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On Periodic Groups Saturated with Finite Frobenius Groups *

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Abstract. A group is called weakly conjugate biprimitively finite if each its element of prime order generates a finite subgroup with any of its conjugate elements. A binary finite group is a periodic group in which any two elements generate a finite subgroup. If $\mathfrak{X}$ is some set of finite groups, then the group G saturated with groups from the set $\mathfrak{X}$ if any finite subgroup of G is contained in a subgroup of G, isomorphic to some group from $\mathfrak{X}$. A group $G = F \leftthreetimes H$ is a Frobenius group with kernel F and a complement H if $H \cap H^f = 1$ for all $f \in F^{\#}$ and each element from $G \setminus F$ belongs to a one conjugated to H subgroup of G. In the paper we prove that a saturated with finite Frobenius groups periodic weakly conjugate biprimitive finite group with a nontrivial locally finite radical is a Frobenius group. A number of properties of such groups and their quotient groups by a locally finite radical are found. A similar result was obtained for binary finite groups with the indicated conditions. Examples of periodic non locally finite groups with the properties above are given, and a number of questions on combinatorial group theory are raised.

Keywords: Frobenius group, weakly conjugate biprimitive finite group, locally finite radical, saturation condition.

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1. Introduction

Let $\mathfrak{X}$ be some set of finite groups. The group G *is saturated with groups from the set* $\mathfrak{X}$ if any finite subgroup from G is contained in a subgroup of the group G isomorphic to some group from $\mathfrak{X}$; the set $\mathfrak{X}$ is called *saturating* for G.

To avoid misunderstandings, clarify that a finite group G saturated with groups from the set $\mathfrak{X}$, by this definition, is isomorphic to one of the groups of the set $\mathfrak{X}$.

Groups saturated with groups from various sets $\mathfrak{X}$ of finite groups have been intensively studied for over 25 years. Several dozen articles are devoted to such studies, including those in the central press (see, for example, [7–9; 22; 24–26; 31; 32]). The set $\mathfrak{X}$ in these papers is typically consisted of finite simple non-Abelian groups, their simple extensions and direct products. They establish either the local finiteness of the studied periodic group, or the existence of a locally finite periodic part in the Shunkov group. There are several works that fall off the specified range (see, for example, [21; 23–29]). In these papers, studied groups are saturated with finite dihedral groups or generalized semidihedral groups. Ultimately, however, the studied periodic groups also turned out to be locally finite. Among the studied in this paper groups there are real periodic non-locally finite groups with the saturation condition considered in this paper.

In the above studies, the properties of the groups of a saturating set, adapted methods of local analysis of the theory of finite groups, and a number of results by Schmidt, Chernikov, Ito, Shunkov, and others on infinite groups were used. In this paper, we also apply the technique of working with Frobenius subgroups from [12].

The well-known results of combinatorial group theory were typically previously used as a justification for the relevance of the saturation condition, since periodic non-locally finite groups without involutions with different saturating sets $\mathfrak{X}$ are widely known: free Burnside groups [2], periodic group products constructed by S.I. Adyan [1], A.Yu. Olshanskii's groups from [10, §35] and others. In this paper, the connection with combinatorial group theory is closer, especially since among the groups under study there are real periodic non-locally finite groups. In the conclusion of the work known to the authors examples of such groups are given, and questions of interest to us that have arisen as a result of research are posed.

We call the group G *a Frobenius group with complement H and kernel F* under the following three conditions: 1) (G, H) is a Frobenius pair, that is, H is a proper subgroup of the group G and $H \cap H^g = 1$ for any element $g \in G \setminus H$; 2) the set $F = G \setminus \bigcup_{g\in G} g^{-1}H^{\#}g$ is a subgroup; 3) $G = F \leftthreetimes H$.

The structure of complements in finite Frobenius groups has been studied in detail (see, for example, [12, §1.2]). The kernel of a finite Frobenius group is nilpotent by the well-known Thompson theorem [12, §1.3]. G.

Higman [30] proved that the nilpotency class of a finite group admitting a regular automorphism of prime order p does not exceed some constant $k(p)$. He also found that $k(p) \geq \frac{p^2-1}{4}$. V.A. Kreknin and A.I. Kostrikin found an upper bound in [6]: $k(p) \leq \frac{(p-1)^{2p-1}-1}{p-1}$ (see, for example, [19], [12, page 18]). For the Frobenius complement H we denote by $k(H)$ the minimum of the values of the function $k(p)$ over $p \in \pi(H)$.

For finite groups the condition 1) of the definition of Frobenius group implies conditions 2) and 3), but in the class of infinite groups all three conditions are independent [12]. An infinite Frobenius group can have a very complex structure. As V.V. Bludov proved [4], any group can be embedded in the kernel of a suitable Frobenius group, and any right-ordered group can be a complement of some Frobenius group (see comments to Questions 6.53 and 13.54 b) from the Kourovka notebook [5]). The kernel of the Frobenius group and the set of its complements constitute its splitting. This property reveals the closeness of Frobenius groups to the objects of combinatorial group theory.

In order to the saturation condition to work effectively, a "dense network" of (locally) finite subgroups with nontrivial intersections is needed, along which "saturation crystallization" could propagate up to a well-defined structure of the whole group [17]. In periodic groups saturated with finite simple non-Abelian groups, the "primary" net is formed by dihedral groups, since in a periodic group every subgroup generated by two involutions is finite. In our studies, the required network is provided by additional finiteness conditions.

2. Main results

An element a is called *finite* in a group G if all subgroups of the form $\langle a, a^g \rangle$ $(g \in G)$ are finite. A group in which every element of prime order is finite is called *weakly conjugately biprimitively finite*. A group all of whose sections along finite subgroups (including the identity one) are weakly conjugately biprimitively finite is called *conjugately biprimitively finite*, or *Shunkov group*. *Binary finite* group is a periodic group in which any two elements generate a finite subgroup. By $\Omega_1(G)$ we denote the subgroup of the group G generated by all elements of prime orders from G.

Theorem 1. *A saturated with finite Frobenius groups periodic weakly conjugate biprimitively finite group G with a nontrivial locally finite radical R is a Frobenius group with kernel F and complement H, moreover, either $F < R$ or $F > R$. If $F < R$, then R is a Frobenius group with kernel F and complement $R \cap H$ containing the subgroup $\Omega_1(H)$. If $F > R$, then $\Omega_1(H)$ is a locally cyclic $\{2,3\}'$-group, any finite subgroup of the quotient group F/R is nilpotent, for each $q \in \pi(F/R)$ the periods of Sylow q-subgroups in F/R*

do not exceed the number $q^{k(H)}$ and for $q > k(H)$ the Sylow q-subgroups have period q.

Theorem 2. *Let the group G satisfy the conditions of Theorem 1, and suppose that its kernel F is not locally finite, and the quotient group G/R is weakly conjugate biprimitively finite. Then $\Omega_1(F/R)$ is the direct product of its non-locally finite Sylow q-subgroups of periods not exceeding $q^{k(H)}$, and for $q > k(H)$ the Sylow q-subgroups of $\Omega_1(F/R)$ have period q.*

We say that mixed group G has a *periodic part* $T(G)$ if all elements of finite order in G constitute the subgroup $T(G)$. In [13, Theorem 2] it was proved that a saturated with finite Frobenius groups Shunkov group with a nontrivial locally finite radical has a periodic part which is a Frobenius group with locally finite complement. Theorems 1, 2 give additional information about the kernels of the periodic parts of the Shunkov groups from [13, Theorem 2].

Theorem 3. *A saturated with finite Frobenius groups binary finite group G with a non-trivial locally finite radical R is a Frobenius group with locally finite complement H and kernel F decomposable into a direct product of Sylow subgroups. If G is not locally finite, then $R < F$, for each $q \in \pi(F/R)$ the periods of Sylow q-subgroups in F/R do not exceed $q^{k(H)}$ and for $q > k(H)$ the Sylow q-subgroups have period q.*

3. Theorem proofs

As in [13], we introduce the notation, assuming that the saturating set $\mathfrak{X}$ of G consists of finite Frobenius groups. By $\mathfrak{X}_G(K) = \mathfrak{X}(K)$ we denote the set of all subgroups of the group G containing a subgroup or set K from G and isomorphic to groups from $\mathfrak{X}$. In particular, $\mathfrak{X}(1)$ is the set of all subgroups of G that are isomorphic to groups from $\mathfrak{X}$. For any (finite) subgroup $L \leq G$, the notation $L = F_L \leftthreetimes H_L$ will mean that $L \in \mathfrak{X}(1)$, where F_L is the kernel and H_L is the complement of the Frobenius group L, and the inclusion $K \leq H_L$ will be understood in the sense that the subgroup K is contained in a suitable complement H_L of the group L.

Let a group G with a nontrivial locally finite radical R satisfy the conditions of Theorem 1. First, we establish some properties of the group G.

Lemma 1. *In the case $G = R$ Theorem 1 is true.*

Proof. By Lemma 1 from [13] $G = F_G \leftthreetimes H_G$ is a Frobenius group, and since the group G is locally finite, Theorem 1 is unconditionally true. The Lemma is proved. □

Lemma 2. *Let $G \neq R$. Then $G = F \rightthreetimes H$ is a Frobenius group with kernel F and complement H, moreover $F \neq R$ and either $F < R$ or $F > R$.*

Proof. If R has a finite Frobenis subgroup, then by [13, Lemma 2] $R = F_R \rightthreetimes H_R$ is a Frobenius group, and according to [13, Lemma 5] G is a Frobenius group with kernel $F = F_R$ and complement $H = N_G(H_R)$.

Let R has no Frobenius subgroups, and let $L \in \mathfrak{X}(1)$. By Schmidt's theorem, the group $M = RL$ is locally finite and by [13, Lemma 1] $M = F_M \rightthreetimes H_M$ is a Frobenis group. Due to the properties of (locally) finite Frobenius groups, $H_M = H_L$, $R \leq F_M$, and R is a nilpotent group by the Thompson-Higman theorem. By the conditions of the theorem, any element of prime order $a \in H_L$ is finite in G, and obviously not Engel. By Theorem 1, [13] either G is a Frobenius group with kernel $F < R$ and complement H, or $a \notin R$, $G = F \rightthreetimes H$, where $H = N_G(\langle a \rangle)$, (G, H) is a Frobenius pair, and $F \rightthreetimes \langle a \rangle$ is a Frobenius group with kernel $F > R$ and complement $\langle a \rangle$. In the first case, the Lemma has already been proved; consider the second case. According to the definition of the Frobenius group, one has to prove the equality $G \setminus F^{\#} = \bigcup_{g \in G} H^g$.

Let b be an arbitrary element of prime order from H. By the conditions of the theorem, all subgroups $L_f = \langle b, b^f \rangle$ ($f \in F^{\#}$) are finite. Since (G, H) is a Frobenius pair and $H \cap F = 1$, then $b^f \notin L \cap H$ and $(L, L \cap H)$ is a pair Frobenius. By Frobenius's theorem $L = F_L \rightthreetimes H_L$, where $H_L = L \cap H$. It is easy to verify that $F_L = L \cap F$ and $H_L = \langle b \rangle$. By the theorem from [14] $F \rightthreetimes \langle b \rangle$ is a Frobenius group with kernel F and complement $\langle b \rangle$, in particular, $Fb = b^F$.

Let t be an element of a composite order from $G \setminus F$, and $|t| = mq$, where q is a prime number. Then $t^m \in Fb$ for some element b of order q from H and, as proved above, $t^m = b^f$ for a suitable element $f \in F$. Since (G, H) is a Frobenius pair, then $t \in H^f$. It follows from this that $G \setminus F^{\#} = \cup_{g \in G} H^g$, and by definition $G = F \rightthreetimes H$ is a Frobenius group with kernel F and complement H. The inequality $F \neq H$ and the incidence of the subgroups F and H were proved above. The Lemma is proved. □

Lemma 3. *For each element b of prime order from H and any $f \in F^{\#}$ the subgroup $L = \langle b, b^f \rangle$ is finite, belongs to $\mathfrak{X}(1)$, $L_F = L \cap F$, $H_L = \langle b \rangle$ and $f \in L$.*

Proof. The lemma is actually proved in the course of the proof of the Lemma 2. □

Lemma 4. *If $F < R$, then $\Omega_1(H) \leq R \cap H$. If $F > R$, then R is a nilpotent group, $A = \Omega_1(H)$ is a locally cyclic $\{2,3\}'$-group, and $F \rightthreetimes A$ is a Frobenius group with kernel F and complement A saturated with finite Frobenius groups.*

Proof. By the Thompson-Higman theorem, G contains an abelian normal subgroup $Z \leq F$ on which H acts freely. By [15, Theorem 1], the subgroup $\Omega_1(H)$ generated by all elements of prime orders is a group of one of the types:

1) $\Omega_1(H)$ is a (locally) cyclic group;
2) $\Omega_1(H) = V \times L$, where V is a (locally) cyclic $\{2, 3\}'$-group, $L \simeq SL_2(3)$.
3) $\Omega_1(H) = V \times L$, where V is a (locally) cyclic $\{2, 3, 5\}'$-group, $L \simeq SL_2(5)$;

If H contains an involution i, then $F \leftthreetimes \langle i \rangle$ is a Frobenius group and by [12, Lemma 2.3] the kernel F is an abelian group. Moreover, $R > F$, R is a Frobenius group with kernel $R \cap F$, complement $H \cap R$, $\Omega_1(H) \leq R \cap H$, and the Lemma is proved. Similarly, if H contains an element a of order 3, then by the Lemma 3 $F \leftthreetimes \langle a \rangle$ is a Frobenius group and by [9; 12, Lemma 2.4] the kernel F is a nilpotent group. For the case $R > F$, the theorem is proved.

Let $R < F$. By what was proved above, $F \leftthreetimes \Omega_1(H)$ is a Frobenius group, and $A = \Omega_1(H)$ is a (locally) cyclic $\{2, 3\}'$-group. It is clear that H does not contain Frobenius subgroups, and the weakly conjugate biprimitively finite Frobenius group $F \leftthreetimes A$ is also saturated with finite Frobenius groups. The Lemma is proved. □

It is not known whether the saturation condition carries over to the groups $F \leftthreetimes A_1$, $1 < A_1 < A$.

Lemma 5. *When $R < F$, the finite subgroups in F and F/R are nilpotent.*

Proof. By the Thompson-Higman theorem, the radical R is nilpotent, and for the group G Lemmas 6-11 from [13] hold. Let K be an arbitrary finite subgroup of F, and $K \leq L \in \mathfrak{X}(K)$. Each element c of prime order q from K is finite in G and by [13, Lemma 7] is Engel, that is, all subgroups $\langle c, c^g \rangle$ in G are finite q-groups. Therefore, $K \leq F_L$, and by Thompson's theorem, K is nilpotent.

Further, let $\overline{K}$ be an arbitrary finite subgroup of F/R. By Schmidt's theorem, its full preimage K is locally finite and, obviously, $K = RM$ for some finite subgroup $M \leq K$. As already proved, the subgroup M is nilpotent. Hence, the subgroup $\overline{K} \simeq M/(M \cap R)$ is nilpotent. The Lemma is proved. □

Lemma 6. *If a is an element of prime order from H, $G_a = F \leftthreetimes \langle a \rangle$ and b, c are permutation elements of coprime orders from F, then the normal closures $B = \langle b^{G_a} \rangle$, $C = \langle c^{G_a} \rangle$ are element-wise permutable and the abelian subgroup $B \cap C$ is contained in R.*

Proof. For any element a of prime order p from A, the subgroup $G_a = F \leftthreetimes \langle a \rangle$ is a Frobenius group with kernel F and complement $\langle a \rangle$, and by the Lemma 3 all subgroups $\langle a, a^g \rangle$ in G_a are finite. Therefore, $G_a \in \mathfrak{A} \cap \mathfrak{G}$ in the notation §5.3 of the monograph [12, p. 172]. Similarly, $G_a/R \in \mathfrak{A} \cap \mathfrak{G}$. In particular, for the groups G_a and G_a/R Lemmas 5.11 - 5.43 from [12] hold. By [12, Lemma 5.21], the subgroups B, C are element-wise permutable, and the abelian subgroup $B \cap C$ is contained in the center of the group BC and is normal in G_a. Therefore, $B \cap C \leq R$, and the Lemma is proved. □

Lemma 7. *If a and G_a from Lemma 6, $\overline{G}_a = G_a/R$ and $\bar{b}, \bar{c}$ are permutation elements of coprime orders from $\overline{F}$, then normal closures $\overline{B} = \langle \bar{b}^{\overline{G}_a} \rangle$, $\overline{C} = \langle \bar{c}^{\overline{G}_a} \rangle$ are element-wise permutable, $\overline{B} \cap \overline{C} = \overline{1}$ and $\overline{B} \cdot \overline{C} = \overline{B} \times \overline{C}$.*

Proof. The proof of the elementwise permutability of the subgroups $\overline{B}$ and $\overline{C}$ carries over word for word from the Lemma 6. And since the locally finite radical in $\overline{F}$ is obviously trivial, then $\overline{B} \cap \overline{C} = \overline{1}$ and $\overline{B} \cdot \overline{C} = \overline{B} \times \overline{C}$. The Lemma is proved. □

Recall that $k(p)$ denotes the value of the Higman function of a prime number p.

Lemma 8. *Let p be the smallest number from $\pi(H)$, $q \in \pi(F)$, and b be an arbitrary q-element from F. Then $b^{k(p)} \in R$ for any q, and $b^q \in R$ for $q \geq k(p)$.*

Proof. As in the Lemma 6, for an element a of order p from H, the group $G_a = F \leftthreetimes \langle a \rangle$ by the Lemma 3 belongs to the class $\mathfrak{A} \cap \mathfrak{G}$ [12, p. 172]. By [12, Lemma 5.22] $b^{k(p)} \in R$, and under the condition $q \geq k(p)$, $b^q \in R$ by [12, Lemma 5.25]. The Lemma is proved. □

Proof of Theorem 1. Under the condition $F \leq R$, the assertions of the theorem follow from the Lemmas 1, 2 and 4. Under the condition $F > R$, the assertions of the theorem, except for the last one, are proved in the Lemmas 2, 4, 5 and 6. The boundedness of the periods of Sylow q-subgroups in F/R is an obvious consequence of the Lemma 8. Theorem 1 is proved.

Proof of Theorem 2. When $\pi(F/R) = \{q\}$, the theorem follows from the Lemma 8. Let $|\pi(F/R)| > 1$ and $q, r \in \pi(F/R)$. Lemma 7 implies that for any q-element b from F/R the normal closure $B = \langle b^{G/R} \rangle$ does not contain q'-elements commuting with b. And if B is not a q-group, then in view of the Lemma 5 this means that an element b with any q'-element from B generates an infinite subgroup. Let b, c be elements of different prime orders q and r from F/R, $L = \langle b, b^c \rangle$ and $M = \langle c, c^b \rangle$. By the hypotheses of the theorem, each element of prime order from F/R is finite in F/R, therefore, the subgroups L and M are finite. By virtue of the Lemma 5, L is a q-group, M is a r-group, and $L \cap M = 1$. Therefore, $bc = cb$, and by the Lemma 7 $B = \langle b^{G/R} \rangle$ is a q-group, $C = \langle c^{G/R} \rangle$ is a q-group. It obviously follows from this that $\Omega_1(F/R)$ is the direct product of its non-locally finite Sylow subgroups. The boundedness of the periods of Sylow subgroups in F/R follows from the Lemma 8. Theorem 2 is proved.

Proof of Theorem 3. By Theorem 1, $G = F \rtimes H$ is a Frobenius group in which the complement H is locally finite in [13, Theorem 2], and all finite subgroups of the kernel F are nilpotent by the Lemma 5. In view of the binary finiteness, the product of any q-elements from F is a q-element, and F is the direct product of its Sylow subgroups. The corresponding boundedness of the Sylow periods in F/R follows from the Lemma 8. Theorem 3 is proved.

4. Discussion of the obtained results

The purpose of this work is to further study periodic non-locally finite Frobenius groups and groups with saturation conditions, as well as to draw boundaries, unconditionally conditional, between domains of (periodic) groups where local methods work effectively, and where the study of groups by local methods is impossible.

Let us first give examples of non-locally finite groups satisfying the conditions of the theorems of the article. If $A = A(m, n)$ is the Adian group [2], $\langle d \rangle = Z(A)$ and k is a natural number, then it is well known that $H = A/\langle d^k \rangle$ is a group of period $\frac{nk}{(n,k)}$ with cyclic non-splittable center $Z = \langle d \rangle / \langle d^k \rangle$ of order k and $H/Z \simeq B(m, n)$. In connection with the conditions of Theorems 1 and 2, we give such examples.

Example 1. If $n = p$, $k = q$ are primes, then the Sylow q-subgroup in H is unique, coincides with Z, and is not distinguished by a direct factor in H.

Example 2. In the case when $\pi(n) \subseteq \pi(k)$, $\Omega_1(H)$ is a central cyclic subgroup of H [16], and the group H is weakly conjugate biprimitive is finite.

Note that in the same year groups similar to the groups H from [16; 17] were constructed by A.Yu. Olshanskiy [11], [10, Theorem 35.1] and V.L. Shirvanyan [20].

Let us give an example of a periodic non-locally finite group satisfying both the conditions and the statements of Theorem 1 (for $F < R$), constructed in [16; 17] using the group H from the example 2.

Example 3. For any prime number $q \notin \pi(H)$, there exists a weakly conjugate biprimitively finite Frobenius group $G = F \leftthreetimes H$ with an (elementary) Abelian kernel F and a periodic not locally finite complement H.

The existence of groups with the condition $F > R$ from Theorems 1–3 has not yet been proved, but has not been disproved either. In particular, even with the additional saturation condition, it has not yet been possible to solve V.P. Shunkov's question 6.56 from the Kourovka notebook [5]:

6.56. *Let $G = F \cdot \langle a \rangle$ be a Frobenius group, and the complement $\langle a \rangle$ has a simple order.*

1) If G is binary finite, will it be locally finite?

2) If the groups $\langle a, a^g \rangle$ are finite for all $g \in G$, then is the kernel F a locally finite group?

In connection with the results of the monograph [12], V.V. Bludov [4], results from [13] and this paper, many questions arise about the structure of infinite Frobenius groups with saturation conditions. As mentioned in the introduction, Frobenius groups are close to the well-known objects of combinatorial group theory in terms of splitting and saturation properties. We will cite only questions with short, clear formulations, which cannot be solved by methods of local analysis (to the authors).

Question 1. *Does there exist infinite simple groups all of whose maximal subgroups are finite Frobenius groups?*

Finite simple groups with this condition exist, that are $L_2(2^p)$ and $Sz(2^p)$ (p is a prime number). According to Dirichlet's theorem in number theory, there are Frobenius groups with cyclic complements and kernels of arbitrarily large prime orders. And the Question 1 may be easier than the Question of A.Yu. Olshanskiy [3, 14]: *Let p be a prime number. Is there an infinite periodic group G in which every maximal subgroup is an elementary abelian group of order p^2?*

Question 2. *It is well known that a finite group with a strongly isolated normal subgroup is a Frobenius group. Is a periodic group with a normal strongly isolated Abelian subgroup a Frobenius group?*

Question 3. *Are there periodic Frobenius groups with abelian kernels and simple infinite complements?*

Let's dwell on a technical issue.

Question 4. *Let the group $G = F \leftthreetimes \langle ab \rangle$ be saturated with finite Frobenius groups, $(G, \langle ab \rangle)$ be a Frobenius pair, $ab = ba$, $|a| = p$, $|b| = q$, where p and q are distinct primes. Is G a Frobenius group? Is G a Frobenius group, provided that a is finite in G?*

Finally, let us cite the questions, the solutions of which the authors hope to receive in the near future. Can Theorems 1–3 of the paper be freed from the condition for the existence of a nontrivially finite radical? More specific:

Question 5. *Is the Frobenius group a binary finite group saturated with finite Frobenius groups?*

Question 6. *Is a periodic (weakly) conjugate biprimitively finite group saturated with finite Frobenius groups a Frobenius group?*

5. Conclusion

The results obtained in [13] and in this paper show that the study of groups saturated with finite Frobenius groups is a complex, interesting and extensive problem, topical both for the theory of groups with finiteness conditions and for combinatorial group theory.

References

1. Adyan S.I. Periodic products of groups. *Proceedings of the Steklov Institute of Mathematics*, 1976, vol. 142, pp. 3–21.
2. Adyan S.I. *The Burnside problem and identities in groups*. Moscow, Nauka Publ., 1975. (in Russian)
3. Belousov I.N., Kondratyev A.S., Rozhkov A.V. XII School-Conference on Group Theory, dedicated to the 65th anniversary of A.A. Mahnev. *Trudy IMM UrO RAN*, 2018, vol. 24, no. 3, pp. 286–295. https://doi.org/10.21538/0134-4889-2018-24-3-286-295 (in Russian)
4. Bludov V.V. On Frobenius groups. *Sib. Math. J.*, 1997, vol. 38, no. 6, pp. 1219–1221. https://doi.org/10.1007/BF02675933
5. Khukhro E.I., Mazurov V.D. Unsolved Problems in Group Theory. *The Kourovka Notebook*, 2020, arXiv:1401.0300.
6. Kreknin V.A., Kostrikin A.I. On Lie algebras with regular automorphism. *Doklady Akademii Nauk*, 1963, vol. 149, pp. 249-251. (in Russian)
7. Lytkina D.V., Mazurov V.D. Periodic groups saturated with finite simple groups of Lie type B_3. *Sib. matem. zhurn.*, 2020, vol. 61, no. 3, pp. 634–640. https://doi.org/10.33048/smzh.2020.61.311 (in Russian)
8. Lytkina D.V., Sozutov A.I., Shlepkin A.A. Periodic groups of 2-rank 2 saturated with finite simple groups. *Sib. electr. math. izv.*, 2018, vol. 15, pp. 786–796. https://doi.org/10.17377/semi.2018.15.064 (in Russian)
9. Lytkina D.V., Shlepkin A.A. Periodic Groups Saturated with the Linear Groups of Degree 2 and the Unitary Groups of Degree 3 over Finite Fields of Odd Characteristic. *Sib. Adv. Math.*, 2018, vol. 28, pp. 175–186. https://doi.org/10.3103/S1055134418030033
10. Olshanskiy A.Yu. Geometry of defining relations in groups. *Springer Science & Business Media*, 2012, vol. 70, 446 p.
11. Olshanskiy A.Yu. An remark about countable non-topologizable group. *Vestnik MGU. Ser. 1, Math., Mech.*, 1980, vol. 3, p. 103. (in Russian)

12. Popov A.M., Sozutov A.I., Shunkov V.P. *Groups with systems of Frobenius subgroups.* Krasnoyarsk, IPC KGTU Publ., 2004, 211 p. (in Russian)
13. Sozutov A.I. On groups saturated with finite Frobenius groups. *Math. Notes.* (Article is accepted for publication.)
14. Sozutov A.I. On groups with Frobenius pairs of conjugated elements. *Algebra i Logika*, 1977, vol. 16, no. 2, pp. 204–212. (in Russian)
15. Sozutov A.I. On the Shunkov groups acting freely on Abelian groups. *Sib. Math. J.*, 2013, vol. 54, pp. 144–151. https://doi.org/10.1134/S0037446613010187
16. Sozutov A.I. On existence of infinite subgroups with non-trivial locally finite radical in the group. *Preprint VC SO AN SSSR.* Krasnoyarsk, 1980, pp. 11–19. (in Russian)
17. Sozutov A.I. An example of infinite finitely generated Frobenius group. *VII Vsesoyuz. simpoz. po teorii grupp.* Krasnoyarsk, 1980, p. 116. (in Russian)
18. Sozutov A.I., Suchkov N.M., Suchkova N.G. Infinite groups with involutions. Krasnoyarsk, SFU Publ., 2011, 149 p. (in Russian)
19. Starostin A.I. On Frobenius groups. *Ukr. Math. J.*, 1971, vol. 23, pp. 518–526. https://doi.org/10.1007/BF01091650
20. Shirvanyan V.L. Non-commutative periodic groups with nontrivial intersections of all cyclic subgroups. *VII Vsesoyuz. simpoz. po teorii grupp.* Krasnoyarsk, 1980, p. 137. (in Russian)
21. Shlepkin A.A. On groups saturated with dihedral groups and linear groups of degree 2. *Sib. elektron. matem. izv.*, 2018, vol. 15, pp. 74–85. https://doi.org/10.17377/semi.2018.15.009 (in Russian)
22. Shlepkin A.A. On Shunkov group saturated with finite simple groups. *The Bulletin of Irkutsk State University. Series Mathematics*, 2018, vol. 24, pp. 51–67. https://doi.org/10.26516/1997-7670.2018.24.51 (in Russian)
23. Shlepkin A.A. On periodic groups and Shunkov groups saturated with dihedral groups and A_5. *The Bulletin of Irkutsk State University. Series Mathematics*, 2017, vol. 20, pp. 96–108. https://doi.org/0.26516/1997-7670.2017.20.96 (in Russian)
24. Shlepkin A.A. On periodic part of a Shunkov group saturated with wreathed groups. *Trudy IMM UrO RAN*, 2018, vol. 24, no. 3, pp. 281–285. https://doi.org/10.21538/0134-4889-2018-24-3-281-285 (in Russian)
25. Shlepkin A.A. On Sylow 2-subgroups of Shunkov groups saturated with groups $L_3(2^m)$. *Trudy IMM UrO RAN*, 2019, vol. 25, no. 4, pp. 275–282. https://doi.org/10.21538/0134-4889-2019-25-4-275-282 (in Russian)
26. Shlepkin A.A. Periodic Groups Saturated with Finite Simple Groups of Lie Type of Rank 1. *Algebra and Logic*, 2018, vol. 57, no. 1, pp. 81–86. https://doi.org/10.1007/s10469-018-9480-y
27. Shlepkin A.K., Rubashkin A.G. A class of periodic groups. *Algebra and Logic*, 2005, vol. 44, pp. 65–71. https://doi.org/10.1007/s10469-005-0008-x
28. Amberg B., Kazarin L. Periodic groups saturated with dihedral subgroups. *Book of abstracts of the international algebraic conference dedicated to 70-th birthday of Anatoly Yakovlev.* Saint Petersburg, 2010, pp. 79–80.
29. Amberg B., Fransman A., Kazarin L. Products of locally dihedral subgroups. *Journal of Algebra*, 2012, vol. 350, no. 1, pp. 308–317. http://doi.org/10.1016/j.jalgebra.2011.11.003
30. Higman G. Groups and ring which have automorphisms without non-trivial fixed elements. *J. London Math. Soc.*, 1957, vol. 32, pp. 321–334. https://doi.org/10.1112/jlms/s1-32.3.321
31. Shlepkin A.A. Groups with a Strongly Embedded Subgroup Saturated with Finite Simple Non-abelian Groups. *The Bulletin of Irkutsk State University. Series Mathematics*, 2020, vol. 31, pp. 132-141. https://doi.org/10.26516/1997-7670.2020.31.132

32. Shlepkin A.A. On the periodic part of the Shunkov group saturated with linear groups of degree 2 over finite fields of even characteristic. *Chebyshevskiy sb*, 2019, vol. 20, no. 4, pp. 399-407. https://doi.org/10.22405/2226-8383-2018-20-4-399-407

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О периодических группах, насыщенных конечными группами Фробениуса

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Аннотация. Группа называется слабо сопряжённо бипримитивно конечной, если каждый её элемент простого порядка порождает с любым своим сопряжённым конечную подгруппу. Бинарно конечная группа — это периодическая группа, в которой любые два элемента порождают конечную подгруппу. Если $\mathfrak{X}$ — некоторое множество конечных групп, то говорят, что группа G насыщена группами из множества $\mathfrak{X}$, если любая конечная подгруппа из G содержится в подгруппе группы G, изоморфной некоторой группе из $\mathfrak{X}$. Группа $G = F \leftthreetimes H$ называется группой Фробениуса с ядром F и дополнением H, если $H \cap H^f = 1$ для любого $f \in F^\#$ и каждый элемент из $G \setminus F$ принадлежит одной из сопряжённых с H подгрупп группы G. В работе доказано, что периодическая слабо сопряженно бипримитивно конечная группа с нетривиальным локально конечным радикалом, насыщенная конечными группами Фробениуса, является группой Фробениуса. Найден ряд свойств таких групп и их фактор-групп по локально конечному радикалу. Аналогичный результат получен для бинарно конечных групп с указанными условиями. Приведены примеры периодических не локально конечных групп, удовлетворяющих условиям теорем, и поставлен ряд вопросов по комбинаторной теории групп.

Ключевые слова: группа Фробениуса, слабо сопряженно бипримитивно конечная группа, локально конечный радикал, насыщенность.

Список литературы

1. Адян С. И. Периодические произведения групп // Труды Математического института АН СССР им. В. А. Стеклова. М. : Наука, 1976. Т. 142. С. 3–21.
2. Адян С. И. Проблема Бернсайда и тождества в группах. М. : Наука, 1975.

3. Белоусов И. Н., Кондратьев А. С., Рожков А. В. XII школа-конференция по теории групп, посвященная 65-летию со дня рождения А. А. Махнева // Труды ИММ УРО РАН. 2018. Т. 24, № 3. С. 286–295. https://doi.org/10.21538/0134-4889-2018-24-3-286-295
4. Блудов В. В. О группах Фробениуса // Сибирский математический журнал. 1997. Т. 38, № 6. С. 1219–1221. https://doi.org/10.1007/BF02675933
5. Коуровская тетрадь. Нерешенные вопросы теории групп. 19-е изд. Новосибирск : Институт математики СО РАН, 2018.
6. Крекнин В. А., Кострикин А. И. Об алгебрах Ли с регулярным автоморфизмом // Доклады Академии наук СССР. 1963. Т. 149. С. 249–251.
7. Лыткина Д. В., Мазуров В. Д. Периодические группы, насыщенные конечными простыми группами лиева типа B_3 // Сибирский математический журнал. 2020. Т. 61, № 3. С. 634–640. https://doi.org/10.33048/smzh.2020.61.311
8. Лыткина Д. В., Созутов А. И., Шлёпкин А. А. Периодические группы 2-ранга два, насыщенные конечными простыми группами // Сибирские электронные математические известия. 2018. Т. 15. С. 786–796. https://doi.org/10.17377/semi.2018.15.064
9. Лыткина Д. В., Шлёпкин А. А. Периодические группы, насыщенные линейными группами степени 2 и унитарными группами степени 3 над конечными полями нечетных характеристик // Математические труды. 2018. Т. 21, № 1. С. 55–72. https://doi.org/10.17377/mattrudy.2018.21.104
10. Ольшанский А. Ю. Геометрия определяющих соотношений в группах. М. : Наука, 1989. 446 с.
11. Ольшанский А. Ю. Замечание о счетной нетопологизируемой группе // Вестник МГУ. Серия 1, Математика, механика. 1980. № 3. С. 103.
12. Попов А. М., Созутов А. И., Шунков В. П. Группы с системами фробениусовых подгрупп. Красноярск : ИПЦ КГТУ, 2004. 211 с.
13. Созутов А. И. О группах, насыщенных конечными группами Фробениуса // Математические заметки. (Статья принята к печати.)
14. Созутов А. И. О группах с фробениусовыми парами сопряженных элементов // Алгебра и логика. 1977. Т. 16, № 2. С. 204–212.
15. Созутов А. И. О группах Шункова, действующих свободно на абелевых группах // Сибирский математический журнал. 2013. Т. 54, № 1. С. 188–198. https://doi.org/10.1134/S0037446613010187
16. Созутов А. И. О существовании в группе бесконечных подгрупп с нетривиальным локально конечным радикалом // Препринт ВЦ СО АН СССР в г. Красноярске. 1980. С. 11–19.
17. Созутов А. И. Пример бесконечной конечнопорожденной группы Фробениуса / VII Всесоюз. симп. по теории групп. Красноярск, 1980. С. 116.
18. Созутов А. И., Сучков Н. М., Сучкова Н. Г. Бесконечные группы с инволюциями. Красноярск : Сибирский федеральный университет, 2011. 149 с.
19. Старостин А. И. О группах Фробениуса // Украинский математический журнал. 1971, Т. 23, № 5. С. 629–639.
20. Ширванян В. Л. Некоммутативные периодические группы с нетривиальными пересечениями всех циклических подгрупп // VII Всесоюзный симпозиум по теории групп. Красноярск, 1980. С. 137.
21. Шлепкин А. А. О группах, насыщенных группами диэдра и линейными группами степени 2 // Сибирские электронные математические известия. 2018. Т. 15. С. 74–85. https://doi.org/10.17377/semi.2018.15.009
22. Шлепкин А. А. О группах Шункова, насыщенных конечными простыми группами // Известия Иркутского государственного университета. Серия Математика. 2018. Т. 24. С. 51–67. https://doi.org/10.26516/1997-7670.2018.24.51

23. Шлепкин А. А. О периодических группах и группах Шункова, насыщенных группами диэдра и $A5$ // Известия Иркутского государственного университета. Серия Математика. 2017. Т. 20. С. 96–108. https://doi.org/0.26516/1997-7670.2017.20.96
24. Шлепкин А. А. О периодической части группы Шункова, насыщенной сплетенными группами // Труды ИММ УрО РАН. 2018. Т. 24, № 3. С. 281–285. https://doi.org/10.21538/0134-4889-2018-24-3-281-285
25. Шлепкин А. А. О силовских 2-подгруппах групп Шункова, насыщенных группами $L_3(2^m)$ // Труды ИММ УрО РАН. 2019. Т. 25, № 4. С. 275–282. https://doi.org/10.21538/0134-4889-2019-25-4-275-282
26. Шлепкин А. А. Периодические группы, насыщенные конечными простыми группами лиева типа ранга 1 // Алгебра и логика. 2018. Т. 57, № 1. С. 118–125. https://doi.org/10.17377/alglog.2018.57.107
27. Шлёпкин А. К., Рубашкин А. Г. Об одном классе периодических групп // Алгебра и логика. 2005. Т. 44, № 1. С. 114–125. https://doi.org/10.1007/s10469-005-0008-x
28. Amberg B., Kazarin L. Periodic groups saturated with dihedral subgroups // Book of abstracts of the international algebraic conference dedicated to 70-th birthday of Anatoly Yakovlev. Saint Petersburg, 2010. P. 79–80.
29. Amberg B., Fransman A., Kazarin L. Products of locally dihedral subgroups // Journal of Algebra. 2012. Vol. 350, N 1. P. 308–317. http://doi.org/10.1016/j.jalgebra.2011.11.003
30. Higman G. Groups and ring which have automorphisms without non-trivial fixed elements// J. London Math. Soc. 1957. Vol. 32. P. 321–334. https://doi.org/10.1112/jlms/s1-32.3.321
31. Shlepkin A. A. Groups with a Strongly Embedded Subgroup Saturated with Finite Simple Non-abelian Groups // Известия Иркутского государственного университета. Серия Математика. 2020. Т. 31. С. 132–141. https://doi.org/10.26516/1997-7670.2020.31.132
32. Shlepkin A. A. On the periodic part of the Shunkov group saturated with linear groups of degree 2 over finite fields of even characteristic // Чебышевский сборник. 2019. Т. 20, № 4. С. 399–407. https://doi.org/10.22405/2226-8383-2018-20-4-399-407

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