r(x, \Delta_0)(\xi_0^j) - P_r\left[\int_0^{\xi_0^j} h(\xi_0^j, \tau) P_r(x, \Delta_0)(\tau) d\tau, \Delta_0\right] = P_r(f, \Delta_0)(\xi_0^j). \tag{2.8}$$

Here for integrals calculation in (2.8) we employ compound Gaussian quadrature rule with $r - 2$ points constructed on the auxiliary mesh linked to the lines $\alpha_i(t)$, $i = \overline{1, n}$, of the kernel discontinuities for each specific value of N. Note also that all values of the unknown function in intermediate points are computed with help of interpolation. The resulting system of linear algebraic equations is then solved by the Jordan-Gauss method.

In order to determine the coefficients $x_1^j,\ j=0,1,\ldots,r-1$, of the local spline $x_N(t)$ on the segment Δ_1 we represent (2.1) in the following form

$$x(t)-\int\limits_{\xi_1^0}^{t} h(t,\tau)x(\tau)d\tau = f_1(t),\ f_1(t)=f(t)+\int\limits_{0}^{\xi_0^{r-1}} h(t,\tau)P_r(x(\tau),\Delta_0)d\tau. \tag{2.9}$$

The equation (2.9) is then solved by analogy, using the scheme (2.8). Repeating this process N times we obtain the approximate solution $x_N(t)$ of the equation (2.1) over all interval $[0,T]$.

The error of approximation of the exact solution by the polynomials constructed in this way at each step of the process can be estimated by the following inequality (here readers may refer to [17])

$$\|x(t)-x_N(t)\|_{C[\Delta_k]} \le \frac{L_r\left(\frac{T}{N}\right)^r}{r!},\ k=0,1,\ldots,N-1, \quad L_r=\max_{t\in\Delta_k}\left|x^r(t)\right|. \tag{2.10}$$

Taking into account the general estimate (2.7) of the error of the method, we obtain the error estimate in this case ($X=C^r[0,T]$)

$$\|x(t)-x_N(t)\|_{C[0,T]} \asymp N^{-r}. \tag{2.11}$$

Boikov and Tynda [18;19] established that such type numerical methods for Volterra integral equations are also optimal with respect to complexity and accuracy order. Thus, an effective projective method for solving equations of the form (1.1) is proposed.

3. Using the Floating Point Arithmetic

In general form when floating point arithmetic (FPA) is employed it is necessary to have the exact and approximate solutions $x(t)$ and $x_N(t)$ and also small value ϵ to use the following conditions

$$|x(t)-x_N(t)|\le\epsilon, \quad or \quad |x_N(t)-x_{N-1}(t)|\le\epsilon. \tag{3.1}$$

But the main problem is that the exact solution and optimal ϵ are unknowns. Thus by putting small values instead of ϵ, the approximate solution will not be accurate and for large values we will have many extra iterations. In order to avoid these problems, the CESTAC method and the CADNA library will be utilized [34]. In this novel method, the accuracy will be obtained using successive iterations $x_N(t)$ and $x_{N-1}(t)$ and the following condition

$$|x_N(t)-x_{N-1}(t)| = @.0. \tag{3.2}$$

We apply this condition to control the accuracy of the method and avoid extra iterations by using number of common significant digits (NCSDs) between $x_N(t)$ and $x_{N-1}(t)$. @.0 in Eq. (3.2) displays the informatical zero which can be produced only in the CESTAC method by the CADNA library. It shows that the NCSDs of two successive approximations and approximate and exact solutions are almost equal to zero. Vignes and La Porte [33] have presented the method for the first time in 1974. In [34] Vignes has described the CESTAC method to evaluate the numerical results of some computational methods. Some conditions of the CESTAC method, applying different tools to write the CADNA codes [35] and also some properties of the SA have been studied by Chesneaux. All the CESTAC evaluations should be accomplished using the CADNA library. Handling this scheme, the optimal results, step and error of the method can be recognized. Lamotte et al. has implemented the CESTAC method using C and C++ codes. Jézéquel et al have discussed the new version of the CADNA library using Fortran programs. Recently applying this method to control the accuracy of the Taylor expansion method to solve the generalized Abel's integral equation [20], mathematical model of Malaria infection [21; 23], nonlinear fractional order model of COVID-19 [22], solving nonlinear shallow water wave equation [24], Adomian decomposition method, homotopy perturbation method and Taylor-collocation method for solving Volterra integral equation [25; 28–30], dynamical control of the reverse osmosis system [26; 27], solving integrals using the numerical methods have been done. Moreover the CESTAC method has been used to find the optimal convergence control parameter of the homotopy analysis method in both fuzzy and crisp forms [31; 32].

4. CESTAC Method

The CESTAC method is a powerful and applicable tool to validate the numerical results of numerical procedures. It should be applied based on the SA. Let B be a set of reproduced values by computer. For real value g^*, we can find a member of set B such as G^* with α mantissa bits of the binary FPA as $G^* = g^* - \rho 2^{E-\alpha}\phi$, where the sign showed by ρ, the missing segment of the mantissa presented by $2^{-\alpha}\phi$ and the binary exponent of the result displayed by E. Replaying 24 and 53 instead of α, the results can be found by single and double accuracies. Assuming ϕ as a stochastic variable and having uniformly distribution on $[-1, 1]$, we will be able to make perturbation on the last mantissa bit of g^*. Thus for the obtained results of G^*, the mean and standard deviation values (μ) and (σ) can be found. Doing the mentioned scheme p-times p samples of G^* can be produced as $\Phi = \left\{G_1^*, G_2^*, ..., G_p^*\right\}$. Thus the mean and standard deviation can be found as follows $\tilde{G}^* = \dfrac{\sum_{k=1}^{p} G_k^*}{p}, \qquad \sigma^2 = \dfrac{\sum_{k=1}^{p}(G_k^* - \tilde{G}^*)^2}{p-1}.$

Using the mentioned computations the NCSDs of G^* and $\tilde{G}^*$ can be generated using the following relation $\mathcal{C}_{\tilde{G}^*,G^*} = \log_{10} \frac{\sqrt{p}\left|\tilde{G}^*\right|}{\tau_\delta \sigma}$, where τ_δ is the value of T distribution as the confidence interval is $1-\delta$, with $p-1$ freedom degree. Showing $G^* = @.0$, the process stopped if we have $\tilde{G}^* = 0$, or $\mathcal{C}_{\tilde{G}^*,G^*} \le 0$.

In this method, the mathematical softwares Mathematica, Maple or MATLAB must be replaced by the CADNA library. This library should be implemented on the LINUX operating system and we all codes should be compiled by C, C++, FORTRAN or ADA. The main benefit of the method is to find the optimal results, step size and error of the method.

Definition 1. *For $\ell_1, \ell_2 \in \mathbb{R}$ the NCSDs can be defined as*

$$\mathcal{C}_{\ell_1,\ell_2} = \begin{cases} \log_{10}\left|\frac{\ell_1+\ell_2}{2(\ell_1-\ell_2)}\right| = \log_{10}\left|\frac{\ell_1}{\ell_1-\ell_2} - \frac{1}{2}\right|, & \ell_1 \ne \ell_2, \\ \infty, & otherwise. \end{cases} \tag{4.1}$$

Theorem 1. *Assume that $x(t)$ and $x_N(t)$ are the exact and approximate solutions of Eq. (1.1). The NCSDs of two successive approximations are almost equal to the NCSDs of exact and approximate solutions and we have*

$$\mathcal{C}_{x_N(t),x_{N+1}(t)} \simeq \mathcal{C}_{x_N(t),x(t)}. \tag{4.2}$$

Proof. Applying Definition 1 and Eq. (2.11) for to iterations $x_N(t)$ and $x_{N+1}(t)$ we can write

$$\begin{aligned} \mathcal{C}_{x_N(t),x_{N+1}(t)} &= \\ &= \log_{10}\left|\frac{x_N(t)+x_{N+1}(t)}{2(x_N(t)-x_{N+1}(t))}\right| = \log_{10}\left|\frac{x_N(t)}{x_N(t)-x_{N+1}(t)} - \frac{1}{2}\right| = \\ &= \log_{10}\left|\frac{x_N(t)}{x_N(t)-x_{N+1}(t)}\right| + \log_{10}\left|1 - \frac{1}{2x_N(t)}(x_N(t)-x_{N+1}(t))\right| = \\ &= \log_{10}\left|\frac{x_N(t)}{x_N(t)-x_{N+1}(t)}\right| + \mathcal{O}\left(x_N(t)-x_{N+1}(t)\right). \end{aligned}$$

We know $x_N(t) - x_{N+1}(t) = x_N(t) - x(t) - (x_{N+1}(t) - x(t)) = E_n(t) - E_{n+1}(t)$, therefore we get

$$\mathcal{O}\left(x_N(t)-x_{N+1}(t)\right) = \mathcal{O}\left(N^{-r}\right) + \mathcal{O}\left((N+1)^{-r}\right) = \mathcal{O}\left(N^{-r}\right).$$

And finally we have

$$\mathcal{C}_{x_N(t),x_{N+1}(t)} = \log_{10}\left|\frac{x_N(t)}{x_N(t)-x_{N+1}(t)}\right| + \mathcal{O}\left(N^{-r}\right). \tag{4.3}$$

Repeating the process for exact and approximate solutions, the following relation can be obtained

$$\mathcal{C}_{x_N(t),x(t)} = \log_{10}\left|\frac{x_N(t)+x(t)}{2(x_N(t)-x(t))}\right| = \log_{10}\left|\frac{x_N(t)}{x_N(t)-x(t)} - \frac{1}{2}\right| =$$
$$= \log_{10}\left|\frac{x_N(t)}{x_N(t)-x(t)}\right| + \mathcal{O}(x_N(t)-x(t)) = \log_{10}\left|\frac{x_N(t)}{x_N(t)-x(t)}\right| + \mathcal{O}\left(N^{-r}\right). \tag{4.4}$$

Based on Eqs. (4.3) and (4.4) we can write

$$\mathcal{C}_{x_N(t),x(t)} - \mathcal{C}_{x_N(t),x_{N+1}(t)} =$$
$$= \log_{10}\left|\frac{x_N(t)}{x_N(t)-x(t)}\right| - \log_{10}\left|\frac{x_N(t)}{x_N(t)-x_{N+1}(t)}\right| + \mathcal{O}\left(N^{-r}\right) =$$
$$= \log_{10}\left|\frac{x_N(t)-x(t)}{x_N(t)-x_{N+1}(t)}\right| + \mathcal{O}\left(N^{-r}\right) =$$
$$= \log_{10}\left|\frac{\mathcal{O}\left(N^{-r}\right)}{\mathcal{O}\left(N^{-r}\right)}\right| + \mathcal{O}\left(N^{-r}\right) = \mathcal{O}\left(N^{-r}\right)$$

and $\mathcal{C}_{x_N(t),x(t)} - \mathcal{C}_{x_N(t),x_{N+1}(t)} = \mathcal{O}\left(N^{-r}\right)$. Clearly by approaching N to infinity, $\mathcal{O}\left(N^{-r}\right)$ tends zero and we obtain $\mathcal{C}_{x_N(t),x(t)} = \mathcal{C}_{x_N(t),x_{N+1}(t)}$. □

5. Numerical results

To illustrate the effectiveness of the suggested polynomial spline-collocation method, we present the results for two test equations.

Example 1. Consider the following integral equation

$$\int_0^{\frac{t}{2}} (t+s)x(s)ds + \int_{\frac{t}{2}}^{\frac{2t}{3}} tsx(s)ds + \int_{\frac{2t}{3}}^{t} e^s x(s)ds = f(t), \quad t \in [0;1], \tag{5.1}$$

where the right side of the equation was chosen so that the exact solution was $x^* = t\sin t$. The following designations are used in the tables below: **N** is the number of segments of the main partition, **r** is the parameter responsible for the order of the spline, $e = ||x_N(t) - x^*(t)||_{C_{[0,T]}}$.

Table 1

The error for (5.1) at the value $\mathbf{r} = 4$.

N	1	5	10	20	50	100	500
e	$6.57{\cdot}10^{-4}$	$2.38{\cdot}10^{-7}$	$7.55{\cdot}10^{-9}$	$2.38{\cdot}10^{-10}$	$2.45{\cdot}10^{-12}$	$7.65{\cdot}10^{-14}$	$2.46{\cdot}10^{-17}$

Clearly, Tables 1 and 2 depend on the exact solution. For Table 3 we apply the CESTAC method and the results are obtained $\mathbf{r} = 5$ and the optimal results are $N_{opt} = 6, x_6(0.05) = 0.2498959, error_{opt} = 0.2328306E - 009$. Table 4 is obtained using the spline-collocation method and the FPA for the same value of $\mathbf{r}$. It is obvious that for $\epsilon = 10^{-2}$ the algorithm will be stopped at $N = 1$ and for $\epsilon = 10^{-10}$ we have $N = 4$.

Table 2

The error for (5.1) at the value $\mathbf{r} = 7$.

N	1	5	10	20	50	100	500
e	$2.95{\cdot}10^{-7}$	$4.71{\cdot}10^{-12}$	$3.73{\cdot}10^{-14}$	$2.94{\cdot}10^{-16}$	$4.82{\cdot}10^{-19}$	$3.77{\cdot}10^{-21}$	$1.39{\cdot}10^{-25}$

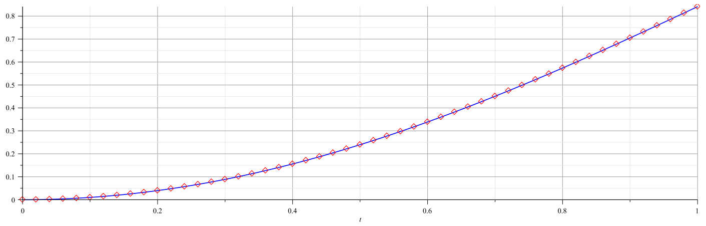

Figure 1. The exact and approximate solution of (5.1) with $N = 5,\ r = 8$.

Table 3

Results of the CESTAC method for $\mathbf{r} = 5$.

N	$x_{N+1}(t)$	$\lvert x_{N+1}(t) - x_N(t)\rvert$	$\lvert x_N(t) - x^*(t)\rvert$
1	0.2474730	0.2474730E-002	0.24227E-004
2	0.2498547	0.23816E-004	0.411E-006
3	0.2498938	0.391E-006	0.19E-007
4	0.2498958	0.19E-007	0.2328306E-009
5	0.2498959	0.6984919E-009	@.0
6	0.2498959	@.0	@.0

For small values of ϵ we will need to provide many extra iterations without improving the accuracy. Tables 5 and 6 present the results for $\mathbf{r} = 10$ using the CESTAC method and we have $N_{opt} = 2, x_6(0.05) = 0.2498958$. It means that we do not need to produce a smaller partition and we can stop at the specified value of N. Thus according to the obtained results, $N = 2$ is enough and we do not need to find more results.

Table 4

The spline-collocation using the FPA for solving Example 1 with $\mathbf{r} = 5$.

N	$x_{N+1}(t)$	$\|x_{N+1}(t) - x^*(t)\|$
1	0.00247473099023539098519400001084329	0.00002422274732985254545492499338416
2	0.0024985471725248365458815553649446	4.112910090798938616946773299e-7
3	0.0024989386743724730905507916941931	1.97891614433491924583480814e-8
4	0.0024989586478083542254432950394024	1.842744377857000449971279e-10
5	0.0024989592632306142870868948525715	7.996966978473436448102970e-10
6	0.0024989588352407427373351401205658	3.717068262975918900782913e-10
7	0.0024989585878160274749598471153271	1.242821110352165970730526e-10
8	0.0024989584860466374183598209171807	2.25127209786165708749062e-11
9	0.0024989584529530046839100658673684	1.05809117558331841749061e-11
10	0.0024989584476966830750021803651006	1.58372333647410696771739e-11

Table 5

Results of the CESTAC method for $\mathbf{r} = 10$.

N	$x_{N+1}(t)$	$\|x_{N+1}(t) - x_N(t)\|$	$\|x_{N+1}(t) - x^*(t)\|$
1	0.2498958	0.2498958E-002	@.0
2	0.2498958	@.0	@.0

Example 2. Consider the following integral equation

$$\int_0^{\frac{t}{3}} (t-s)^2 x(s)ds + \int_{\frac{t}{3}}^{\frac{3t}{4}} \cos(s)x(s)ds + \int_{\frac{3t}{4}}^{t} (1+\sin(2s))x(s)ds = f(t), \quad t \in [0;2], \tag{5.2}$$

where the right side of the equation was chosen so that the exact solution was $x^* = e^{2-t}t^2$.

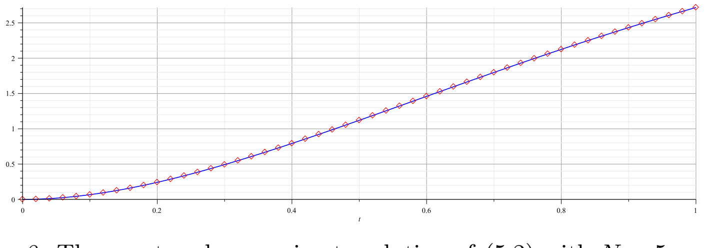

Figure 2. The exact and approximate solution of (5.2) with $N = 5,\ r = 8$.

All calculations were performed in the Maple system with parameter `Digits:=30;` (the number of digits that Maple uses when making calculations with software floating-point numbers). As we can see from Tables

Table 6

The spline-collocation using the FPA for solving Example 1 with $\mathbf{r} = 10$.

N	$x_{N+1}(t)$	$\lvert x_{N+1}(t) - x^*(t)\rvert$
1	0.0024989584688525888359310360465249	5.3186723961877860042504e-12
2	0.0024989584635324124519264394003959	1.5039878168106418786e-15
3	0.0024989584635338888159300590349018	2.76238131910073727e-17
4	0.0024989584635339165728796529239866	1.331364028817121e-19
5	0.0024989584635339164599105607342320	2.01673106919575e-20
6	0.0024989584635339164307566189869647	8.9866310553098e-21
7	0.0024989584635339164379496352522859	1.7936147899886e-21
8	0.0024989584635339164399427718497446	1.995218074701e-22
9	0.0024989584635339164398662187644619	1.229687221874e-22
10	0.0024989584635339164397061697372937	3.70803049808e-23

Table 7

The error for (5.2) at the value $\mathbf{r} = 5$.

N	1	5	10	20	50	100	500
e	$7.67{\cdot}10^{-3}$	$4.89{\cdot}10^{-6}$	$1.70{\cdot}10^{-7}$	$5.61{\cdot}10^{-9}$	$5.96{\cdot}10^{-11}$	$1.88{\cdot}10^{-12}$	$6.09{\cdot}10^{-16}$

7 and 8, the practical error of the method corresponds to the theoretical estimate (2.11). All the results of Tables 9 and 10, are obtained using the CESTAC method. For $\mathbf{r} = 6$ we get $N_{opt} = 6, x_6(0.05) = 0.1757171, error_{opt} = 0.9313225E - 008$, and for $\mathbf{r} = 12$ we have $N_{opt} = 2, x_6(0.05) = 0.1757171$. According to the results for large values of $\mathbf{r}$ the results are more accurate. Tables 11 and 12 are obtained for the spline-collocation method using the FPA. Comparing the results of the FPA and the SA, we can introduce the CESTAC method as a good tool to control the accuracy and the step size of the spline-collocation method for solving the mentioned problem.

5.1. Stability experiments

To illustrate the stability of suggested numerical method, we introduced a random error in calculating the values of free term $f(t)$ of the equations (5.1) and (5.2). The range of introduced random errors is $(-\delta, \delta)$.

Depending on the δ, the following results are obtained.

From the results proposed in the tables 13 and 14 it is possible to judge the continuous dependence of the solution on the initial data and conclude about the stability of the numerical method. This result is not surprising: despite the fact that the initial equation is an equation of the first kind, it is solved in such spaces in which the problem is well-posed. The case

Table 8

The error for (5.2) at the value $\mathbf{r} = 10$.

N	1	5	10	20	50	100	500
e	$8.61 \cdot 10^{-9}$	$1.41 \cdot 10^{-15}$	$1.46 \cdot 10^{-18}$	$1.47 \cdot 10^{-21}$	$1.49 \cdot 10^{-23}$	$1.43 \cdot 10^{-24}$	$7.47 \cdot 10^{-27}$

Table 9

Results of the CESTAC method for $\mathbf{r} = 6$.

N	$x_{N+1}(t)$	$\lvert x_{N+1}(t) - x_N(t)\rvert$	$\lvert x_N(t) - x^*(t)\rvert$
1	0.1762535	0.1762535E-001	0.5363E-004
2	0.1757219	0.5316E-004	0.47E-006
3	0.1757169	0.49E-006	0.2E-007
4	0.1757170	0.1E-007	0.9313225E-008
5	0.1757171	0.7450580E-008	@.0
6	0.1757171	@.0	@.0

Table 10

Results of the CESTAC method for $\mathbf{r} = 12$.

N	$x_{N+1}(t)$	$\lvert x_{N+1}(t) - x_N(t)\rvert$	$\lvert x_N(t) - x^*(t)\rvert$
1	0.1757171	0.1757171E-001	@.0
2	0.1757171	@.0	@.0

Table 11

The spline-collocation using the FPA for solving Example 2 with $\mathbf{r} = 6$.

N	$x_{N+1}(t)$	$\lvert x_{N+1}(t) - x^*(t)\rvert$
1	0.017625357759763366370583965414699	0.000053638808290133034856760580788
2	0.017572191765083977687409203731931	4.72813610744351681998898020e-7
3	0.017571692863117541262788927971815	2.6088355692072938276862096e-8
4	0.017571708598170673074341558874276	1.0353302560261385645959635e-8
5	0.017571717076595299740419367871963	1.874877933595307836961948e-9
6	0.017571719104941762543174921162397	1.53468529207447716328486e-10
7	0.017571719329605570787973205575224	3.78132337452246000741313e-10
8	0.017571719187368398597882888516563	2.35895165262155683682652e-10
9	0.017571719040773236192761953789278	8.9300002857034748955367e-11
10	0.017571718954164366107614229639737	2.691132771887024805826e-12

of an ill-posedness (when noisy initial data may lead to instability and regularization will be required) will be studied in future works.

Table 12

The spline-collocation using the FPA for solving Example 2 with $\mathbf{r} = 12$.

N	$x_{N+1}(t)$	$\|x_{N+1}(t) - x^*(t)\|$
1	0.017571718951506900943716697361262	3.3667607989492527351e-14
2	0.017571718951473218595173777454132	1.4740553427379779e-17
3	0.017571718951473233466169941902544	1.30442737068633e-19
4	0.017571718951473233336814972540526	1.087767706615e-21
5	0.017571718951473233335480404651884	2.46800182027e-22
6	0.017571718951473233335711266248458	1.5938585453e-23
7	0.017571718951473233335728475858345	1.271024434e-24
8	0.017571718951473233335727478289445	2.73455534e-25
9	0.017571718951473233335727218074944	1.3241033e-26
10	0.017571718951473233335727227475141	2.2641230e-26

Table 13

The error for (5.1) with $\mathbf{N} = 5,\ \mathbf{r} = 5$.

δ	0	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
ε	$7.17{\cdot}10^{-7}$	$2.74{\cdot}10^{-5}$	$2.75{\cdot}10^{-4}$	0.00275	0.0275	0.2746

Table 14

The error for (5.2) with $\mathbf{N} = 5,\ \mathbf{r} = 5$.

δ	0	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
ε	$4.89{\cdot}10^{-6}$	$5.71{\cdot}10^{-5}$	$5.28{\cdot}10^{-4}$	0.00524	0.05235	0.52351

6. Conclusion

We have applied the spline-collocation method for solving the Volterra integral equations of the first kind with discontinuous kernel. The convergence of the method and the smoothness of the solution have been discussed. Using the CESTAC method we have tried to control the accuracy and step size of the method. The principle theorem of the CESTAC method will help us to apply the condition (3.2) instead of (3.1). Thus we will be able to find the optimal results, optimal error and optimal step of the method.

References

1. Sidorov D.N. On parametric families of solutions of Volterra integral equations of the first kind with piecewise smooth kernel. *Differential Equations*, 2013, vol. 49, pp. 210–216. https://doi.org/10.1134/S0012266113020079
2. Lorenzi A. Operator equations of the first kind and integro-differential equations of degenerate type in Banach spaces and applications of integro-differential PDE's. *Eurasian Journal of Mathematical and Computer Applications*, 2013, vol. 1, no. 2, pp. 50–75.

3. Liang H., Brunner H. The Fine Error Estimation of Collocation Methods on Uniform Meshes for Weakly Singular Volterra Integral Equations. *Journal of Scientific Computing*, 2020, vol. 84, no. 12. https://doi.org/10.1007/s10915-020-01266-1
4. Brunner H. *Volterra integral equations: an introduction to theory and applications.* Cambridge University Press, Cambridge, 2017, 405 p.
5. Apartsyn A.S. *Nonclassical linear Volterra equations of the first kind.* Walter De Gruyter, Berlin, 2003, 177 p.
6. Tynda A.N. Review of the article “To the theory of Volterra integral equations of the first kind with discontinuous kernels” by A.S. Apartsyn. *zbMATH, FIZ Karlsruhe GmbH*, 2017, zbl:1361.65096.
7. Sidorov D. *Integral Dynamical Models: Singularities, Signals And Control*, In: L.O. Chua, ed. World Scientific Series on Nonlinear Sciences Series A, vol. 87. Singapore World Scientific Press, 2015, 258 p.
8. Sidorov N.A., Sidorov D.N. On the solvability of a class of Volterra operator equations of the first kind with piecewise continuous kernels. *Math. Notes*, 2014, vol. 96, pp. 811–826. https://doi.org/10.1134/S0001434614110170
9. Muftahov I., Tynda A., Sidorov D. Numeric solution of Volterra integral equations of the first kind with discontinuous kernels. *Journal of Computational and Applied Mathematics*, 2017, vol. 313, pp. 119–128. https://doi.org/10.1016/j.cam.2016.09.003
10. Muftahov I.R., Sidorov D.N. Solvability and numerical solutions of systems of nonlinear Volterra integral equations of the first kind with piecewise continuous kernels. *Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programming and Computer Software*, 2016, vol. 9, 1. pp. 130–136.
11. Aghaei A.S., Mahmoudi Y., Salimi S. A. Legendre polynomials approximation method for solving Volterra integral equations of the first kind with discontinuous kernels. *Indian Journal of Pure and Applied Mathematics*, 2021. https://doi.org/10.1007/s13226-021-00109-5
12. Davies P.J., Duncan D.B. Numerical approximation of first kind Volterra convolution integral equations with discontinuous kernels. *J. Integral Equations Applications*, 2017, vol. 29, no. 1, pp. 41–73. https://doi.org/10.1216/JIE-2017-29-1-41.
13. Sidorov D., Panasetsky D., Tomin N., Karamov D., Zhukov A., Muftahov I., Dreglea A., Liu F., Li Y. Toward Zero-Emission Hybrid AC/DC Power Systems with Renewable Energy Sources and Storages: A Case Study from Lake Baikal Region. *Energies*, 2020, vol. 13, 1226. https://doi.org/10.3390/en13051226
14. Sidorov D., Tynda A., Muftahov I., Dreglea A., Liu F. Nonlinear Systems of Volterra Equations with Piecewise Smooth Kernels: Numerical Solution and Application for Power Systems Operation. *Mathematics*, 2020, vol. 8, 1257. https://doi.org/10.3390/math8081257
15. Sizikov V., Sidorov D. Generalized quadrature for solving singular integral equations of Abel type in application to infrared tomography. *Applied Numerical Mathematics*, 2016, vol. 106, pp. 69–78. https://doi.org/10.1016/j.apnum.2016.03.004.
16. Kantorovich L. V., Akilov G. P. *Functional Analysis.* 2nd ed. New York, Pergamon Press, Oxford – Elmsford, 1982, 589 p.
17. Dziadyk V.K. *Introduction in Theory of Uniform Approximation of the Functions by Polynomials.* Moscow, Nauka Publ., 1977, 512 p. (in Russian)
18. Tynda A.N. Numerical Algorithms of Optimal Complexity for Weakly Singular Volterra Integral Equations. *Comp. Meth. Appl. Math.*, 2006, vol. 6, no. 4, pp. 436–442.

19. Boykov I.V., Tynda A.N. Numerical Methods of Optimal Accuracy for Weakly Singular Volterra Integral Equations. *Annals of Functional Analysis*, 2015, vol. 6, pp. 114–133.
20. Zarei S., Noeiaghdam S. Advantages of the Discrete Stochastic Arithmetic to Validate the Results of the Taylor Expansion Method to Solve the Generalized Abel's Integral Equation. *Symmetry*, 2021, vol. 13, no. 1370. https://doi.org/10.3390/sym13081370
21. Noeiaghdam S., Dreglea A., Isik H., Suleman M. Comparative Study between Discrete Stochastic Arithmetic and Floating-Point Arithmetic to Validate the Results of Fractional Order Model of Malaria Infection. *Mathematics*, 2021, vol. 9, no. 1435. https://doi.org/10.3390/math9121435
22. Noeiaghdam S., Micula S., Nieto J.J. Novel Technique to Control the Accuracy of a Nonlinear Fractional Order Model of COVID-19: Application of the CESTAC Method and the CADNA Library. *Mathematics*, 2021, vol. 9, no. 1321. https://doi.org/10.3390/math9121321
23. Noeiaghdam S., Micula S. Dynamical Strategy to Control the Accuracy of the Nonlinear Bio-mathematical Model of Malaria Infection. *Mathematics*, 2021, vol. 9 (9), no. 1031. https://doi.org/10.3390/math9091031
24. Noeiaghdam L., Noeiaghdam S., Sidorov D. Dynamical Control on the Homotopy Analysis Method for Solving Nonlinear Shallow Water Wave Equation. *J. Phys.: Conf. Ser.*, 2021, vol. 1847. 012010. https://dx.doi.org/10.1088/1742-6596/1847/1/012010
25. Noeiaghdam S., Sidorov D., Wazwaz A.M., Sidorov N., Sizikov V. The numerical validation of the Adomian decomposition method for solving Volterra integral equation with discontinuous kernel using the CESTAC method. *Mathematics*, 2021, vol. 9(3), no. 260. https://doi.org/10.3390/math9030260
26. Noeiaghdam S., Sidorov D., Zamyshlyaeva A., Tynda A., Dreglea A. A valid dynamical control on the reverse osmosis system using the CESTAC method. *Mathematics*, 2021, vol. 9(1), no. 48. https://dx.doi.org/10.3390/math9010048
27. Noeiaghdam S., Fariborzi Araghi M. A. A novel algorithm to evaluate definite integrals by the Gauss-Legendre integration rule based on the stochastic arithmetic: Application in the model of osmosis system. *Mathematical Modelling of Engineering Problems*, 2020, vol. 7 (4), pp. 577–586. https://doi.org/10.18280/mmep.070410
28. Noeiaghdam S., Dreglea A., He J. H., Avazzadeh Z., Suleman M., Fariborzi Araghi M.A., Sidorov D., Sidorov N. Error estimation of the homotopy perturbation method to solve second kind Volterra integral equations with piecewise smooth kernels: Application of the CADNA library. *Symmetry*, 2020, vol. 12(10), no. 1730. https://doi.org/10.3390/sym12101730
29. Noeiaghdam S., Sidorov D., V. Sizikov, N. Sidorov, Control of accuracy on Taylor-collocation method to solve the weakly regular Volterra integral equations of the first kind by using the CESTAC method. *Applied and Computational Mathematics an International Journal*, 2020. vol. 19 (1). pp. 81–105.
30. Noeiaghdam S., Sidorov D.N., Muftahov I.R., Zhukov A.V. Control of accuracy on Taylor-collocation method for load leveling problem. *The Bulletin of Irkutsk State University. Series Mathematics*, 2019, vol. 30 (2019), pp. 59–72. https://doi.org/10.26516/1997-7670.2019.30.59
31. Noeiaghdam S., Fariborzi M. A.A. Application of the CESTAC Method to Find the Optimal Iteration of the Homotopy Analysis Method for Solving Fuzzy Integral Equations. In: Allahviranloo T., Salahshour S., Arica N. (eds.) Progress in Intelligent Decision Science. IDS 2020. *Advances in Intelligent Systems and Computing*, 2021, vol. 1301. Springer, Cham. https://doi.org/10.1007/978-3-030-66501-2_49

32. Fariborzi Araghi M.A., Noeiaghdam S. Finding Optimal Results in the Homotopy Analysis Method to Solve Fuzzy Integral Equations. In: Allahviranloo T., Salahshour S. (eds) Advances in Fuzzy Integral and Differential Equations. *Studies in Fuzziness and Soft Computing*, 2022, vol. 412. Springer, Cham. https://doi.org/10.1007/978-3-030-73711-5_7
33. Vignes J., M. La Porte. Error analysis in computing. *Information Processing*, North- Holland, 1974, pp. 610–614.
34. Vignes J., New Methods For Evaluating the Validity of the Results of Mathematical Computations. *Mathematics and Computers in Simulation*, IMACS, XX. 1978, vol. 4, pp. 227–249.
35. Chesneaux J.M. CADNA, an ADA tool for round-off error analysis and for numerical debugging. *Proc. Congress on ADA in Aerospace*. Barcelona, 1990.

Об авторах

Тында Александр Николаевич, канд. физ.-мат. наук, доц., Пензенский государственный университет, Российская Федерация, 440026, г. Пенза, tyndaan@mail.ru, https://orcid.org/0000-0001-6023-9847

Нойягдам Самад, д-р прикладной математики (PhD), доц., Иркутский национальный исследовательский технический университет, Российская Федерация, 664074, г. Иркутск;
Южно-Уральский государственный университет, Российская Федерация, 454080, г. Челябинск, snoei@istu.edu, noiagdams@susu.ru
https://orcid.org/0000-0002-2307-0891

Сидоров Денис Николаевич, д-р физ.-мат. наук, проф., Институт систем энергетики им. Л. А. Мелентьева СО РАН, Российская Федерация, 664033, г. Иркутск;
Иркутский национальный исследовательский технический университет, Российская Федерация, 664074, г. Иркутск;
Иркутский государственный университет, Российская Федерация, 664003, г. Иркутск, contact.dns@gmail.com, https://orcid.org/0000-0002-3131-1325

About the authors

Aleksandr N. Tynda, Cand. Sci. (Phys.–Math.), Assoc. Prof., Penza State University, Penza, 440026, Russian Federation, tyndaan@mail.ru, https://orcid.org/0000-0001-6023-9847

Samad Noeiaghdam, Dr. Sci. (PhD), Assoc. Prof., Irkutsk National Research Technical University, Irkutsk, 664074, Russian Federation; South Ural State University, Chelyabinsk, 454080, Russian Federation, noiagdams@susu.ru, https://orcid.org/0000-0002-2307-0891

Denis Sidorov, Dr. Sci. (Phys.–Math.), Prof., Energy Systems Institute SB RAS, Irkutsk, 664033, Russian Federation;
Irkutsk National Research Technical University, Irkutsk, 664074, Russian Federation;
Irkutsk State University, Irkutsk, 664003, Russian Federation, sidorovdn@istu.edu, https://orcid.org/0000-0002-3131-1325

Поступила в редакцию / Received 25.12.2021
Поступила после рецензирования / Revised 17.01.2022
Принята к публикации / Accepted 21.02.2022