Серия «Математика»
2021. Т. 37. С. 31—46

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

УДК 517.957
MSC 35K40, 35K57
DOI https://doi.org/10.26516/1997-7670.2021.37.31

Analytical Diffusion Wave-type Solutions to a Nonlinear Parabolic System with Cylindrical and Spherical Symmetry*

A. L. Kazakov[1,2], P. A. Kuznetsov[1]

[1] *Matrosov Institute for System Dynamics and Control Theory SB RAS, Irkutsk, Russian Federation*

[2] *Institute of Engineering Science UB RAS, Ekaterinburg, Russian Federation*

Abstract. The paper deals with a second-order nonlinear parabolic system that describes heat and mass transfer in a binary liquid mixture. The nature of nonlinearity is such that the system has a trivial solution where its parabolic type degenerates. This circumstance allows us to consider a class of solutions having the form of diffusion waves propagating over a zero background with a finite velocity. We focus on two spatially symmetric cases when one of the two independent variables is time, and the second is the distance to a certain point or line. The existence and uniqueness theorem of the diffusion wave-type solution with analytical components is proved. The solution is constructed as a power series with recursively determined coefficients, which convergence is proved by the majorant method. In one particular case, we reduce the considered problem to the Cauchy problem for a system of ordinary differential equations that inherits all the specific features of the original one. We present the form of exact solutions for exponential and power fronts. Thus, we extend the results previously obtained for a nonlinear parabolic reaction-diffusion system in the plane-symmetric form to more general cylindrical and spherical symmetry cases. Parabolic equations and systems often underlie population dynamics models. Such modeling allows one to determine the properties of populations and predict changes in population size. The results obtained, in particular, may be useful for mathematical modeling of the population dynamics of Baikal microorganisms.

* The study was funded by RFBR (project No. 20-07-00407); RFBR and the Government of the Irkutsk Region (project No. 20-41-385002); RFBR and MOST (project No. 20-51-S52003)

Keywords: parabolic partial differential equations, analytical solution, diffusion wave, existence theorem, exact solution.

1. Introduction

We consider the system of nonlinear second-order parabolic equations that describes heat and mass transfer in a binary liquid mixture [2]

$$\begin{cases} T_t = \Delta\left[\Phi(T)\right] + \Gamma(T,S), \\ S_t = \Delta\left[\Psi(S)\right] + \Lambda(S,T). \end{cases} \tag{1.1}$$

In literature, system (1.1), having the form

$$\begin{cases} T_t = \operatorname{div}\left[\Phi'(T)\nabla T\right] + \Gamma(T,S), \\ S_t = \operatorname{div}\left[\Psi'(S)\nabla S\right] + \Lambda(S,T), \end{cases} \tag{1.2}$$

is also used to describe various reaction-diffusion processes [2;4]. The equations included in system (1.2) are widely presented in the scientific literature and allow to describe the mechanisms of radiant thermal conductivity [13], filtration of liquids and gases [16], migration of biological populations [11].

Note that the case of power functions is most often found in the literature

$$\Phi'(T) = T^{\sigma}, \quad \Psi'(S) = S^{\delta}, \ \sigma, \delta > 0 \ - \ \text{const}. \tag{1.3}$$

It provides a good approximation to real processes with a comparative simplicity of research. Such systems are often applied in chemical kinetics to model the reaction-diffusion processes [4; 17].

Let $\Gamma(0,0) = \Lambda(0,0) = 0$. It can be easily seen that system (1.3) has the trivial solution $T \equiv S \equiv 0$. On the other hand, it follows from (1.3) that $\Phi'(0) = \Psi'(0) = 0$, i.e. the parabolic type of the system degenerates at $T = S = 0$. These facts allow us to consider a class of solutions having the form of diffusion waves propagating over a zero background with a finite velocity. For system 1.2), (1.3), such a wave consists of two solutions: the trivial and the perturbed ($T, S \geq 0$) continuously joined along a certain line called the wave front. Previously, such solutions were studied only for single equations. We emphasize here the classic monograph by A. A. Samarsky with co-authors [13], as well as the works of A. F. Sidorov and his followers [3;14]. These papers propose some formulations of boundary value problems on the initiation of filtration waves, as well as effective methods for constructing solutions to these problems in the class of analytical functions. Among them, the method of special series [3] plays a significant role, which is relatively easy to use and allows to eliminate singularity.

Our study of the analytical solvability of problems on the initiation of diffusion (heat, filtration) waves for the nonlinear heat (filtration) equation has a long history. Thus, we considered the case of the plane [6; 10], cylindrical, and spherical symmetry [5; 7]. To construct solutions, we used the power series, the convergence of which was proved by the majorant method.

A separate issue is constructing an exact solution to nonlinear equations of mathematical physics with predefined properties (ansatz). Such solutions, for example, can be useful for the verification of numerical calculations. Today there are many known exact solutions to the equations and systems under consideration [12; 15], but they, as a rule, are not diffusion waves. As far as we know, similar solutions can be found only in the monograph [13]. Exact solutions having the heat (diffusion) waves type are also constructed in our papers [5; 9]. We consider cases with symmetries [8] because then the original partial differential equation is reduced to an ordinary differential equation.

In [6], for the first time, the construction and study of diffusion waves type solutions for systems having the form (1.2), (1.3) are carried out. This paper continues the study and generalizes the results to cylindrical and spherical symmetry cases. Here we prove the existence and uniqueness theorem of a diffusion waves type solution with analytical components. In one particular case, we reduce the problem considered to the Cauchy problem for a system of ordinary differential equations that inherits all the specific features of the original formulation. The forms of exact solutions for exponential and power-law fronts are found.

2. Formulation

Using the standard [14] substitution $u = T^\sigma$, $v = S^\delta$, system (1.2), (1.3) can be brought to the more convenient form

$$\begin{cases} u_t = u\Delta u + \frac{1}{\sigma}(\nabla u)^2 + F(u,v), \\ v_t = v\Delta v + \frac{1}{\delta}(\nabla v)^2 + G(v,u), \end{cases} \tag{2.1}$$

where $F(u,v) = \sigma\Gamma(u^{\frac{1}{\sigma}}, v^{\frac{1}{\delta}})u^{1-1/\sigma}, G(v,u) = \delta\Lambda(v^{\frac{1}{\delta}}, u^{\frac{1}{\sigma}})v^{1-1/\delta}$. Here and further we assume that $F(0,0) = G(0,0) = 0$ and F, G are sufficiently smooth . In the presence of the spatial symmetries, (2.1) takes the form

$$\begin{cases} u_t = uu_{xx} + \frac{1}{\sigma}u_x^2 + \frac{\mu}{x}uu_x + F(u,v), \\ v_t = vv_{xx} + \frac{1}{\delta}v_x^2 + \frac{\mu}{x}vv_x + G(v,u). \end{cases} \tag{2.2}$$

The parameter μ takes the values $0, 1, 2$, which corresponds to plane, cylindrical and spherical symmetry. Then x is the distance to some plane, a straight line, and a point, respectively.

Let us consider system (2.2) with the boundary conditions

$$u(t,x)|_{x=a(t)} = v(t,x)|_{x=a(t)} = 0. \tag{2.3}$$

A curve $x = a(t)$ specifies a front of a diffusion wave, where $a(t)$ is a sufficiently smooth function and obeys inequalities $a(0) > 0, a'(0) \neq 0$. The plane-symmetric case of (2.2), (2.3) was considered in [6], but without restrictions on $a(0)$.

3. Main theorem

Here and further, an analytical function at a point means a function that coincides with its Taylor expansion in some neighborhood.

Theorem 1. *Let functions $F(u,v)$, $G(v,u)$, and $a(t)$ are analytical if $u = v = 0$ and $t = 0$, respectively, and one of the following relations hold:*
a) $u_x(t,a(t)), v_x(t,a(t)) \not\equiv 0$;
b) $u_x(t,a(t)), v_x(t,a(t)) \equiv 0$.
Then problem (2.2), (2.3) has a unique analytical solution for $t = 0$, $x = a(0)$, which is nontrivial for the case a) and trivial for the case b).

Remark 1. The pointed nontrivial and trivial solutions, joining at the front $x = a(t)$, form a diffusion wave. Thus, the theorem states that problem (2.2), (2.3) has a unique piecewise analytical solution of diffusion waves type that propagates over a zero background with a finite velocity.

Proof. We prove the theorem in two stages. At the first stage, we construct a solution to the problem in the form of the Taylor series with respect to powers of $x - a(t)$. At the second stage, the local convergence of the constructed series is proved by the majorant method.

First, we simplify the boundary conditions in problem (2.2), (2.3) by introducing a new variable $z = x - a(t)$ instead of x. The problem takes the form

$$\begin{cases} u_t - a'u_z = uu_{zz} + \frac{1}{\sigma}u_z^2 + \frac{\mu}{z+a}uu_z + F(u,v), \\ v_t - a'v_z = vv_{zz} + \frac{1}{\delta}v_z^2 + \frac{\mu}{z+a}vv_z + G(v,u), \end{cases} \tag{3.1}$$

$$u(t,z)|_{z=0} = v(t,z)|_{z=0} = 0. \tag{3.2}$$

Let us construct the solution to (3.1), (3.2) in the form of Taylor series

$$u(t,z) = \sum_{n=0}^{\infty} u_n(t)\frac{z^n}{n!}, \quad v(t,z) = \sum_{n=0}^{\infty} v_n(t)\frac{z^n}{n!}. \tag{3.3}$$

It follows from boundary condition (3.2) that $u_0 = v_0 = 0$. Assuming in (3.1) $z = 0$, we obtain the system

$$-a'u_1 = \frac{1}{\sigma}u_1^2, \quad -a'v_1 = \frac{1}{\delta}v_1^2. \tag{3.4}$$

System (3.4) has four solutions. It is easy to see that $u_1, v_1 \equiv 0$, corresponding to condition b) of the theorem, leads to a trivial solution to the problem. So, we consider the case

$$u_1 = -\sigma a', \quad v_1 = -\delta a', \tag{3.5}$$

that correcponds to condition a). The cases $u_1 \equiv 0$, $v_1 \not\equiv 0$ and $u_1 \not\equiv 0$, $v_1 \equiv 0$ require separate consideration, which is beyond the scope of the theorem. We denote

$$F_n = \frac{\partial^n F(u,v)}{\partial z^n}\Big|_{z=0}, \quad G_n = \frac{\partial^n G(v,u)}{\partial z^n}\Big|_{z=0}, \quad n = 0,1,2...$$

To find the coefficients u_2, v_2, we differentiate the equations of (3.1) with respect to z and set $z = 0$. Then we obtain the formulas

$$u_2 = \frac{1}{(1+\sigma)a'}\left(\frac{\mu(a')^2\sigma^2}{a} + F_1 + \sigma a''\right), \tag{3.6}$$

$$v_2 = \frac{1}{(1+\delta)a'}\left(\frac{\mu(a')^2\delta^2}{a} + G_1 + \delta a''\right). \tag{3.7}$$

The rest coefficients of series (3.4) are determined by n-fold differentiation of (3.1). Applying the operator $\partial^n[.]/\partial z^n|_{z=0}$, $n \geq 2$ to the first equation, we arrive to

$$u'_n - a'u_{n+1} = \sum_{k=0}^{n} C_n^k u_k u_{n+2-k} + \frac{1}{\sigma}\sum_{k=0}^{n} C_n^k u_{k+1}u_{n+1-k} +$$
$$+\mu\sum_{k=0}^{n} C_n^k \frac{(-1)^{n-k}(n-k)!}{a^{n+1-k}}\left(\sum_{l=0}^{k} C_k^l u_l u_{k+1-l}\right) + F_n. \tag{3.8}$$

Expressing the coefficient u_{n+1} from (3.8), we get the formula

$$u_{n+1} = \frac{1}{a'(1+n\sigma)}\Big[\sum_{k=2}^{n}\left(C_n^k + \frac{1}{\sigma}C_n^{k-1}\right)u_k u_{n+2-k} +$$
$$+\mu\sum_{k=0}^{n} C_n^k \frac{(-1)^{n-k}(n-k)!}{a^{n+1-k}}\left(\sum_{l=0}^{k} C_k^l u_l u_{k+1-l}\right) + F_n - u'_n\Big], \quad n \geq 2. \tag{3.9}$$

A similar formula can be derived by differentiating the second equation:

$$v_{n+1} = \frac{1}{a'(1+n\delta)}\Big[\sum_{k=2}^{n}\left(C_n^k + \frac{1}{\delta}C_n^{k-1}\right)v_k v_{n+2-k} +$$
$$+\mu\sum_{k=0}^{n} C_n^k \frac{(-1)^{n-k}(n-k)!}{a^{n+1-k}}\left(\sum_{l=0}^{k} C_k^l v_l v_{k+1-l}\right) + G_n - v'_n\Big], \quad n \geq 2. \tag{3.10}$$

Thus, we obtain that the coefficients of series (3.3) are determined uniquely by the formulas $u_0 = v_0 = 0$, (3.5)–(3.7), (3.9), (3.10). This completes the first stage of the proof.

The proof of convergence is carried out by the majorant method using the Cauchy-Kovalevskaya theorem. First, we perform the necessary preparatory transformations of the original problem, and then we construct the majorant problem.

Before constructing the majorant problem, we make the following substitution in (3.1), (3.2) $u(t,z) = zu_1(t)+z^2U(t,z), v(t,z) = zv_1(t)+z^2V(t,z)$, which is a partial Taylor expansion of the unknown functions (3.3). Note that condition (3.2) is satisfied automatically. The first equation of (3.1) takes the form

$$u_1'z + z^2U_t - a'(u_1 + 2zU + z^2U_z) =$$
$$= (u_1z + z^2U)(2U + 4zU_z + z^2U_{zz}) + \frac{1}{\sigma}(u_1 + 2zU + z^2U_z)^2+$$
$$+\frac{\mu}{z+a}(u_1z + z^2U)(u_1 + 2zU + z^2U_z) + F(u_1z + z^2U, v_1z + z^2V). \quad (3.11)$$

After collecting the terms and dividing by z, equation (3.11) becomes

$$u_1' + zU_t - a'(2U + zU_z) =$$
$$= (u_1 + zU)(2U + 4zU_z + z^2U_{zz}) + \frac{2}{\sigma}u_1(2U + zU_z) + \frac{z}{\sigma}(2U + zU_z)^2+$$
$$+\frac{\mu}{z+a}(u_1 + zU)(u_1 + 2zU + z^2U_z) + \frac{1}{z}F(u_1z + z^2U, v_1z + z^2V). \quad (3.12)$$

Since $a(0) \neq 0$, $F_0 = F(0,0) = 0$ and the functions $1/(z+a)$, $F(u_1z + z^2U, v_1z + z^2V)$ are analytical, then the following expansions are valid

$$\frac{1}{z+a} = \sum_{n=0}^{\infty}\frac{(-1)^n}{a^{n+1}}z^n,\ \frac{1}{z}F(u_1z+z^2U, v_1z+z^2V) = \frac{1}{z}\sum_{n=0}^{\infty}F_n\frac{z^n}{n!} = \sum_{n=1}^{\infty}F_n\frac{z^{n-1}}{n!}.$$

They help us to bring (3.11) to the equation

$$2U\left(1+\frac{1}{\sigma}\right) + \left(4+\frac{1}{\sigma}\right)zU_z + z^2U_{zz} =$$
$$= f_0(t) + zf_1(t,U,U_t,V) + z^2f_2(t,U,V,U_z) + z^3f_3(t,z,U,V,U_z,U_{zz}), \quad (3.13)$$

where f_0, f_1, f_2, f_3 are known analytical functions of their variables. We don't present them here due to their bulkiness.

The second equation of (3.1) is transformed similarly:

$$2V\left(1+\frac{1}{\delta}\right) + \left(4+\frac{1}{\delta}\right)zV_z + z^2V_{zz} =$$

$$= g_0(t) + zg_1(t, V, V_t, U) + z^2 g_2(t, V, U, V_z) + z^3 g_3(t, z, V, U, V_z, V_{zz}). \quad (3.14)$$

Here g_0, g_1, g_2, g_3 also known analytical functions.

Thus, problem (3.1), (3.2) is reduced to two equations (3.13) and (3.14). The solutions to the equations can be constructed in the form of Taylor series

$$U(t, z) = \sum_{n=0}^{\infty} U_n(t) \frac{z^n}{n!}, \quad V(t, z) = \sum_{n=0}^{\infty} V_n(t) \frac{z^n}{n!}. \quad (3.15)$$

The coefficients of (3.15) are determined according to the already known procedure by the formulas

$$U_0 = \frac{f_0(t)}{2(1+\frac{1}{\sigma})}, \; V_0 = \frac{g_0(t)}{2(1+\frac{1}{\delta})}, \quad U_1 = \frac{f_1(t, U_0, U_0', V_0)}{3(2+\frac{1}{\sigma})},$$

$$V_1 = \frac{g_1(t, V_0, V_0', U_0)}{3(2+\frac{1}{\delta})}, \; U_2 = \frac{\frac{\partial f_1}{\partial z}\Big|_{z=0} + f_2\Big|_{z=0}}{2(3+\frac{1}{\sigma})}, \quad V_2 = \frac{\frac{\partial g_1}{\partial z}\Big|_{z=0} + g_2\Big|_{z=0}}{2(3+\frac{1}{\delta})},$$

$$U_n = n\alpha_n \frac{\partial^{n-1} f_1}{\partial z^{n-1}}\Big|_{z=0} + n(n-1)\alpha_n \frac{\partial^{n-2} f_2}{\partial z^{n-2}}\Big|_{z=0} +$$

$$+ n(n-1)(n-2)\alpha_n \frac{\partial^{n-3} f_3}{\partial z^{n-3}}\Big|_{z=0}, \quad \alpha_n = \frac{1}{2(1+\frac{1}{\sigma}) + (4+\frac{1}{\sigma})n + n(n-1)},$$

$$V_n = n\beta_n \frac{\partial^{n-1} g_1}{\partial z^{n-1}}\Big|_{z=0} + n(n-1)\beta_n \frac{\partial^{n-2} g_2}{\partial z^{n-2}}\Big|_{z=0} +$$

$$+ n(n-1)(n-2)\beta_n \frac{\partial^{n-3} g_3}{\partial z^{n-3}}\Big|_{z=0}, \quad \beta_n = \frac{1}{2(1+\frac{1}{\delta}) + (4+\frac{1}{\delta})n + n(n-1)},$$

where $n \geq 3$. Since all these coefficients, as well as the functions f_i, g_i, $i = 1, 2, 3$, are analytical, it is possible to construct majorants for them.

If the majorant estimates

$$U_0(t), V_0(t) \ll W_0(t); \quad U_1(t), V_1(t) \ll W_1(t);$$

$$f_1(t, U, U_t, V), g_1(t, V, V_t, U) \ll h_1(t, W, W_t, W);$$

$$f_2(t, U, V, U_z), g_2(t, U, V, U_z) \ll h_2(t, W, W, W_z);$$

$$f_3(t, z, U, V, U_z, U_{zz}), g_3(t, z, U, V, U_z, U_{zz}) \ll h_3(t, z, W, W, W_z, W_{zz})$$

hold, then the solution to the problem

$$W_{zz} = \frac{\partial h_1(t, W, W_t, W)}{\partial z} + h_2(t, W, W, W_z) + zh_3(t, z, W, W, W_z, W_{zz}), \quad (3.16)$$

$$W(t, z)|_{z=0} = W_0(t), \quad W_z(t, z)|_{z=0} = W_1(t) \quad (3.17)$$

majorizes the solution to problem (3.14), (3.15). This can be verified by constructing the solution to (3.16), (3.17) in the form of the Taylor series

$$W(t,z)=\sum_{n=0}^{\infty}W_n(t)\frac{z^n}{n!}.$$

The obtained coefficients satisfy the majorant estimates

$$U_2=\frac{\frac{\partial f_1}{\partial z}\Big|_{z=0}+f_2\Big|_{z=0}}{2(3+\frac{1}{\sigma})}\ll\frac{\partial h_1}{\partial z}\Big|_{z=0}+h_2\Big|_{z=0}=W_2,$$

$$V_2=\frac{\frac{\partial g_1}{\partial z}\Big|_{z=0}+g_2\Big|_{z=0}}{2(3+\frac{1}{\delta})}\ll\frac{\partial h_1}{\partial z}\Big|_{z=0}+h_2\Big|_{z=0}=W_2,$$

$$U_n=n\alpha_n\frac{\partial^{n-1}f_1}{\partial z^{n-1}}\Big|_{z=0}+$$

$$+n(n-1)\alpha_n\frac{\partial^{n-2}f_2}{\partial z^{n-2}}\Big|_{z=0}+n(n-1)(n-2)\alpha_n\frac{\partial^{n-3}f_3}{\partial z^{n-3}}\Big|_{z=0}\ll$$

$$\ll\frac{\partial^{n-1}h_1}{\partial z^{n-1}}\Big|_{z=0}+\frac{\partial^{n-2}h_2}{\partial z^{n-2}}\Big|_{z=0}+(n-2)\frac{\partial^{n-3}h_3}{\partial z^{n-3}}\Big|_{z=0}=W_n,\quad n\geq 3,$$

$$V_n=n\beta_n\frac{\partial^{n-1}g_1}{\partial z^{n-1}}\Big|_{z=0}+$$

$$+n(n-1)\beta_n\frac{\partial^{n-2}g_2}{\partial z^{n-2}}\Big|_{z=0}+n(n-1)(n-2)\beta_n\frac{\partial^{n-3}g_3}{\partial z^{n-3}}\Big|_{z=0}\ll$$

$$\ll\frac{\partial^{n-1}h_1}{\partial z^{n-1}}\Big|_{z=0}+\frac{\partial^{n-2}h_2}{\partial z^{n-2}}\Big|_{z=0}+(n-2)\frac{\partial^{n-3}h_3}{\partial z^{n-3}}\Big|_{z=0}=W_n,\quad n\geq 3.$$

It follows that (3.16), (3.17) is majorant problem for (3.1), (3.2). Next, we reduce (3.16), (3.17) to a Kovalevskaya-type problem. To do this, we differentiate equation (3.16) by z, resolve it with respect to W_{zzz} and add the third boundary condition $W_{zz}(t,0)=W_2(t)$. To avoid confusion in determining with respect to which variable the differentiation was performed, we use the notation $h_3=h_3(t,y_1,y_2,y_3,y_4,y_5)$. Problem (3.16), (3.17) takes the form

$$W_{zzz}=\frac{1}{1-z\frac{\partial h_3}{\partial y_5}}\Big(\frac{\partial^2 h_1}{\partial z^2}+\frac{\partial h_2}{\partial z}+h_3+$$

$$+z\frac{\partial h_3}{\partial y_1}+z\frac{\partial h_3}{\partial y_2}W_z+z\frac{\partial h_3}{\partial y_3}W_z+z\frac{\partial h_3}{\partial y_4}W_{zz}\Big),\tag{3.18}$$

$$W(t,z)|_{z=0}=W_0(t),\ W_z(t,z)|_{z=0}=W_1(t),\ W_{zz}(t,z)|_{z=0}=W_2(t).\tag{3.19}$$

Now we have problem (3.18), (3.19) of the Kovalevskaya type with analytical input data. By the Cauchy-Kovalevskaya theorem, we obtain that series (3.15) have a non-zero convergence radius. Thus, the second stage of the proof is completed, and the theorem is proved. □

Remark 2. A counterexample constructed in [6] shows that if both conditions of this theorem are violated, the original problem may or may not have an analytical solution. In the latter case, the radii of convergence of the constructed series can be equal to zero.

4. Reduction to the Cauchy problem for the ODEs

For constructing exact solutions of systems of nonlinear partial differential equations, it becomes necessary to reduce them to systems of ordinary differential equations (see, for example, [6; 15]). In this section, a similar reduction is performed for problem (3.1), (3.2) with the following functions F and G:

$$F(u,v) = \alpha v^{\gamma-\xi} u^{\xi}, G(v,u) = \beta u^{\gamma-\eta} v^{\eta},$$

where $\alpha, \beta \in \mathbb{R},\ \gamma \in \mathbb{N},\ \ \xi, \eta \in \mathbb{N} \cup \{0\},\ \ \xi, \eta \leq \gamma$.

The problem in this case has the form

$$\begin{cases} u_t - a' u_z = u u_{zz} + \frac{1}{\sigma} u_z^2 + \frac{\mu}{z+a} u u_z + \alpha v^{\gamma-\xi} u^{\xi}, \\ v_t - a' v_z = v v_{zz} + \frac{1}{\delta} v_z^2 + \frac{\mu}{z+a} v v_z + \beta u^{\gamma-\eta} v^{\eta}, \end{cases} \tag{4.1}$$

$$u(t,z)|_{z=0} = v(t,z)|_{z=0} = 0. \tag{4.2}$$

We also assume that $u_z(t,0), v_z(t,0) \not\equiv 0$. In this case, the problem satisfies the conditions of the proved theorem and has an analytical solution representable in the form of the Taylor series (3.3).

Reduction (4.1), (4.2) to the Cauchy problem for a system of ordinary differential equations is performed in accordance with the technique proposed in [6], where it was described for $\mu = \xi = \eta = 0$. The essence of this technique is to transform the original problem into such a form which allows the separation of variables.

Let us introduce a new variable $s = -z/a$. We get the problem

$$\begin{cases} u_t + \frac{(1-s)a'}{a} u_s = \frac{1}{a^2} u u_{ss} + \frac{1}{\sigma a^2} u_s^2 - \frac{\mu}{a^2(1-s)} u u_s + \alpha v^{\gamma-\xi} u^{\xi}, \\ v_t + \frac{(1-s)a'}{a} v_s = \frac{1}{a^2} v v_{ss} + \frac{1}{\delta a^2} v_s^2 - \frac{\mu}{a^2(1-s)} v v_s + \beta u^{\gamma-\eta} v^{\eta}, \end{cases} \tag{4.3}$$

$$u(t,s)|_{s=0} = v(t,s)|_{s=0} = 0. \tag{4.4}$$

Then we change unknown functions in problem (4.3), (4.4) as $u = a^{\theta}(t)p(t,s)$, $v = a^{\theta}(t)q(t,s)$. The parameter $\theta = \text{const}$ and the function $a(t)$ will be specified further. The problem takes the form

$$\begin{cases} a^{\theta}p_t + \theta a^{\theta-1}a'p + (1-s)a^{\theta-1}a'p_s = \\ = a^{2\theta-2}pp_{ss} + \frac{1}{\sigma}a^{2\theta-2}p_s^2 - \frac{\mu}{1-s}a^{2\theta-2}pp_s + \alpha a^{\theta\gamma}q^{\gamma-\xi}p^{\xi}, \\ a^{\theta}q_t + \theta a^{\theta-1}a'q + (1-s)a^{\theta-1}a'q_s = \\ = a^{2\theta-2}qq_{ss} + \frac{1}{\delta}a^{2\theta-2}q_s^2 - \frac{\mu}{1-s}a^{2\theta-2}qq_s + \beta a^{\theta\gamma}p^{\gamma-\eta}q^{\eta}, \end{cases} \tag{4.5}$$

$$p(t,s)|_{s=0} = q(t,s)|_{s=0} = 0. \tag{4.6}$$

The right-hand side of (4.5) can be significantly simplified. We set $\theta = 2/(2-\gamma)$, $\gamma \neq 2$ and divide both equations by $a^{2\theta-2}$. So, we obtain the system

$$\begin{cases} a^{\frac{2-2\gamma}{2-\gamma}}p_t + \frac{2}{2-\gamma}a^{\frac{\gamma}{\gamma-2}}a'p + (1-s)a^{\frac{\gamma}{\gamma-2}}a'p_s = \\ = pp_{ss} + \frac{1}{\sigma}p_s^2 - \frac{\mu}{1-s}pp_s + \alpha q^{\gamma-\xi}p^{\xi}, \\ a^{\frac{2-2\gamma}{2-\gamma}}q_t + \frac{2}{2-\gamma}a^{\frac{\gamma}{\gamma-2}}a'q + (1-s)a^{\frac{\gamma}{\gamma-2}}a'q_s = \\ = qq_{ss} + \frac{1}{\delta}q_s^2 - \frac{\mu}{1-s}qq_s + \beta p^{\gamma-\eta}q^{\eta}. \end{cases} \tag{4.7}$$

One can see that the functions a and a' are present only in the left-hand side of (4.7). Moreover, the term $a^{\frac{\gamma}{\gamma-2}}a'$ can be made constant by choosing the function $a(t)$. It is possible if

$$a^{\frac{\gamma}{\gamma-2}}a' = c, \ c \neq 0 \ - \ \text{const}.$$

Therefore, $a(t) = c_1 e^{c_2 t}$ for $\gamma = 1$; and $a(t) = (c_1 t + c_2)^{(\gamma-2)/(2\gamma-2)}$ for $\gamma \geq 3$.

System (4.7) in this case takes the form

$$\begin{cases} a^{\frac{2-2\gamma}{2-\gamma}}p_t + \frac{2}{2-\gamma}cp + (1-s)cp_s = pp_{ss} + \frac{1}{\sigma}p_s^2 - \frac{\mu}{1-s}pp_s + \alpha q^{\gamma-\xi}p^{\xi}, \\ a^{\frac{2-2\gamma}{2-\gamma}}q_t + \frac{2}{2-\gamma}cq + (1-s)cq_s = qq_{ss} + \frac{1}{\delta}q_s^2 - \frac{\mu}{1-s}qq_s + \beta p^{\gamma-\eta}q^{\eta}, \end{cases} \tag{4.8}$$

We show now that in problem (4.8), (4.6), the analytical functions p and q don't depend on the variable t, thereby completing its reduction to the Cauchy problem for the ODEs. To do this, we construct a solution in the form of Taylor series

$$p(t,s) = \sum_{n=0}^{\infty} p_n(t)\frac{s^n}{n!}, \ q(t,s) = \sum_{n=0}^{\infty} q_n(t)\frac{s^n}{n!}, \tag{4.9}$$

According to the proved theorem, series (4.9) have nonzero convergence radii. Following the well-known procedure, we obtain the formulas for the coefficients of the series:

$$p_0 = q_0 = 0,\ p_1 = c\sigma,\ q_1 = c\delta,$$

$$p_2 = \frac{1}{c(1+\sigma)}\left(\frac{c^2\sigma\gamma}{2-\gamma} + \mu c^2\sigma^2 - \widetilde{F}_1\right),\ q_2 = \frac{1}{c(1+\delta)}\left(\frac{c^2\delta\gamma}{2-\gamma} + \mu c^2\delta^2 - \widetilde{G}_1\right),$$

$$p_{n+1} = \frac{1}{c(1+n\sigma)}\Big[p_n' a^{\frac{2-2\gamma}{2-\gamma}} + \frac{2c}{2-\gamma}p_n - cnp_n - \sum_{k=2}^{n}\left(C_n^k + \frac{1}{\sigma}C_n^{k-1}\right)p_k p_{n+2-k} +$$

$$+\mu\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{n!}{(k-l)!l!}p_l p_{k+1-l} - \widetilde{F}_n\Big],$$

$$q_{n+1} = \frac{1}{c(1+n\delta)}\Big[q_n' a^{\frac{2-2\gamma}{2-\gamma}} + \frac{2c}{2-\gamma}q_n - cnq_n - \sum_{k=2}^{n}\left(C_n^k + \frac{1}{\delta}C_n^{k-1}\right)q_k q_{n+2-k} +$$

$$+\mu\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{n!}{(k-l)!l!}q_l q_{k+1-l} - \widetilde{G}_n\Big],$$

$$\widetilde{F}_n = \frac{\partial^n(\alpha q^{\gamma-\xi}p^{\xi})}{\partial s^n}\Big|_{s=0},\ \widetilde{G}_n = \frac{\partial^n(\beta p^{\gamma-\eta}q^{\eta})}{\partial s^n}\Big|_{s=0}.$$

These formulas show that all the coefficients of the series (4.9) do not depend on the variable t and are constants. Hence, $p = p(s)$, $q = q(s)$, and problem (4.8), (4.6) is the Cauchy problem for ODEs

$$\begin{cases} pp'' + \frac{1}{\sigma}(p')^2 + \frac{\mu}{s-1}pp' + (s-1)cp' + \frac{2}{\gamma-2}cp + \alpha q^{\gamma-\xi}p^{\xi} = 0, \\ qq'' + \frac{1}{\delta}(q')^2 + \frac{\mu}{s-1}qq' + (s-1)cq' + \frac{2}{\gamma-2}cq + \beta p^{\gamma-\eta}q^{\eta} = 0, \end{cases}$$

$$p(0) = q(0) = 0,\ \ p'(0) = c\sigma,\ q'(0) = c\delta.$$

The given below theorem follows from the above reasoning.

Theorem 2. *Let $c_1, c_2, c_3 \in \mathbb{R}$, $c_1, c_3 \neq 0$, $c_2 > 0$. Then problem* (4.1), (4.2) *for $\gamma \neq 2$ can be reduced to the Cauchy problem for a system of ordinary differential equations for a) $a(t) = c_1 e^{c_3 t}$, if $\gamma = 1$; b) $a(t) = (c_1 t + c_2)^{(\gamma-2)/(2\gamma-2)}$, if $\gamma \geq 3$.*

In this case, the solution to problem (4.1), (4.2) can be found as

$$u(t,z) = a^{\frac{2}{2-\gamma}}(t)p(-z), v(t,z) = a^{\frac{2}{2-\gamma}}(t)q(-z),$$

where $a(t) = c_1 e^{c_2 t}$ for $\gamma = 1$, $a(t) = (c_1 t + c_2)^{(\gamma-2)/(2\gamma-2)}$ for $\gamma \in \mathbb{N}, \gamma \geq 3$.

Remark 3. We don't discuss the case $\gamma = 2$ since for $\mu \neq 0$, the technique used in the proof of Theorem 2 does not allow the reduction of problem (4.3), (4.4) to the Cauchy problem for ODEs. If $\gamma = 2$, $\mu = 0$, then, as shown in [6], the reduction is possible for $\xi = \eta = 0$. It is easy to show that such a reduction is also possible in a somewhat more general case, when ξ and η take the values $0, 1, 2$.

5. Conclusion

Summing up the research, we note that the main result is the extending the results previously obtained for a nonlinear parabolic reaction-diffusion system [6] in the plane-symmetric form to more general cylindrical and spherical symmetry cases. It is not automatic expansion, which, in particular, is corroborated by the fact that not all previously found solutions can be generalized. We also point out that both theorems formulated and all the transformations performed are valid for any $\mu \geq 0$. Moreover, in principle, you can take two different $\mu_1 \geq 0$ and $\mu_2 \geq 0$, it does not matter from a mathematical point of view. However, the physical meaning of such a generalization is not obvious. The singular parabolic equations and systems we consider, along with hyperbolic ones, often underlie population dynamics models [1; 11]. Such modeling allows one to determine the properties of populations and predict changes in population size. The results obtained, in particular, may be useful for mathematical modeling of the population dynamics of Baikal microorganisms.

Further research in this direction can be associated with studying the initiation of a diffusion wave by given boundary conditions for the considered system. Such a problem is, in a certain sense, more natural. However, as the history of studying such issues in the scientific school of A. F. Sidorov [14] shows, it is much more difficult from a mathematical point of view.

References

1. Arguchintsev A.V., Poplevko V.P. An optimal control problem by a hyperbolic system with boundary delay. *The Bulletin of Irkutsk State University. Series Mathematics*, 2021, vol. 35, pp. 3-17. https://doi.org/10.26516/1997-7670.2021.35.3
2. Bergman T.L., Lavine A.S., Incropera F.P., DeWitt D.P. *Fundamentals of heat and mass transfer*. John Wiley & Sons, 2011, 992 p.
3. Filimonov M.Yu. Representation of solutions of boundary value problems for nonlinear evolution equations by special series with recurrently caculated coefficients. *Journal of Physics: Conference Series*, 2019, art. no. 012071. https://doi.org/10.1088/1742-6596/1268/1/012071
4. Gambino G., Lombardo M.C., Sammartino M., Sciacca V. Turing pattern formation in the Brusselator system with nonlinear diffusion. *Physical Review E*, 2013, vol. 88, art. no. 042925. https://doi.org/10.1103/PhysRevE.88.042925

5. Kazakov A.L. On exact solutions to a heat wave propagation boundary-value problem for a nonlinear heat equation. *Siberian Electronic Mathematical Reports*, 2019, vol. 16, pp. 1057-1068. https://doi.org/10.33048/semi.2019.16.073 (in Russian)
6. Kazakov A.L., Kuznetsov P.A., Lempert A.A. Analytical solutions to the singular problem for a system of nonlinear parabolic equations of the reaction-diffusion type. *Symmetry*, 2020, vol. 12, no. 6, art. no. 999. https://doi.org/10.3390/SYM12060999
7. Kazakov A.L., Kuznetsov P.A., Lempert A.A. On construction of heat wave for nonlinear heat equation in symmetrical case. *The Bulletin of Irkutsk State University. Series Mathematics*, 2015, vol. 11, pp. 39-53. (in Russian)
8. Kazakov A.L., Orlov Sv.S. On some exact solutions of the nonlinear heat equation. *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2016, vol. 22, no. 1, pp. 112-123. (in Russian)
9. Kazakov A.L., Orlov Sv.S., Orlov S.S. Construction and study of exact solutions to a nonlinear heat equation. *Siberian Mathematical Journal*, 2018, vol. 59, no. 3, pp. 427–441. https://doi.org/10.1134/S0037446618030060
10. Kazakov A.L., Spevak L.F. Boundary element method and power series method for onedimensional non-linear filtration problems. *The Bulletin of Irkutsk State University. Series Mathematics*, 2012, vol. 5, no. 2, pp. 2-17. (in Russian)
11. Murray J. *Mathematical biology: I. An introduction.* Interdisciplinary applied mathematics, vol. 17. New York, Springer, 2002, 575 p.
12. Polyanin A.D., Zaitsev V.F. *Handbook of nonlinear partial differential equations.* New York, Chapman and Hall/CRC, 2003, 840 p. https://doi.org/10.1201/9780203489659
13. Samarskii A.A., Galaktionov V.A., Kurdyumov S.P., Mikhailov A.P. *Blow-up in quasilinear parabolic equations.* Walter de Gruyte Berlin, New-York, 1995, 554 p. https://doi.org/10.1515/9783110889864
14. Sidorov A.F. *Izbrannye trudy: Matematika. Mekhanika* [Selected Works: Mathematics. Mechanics]. Moscow, Fizmatlit Publ., 2001, 576 p. (in Russian)
15. Stepanova I.V. Group analysis of variable coefficients heat and mass transfer equations with power nonlinearity of thermal diffusivity. *Applied Mathematics and Computation*, 2019, vol. 343, pp. 57-66. https://doi.org/10.1016/j.amc.2018.09.036
16. Vazquez J.L. *The porous medium equation: mathematical theory.* Oxford, Clarendon Press, 2007, 648 p.
https://doi.org/10.1093/acprof:oso/9780198569039.001.0001
17. Zemskov E.P. Turing instability in reaction–diffusion systems with nonlinear diffusion. *Journal of Experimental and Theoretical Physics*, 2013, vol. 117, no. 4, pp. 764-769. https://doi.org/10.1134/S1063776113120194

Alexander Kazakov, Doctor of Sciences (Physics and Mathematics), Professor RAS, Principal Research Scientist, Matrosov Institute for System Dynamics and Control Theory SB RAS, 134, Lermontov st., Irkutsk, 664033, Russian Federation; Lead Researcher, Institute of Engineering Science UB RAS, 34, Komsomolskaya st., Ekaterinburg, 620049, Russian Federation, tel.: (3952)453033, email: kazakov@icc.ru,
ORCID iD https://orcid.org/0000-0002-3047-1650

Pavel Kuznetsov, Candidate of Sciences (Physics and Mathematics), Junior Researcher, Matrosov Institute for System Dynamics and Control Theory SB RAS, 134, Lermontov st., Irkutsk, 664033, Russian Federation,

tel.: (3952)453107, email: kuznetsov@icc.ru,
ORCID iD https://orcid.org/0000-0002-9118-5236

Received 08.07.2021

Аналитические решения нелинейной параболической системы, имеющие тип диффузионной волны, при наличии цилиндрической и сферической симметрии

А. Л. Казаков[1,2], П. А. Кузнецов[1]

[1] *Институт динамики систем и теории управления им. В. М. Матросова СО РАН, Иркутск, Российская Федерация*

[2] *Институт машиноведения УрО РАН, Екатеринбург, Российская Федерация*

Аннотация. Рассматривается система нелинейных параболических уравнений второго порядка, описывающая тепломассоперенос в бинарной жидкой смеси. Специфика нелинейности такова, что система имеет тривиальное решение, на котором ее параболический тип вырождается. Данное обстоятельство позволяет рассматривать класс решений типа диффузионных волн, распространяющихся по нулевому фону с конечной скоростью. В работе основное внимание уделено двум пространственно-симметричным случаям, когда одна из двух независимых переменных есть время, а вторая — расстояние до некоторой точки или прямой. Доказана теорема существования и единственности решения типа диффузионной волны с аналитическими составляющими. Решение строится в виде степенного ряда с рекуррентно определяемыми коэффициентами. Сходимость рядов доказывается методом мажорант. В одном частном случае проведена редукция рассматриваемой задачи к задаче Коши для системы обыкновенных дифференциальных уравнений, наследующей все специфические особенности исходной. Выписана форма точных решений при экспоненциальном и степенном фронтах. Таким образом, удалось распространить результаты, ранее полученные для нелинейной параболической системы «реакция – диффузия» в плоскосимметричном виде, на более общие случаи цилиндрической и сферической симметрии. Параболические уравнения и системы часто лежат в основе моделей популяционной динамики. Такое моделирование позволяет выявлять свойства популяций и прогнозировать изменение численности. Полученные результаты, в частности, могут быть интересны с точки зрения математического моделирования популяционной динамики байкальских микроорганизмов.

Ключевые слова: параболические уравнения с частными производными, аналитическое решение, диффузионная волна, теорема существования, точное решение.

Список литературы

1. Arguchintsev A. V., Poplevko V. P. An optimal control problem by a hyperbolic system with boundary delay // The Bulletin of Irkutsk State University. Series Mathematics. 2021. Vol. 35. P. 3–17. https://doi.org/10.26516/1997-7670.2021.35.3
2. Fundamentals of heat and mass transfer / T. L. Bergman, A. S. Lavine, F. P. Incopera, D. P. Dewitt. John Wiley & Sons, 2011. 992 p.

3. Filimonov M. Yu. Representation of solutions of boundary value problems for nonlinear evolution equations by special series with recurrently caculated coefficients // Journal of Physics: Conference Series. 2019. Art. N 012071. https://doi.org/10.1088/1742-6596/1268/1/012071
4. Gambino G., Lombardo M. C., Sammartino M., Sciacca V. Turing pattern formation in the Brusselator system with nonlinear diffusion // Physical Review E. 2013. Vol. 88. Art. N 042925. https://doi.org/10.1103/PhysRevE.88.042925
5. Казаков А. Л. О точных решениях краевой задачи о движении тепловой волны для уравнения нелинейной теплопроводности // Сибирские электронные математические известия. 2019. Т. 16. С. 1057–1068. https://doi.org/10.33048/semi.2019.16.073
6. Kazakov A. L., Kuznetsov P. A., Lempert A. A. Analytical solutions to the singular problem for a system of nonlinear parabolic equations of the reaction-diffusion type // Symmetry. 2020. Vol. 12, N 6. 999. https://doi.org/10.3390/SYM12060999
7. Казаков А. Л., Кузнецов П. А., Лемперт А. А. О построении тепловой волны для нелинейного уравнения теплопроводности в симметричном случае // Известия Иркутского государственного университета. Серия Математика. 2015. Т. 11. С. 39–53.
8. Казаков А. Л., Орлов Св. С. О некоторых точных решениях нелинейного уравнения теплопроводности // Труды института математики и механики УрО РАН. 2016. Т. 22, № 1. С. 112–123.
9. Казаков А. Л., Орлов Св. С., Орлов С. С. Построение и исследование некоторых точных решений нелинейного уравнения теплопроводности // Сибирский математический журнал. 2018. Т. 59, № 3. С. 544–560. https://doi.org/10.17377/smzh.2018.59.306
10. Казаков А. Л., Спевак Л. Ф. Методы граничных элементов и степенных рядов в одномерных задачах нелинейной фильтрации // Известия Иркутского государственного университета. Серия Математика. 2012. Т. 5, № 2. С. 2–17.
11. Murray J. Mathematical biology: I. An introduction, third edition. Interdisciplinary applied mathematics. Vol. 17. New York : Springer, 2002. 575 p.
12. Полянин А. Д., Зайцев В. Ф. Справочник по нелинейным уравнениям математической физики. Точные решения. М. : Физматлит, 2002. 432 с.
13. Режимы с обострением в задачах для квазилинейных параболических уравнений / А. А. Самарский, В. А. Галактионов, С. П. Курдюмов, А. П. Михайлов. М. : Наука, 1987. 534 p.
14. Сидоров А. Ф. Избранные труды: Математика. Механика. М. : Физматлит, 2001. 576 с.
15. Stepanova I. V. Group analysis of variable coefficients heat and mass transfer equations with power nonlinearity of thermal diffusivity // Applied Mathematics and Computation. 2019. Vol. 343. P. 57–66. https://doi.org/10.1016/j.amc.2018.09.036
16. Vazquez J. L. The porous medium equation: mathematical theory. Oxford : Clarendon Press, 2007. 648 p. https://doi.org/10.1093/acprof:oso/9780198569039.001.0001
17. Земсков Е. П. Неустойчивость Тьюринга в реакционно-диффузионных системах с нелинейной диффузией // Журнал экспериментальной и теоретической физики. 2013. Т. 144, № 4. С. 878–884.

Александр Леонидович Казаков, доктор физико-математических наук, профессор РАН, главный научный сотрудник, Институт динамики систем и теории управления им. В. М. Матросова СО РАН, Российская Федерация, 664033, г. Иркутск, ул. Лермонтова, 134; ведущий научный сотрудник, Институт машиноведения УрО РАН, Российская Федерация, 620049, г. Екатеринбург, ул. Комсомольская, 34, тел.: (3952)453033, email: kazakov@icc.ru, ORCID iD https://orcid.org/0000-0002-3047-1650

Павел Александрович Кузнецов, кандидат физико-математических наук, младший научный сотрудник, Институт динамики систем и теории управления им. В. М. Матросова СО РАН, Российская Федерация, 664033, г. Иркутск, ул. Лермонтова, 134, тел.: (3952)453107, email: kuznetsov@icc.ru, ORCID iD https://orcid.org/0000-0002-9118-5236

Поступила в редакцию 08.07.2021