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S-acts over a Well-ordered Monoid with Modular Congruence Lattice *

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Abstract. This work relates to the structural act theory. The structural theory includes the description of acts over certain classes of monoids or having certain properties, for example, satisfying some requirement for the congruence lattice. The congruences of universal algebra is the same as the kernels of homomorphisms from this algebra into other algebras. Knowledge of all congruences implies the knowledge of all the homomorphic images of the algebra. A left S–act over monoid S is a set A upon which S acts unitarily on the left. In this paper, we consider S–acts over linearly ordered and over well-ordered monoids, where a linearly ordered monoid S is a linearly ordered set with a minimal element and with a binary operation max, with respect to which S is obviously a commutative monoid; a well-ordered monoid S is a well-ordered set with a binary operation max, with respect to which S is also a commutative monoid. The paper is a continuation of the work of the author in co-authorship with M.S. Kazak, which describes S–acts over linearly ordered monoids with a linearly ordered congruence lattice and S-acts over a well-ordered monoid with distributive congruence lattice. In this article, we give the description of S-acts over a well-ordered monoid such that the corresponding congruence lattice is modular.

Keywords: act over monoid, congruence lattice of algebra, modular lattice.

1. Introduction

A significant number of works are devoted to the study of S-acts with given conditions on their congruence lattices. In particular, in [1], unars,

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that are the unary algebras with one unary operation, with linearly ordered, distributive or modular congruence lattices are described. A description of commutative unary algebras whose congruence lattices are a chain is obtained in [4]. The congruence lattices of disconnected S-acts over monoids are studied in [8]. A description of acts over certain classes of semigroups (such that semigroups of right and left zeros, rectangular bands, linearly ordered monoids) that have modular, distributive or linearly ordered congruence lattice is obtained in [2; 7; 10]. For commutative S-acts the conditions of modularity and distributivity of the congruence lattice are investigated in [3]. In this paper, we describe S-acts over a well-ordered monoid with modular congruence lattices.

2. Preliminaries

Let us recall some definitions and facts from act theory and universal algebra (see [5; 6; 9]). Throughout this paper S will denote a monoid with identity 1. An algebraic system $\langle A; s\rangle_{s\in S}$ of the language $L_S = \{s \mid s \in S\}$ consisting of unary operation symbols is a (left) *S-act* if $s_1(s_2a) = (s_1s_2)a$ and $1a = a$ for all $s_1, s_2 \in S$ and $a \in A$. An S-act $\langle A; s\rangle_{s\in S}$ is denoted by ${}_SA$. Let ${}_SA$ be an S-act and ${}_SB$ be a subact of ${}_SA$. An equivalence relation θ on ${}_SA$ is called *a congruence* on ${}_SA$, if $(a, b) \in \theta$ implied $(sa, sb) \in \theta$ for $a, b \in A$, $s \in S$. Any subact ${}_SB \subset {}_SA$ defines *the Rees congruence* $\rho(B)$ on ${}_SA$, by setting $(a, b) \in \rho(B)$ if $a, b \in B$ or $a = b$.

Elements x, y of an S-act ${}_SA$ are called *connected* (denoted by $x \sim y$) if there exist $n \in \omega$, $a_0, \ldots, a_n \in A$, $s_1, \ldots, s_n \in S$ such that $x = a_0$, $y = a_n$, and $a_i = s_ia_{i-1}$ or $a_{i-1} = s_ia_i$. An S-act ${}_SA$ is called *connected* if we have $x \sim y$ for any $x, y \in {}_SA$. It is easy to check that $\sim$ is a congruence relation on the S-act ${}_SA$. The classes of this relation are called *connected components* of the S-act ${}_SA$. *A coproduct* of S-acts ${}_SA_i$ is a disjunctive union of this S-acts. The coproduct of S-acts ${}_SA_i$ is denoted by $\coprod\limits_{i\in I} {}_SA_i$. It is known (see [5]) that that every S-act ${}_SA$ can be uniquely represented as a coproduct of connected components.

For a congruence θ on ${}_SA$ and a subact ${}_SB \subset {}_SA$ we define *the restriction* $\theta \restriction B$ of θ for ${}_SB$ by $\theta \restriction B = \theta \cap (B \times B)$. Instead $(a, b) \in \theta$ we will write sometimes $a\theta b$. *The class of $a \in A$ with respect to congruence θ* is a set $\theta(a) = \{b \in A \mid a\theta b\}$. Note that the set of all congruences on ${}_SA$ forms a lattice according to the relation $\subseteq$, which is called the lattice of congruences on the S-act ${}_SA$, and denoted by $Con({}_SA)$.

Theorem 1. *[5] Let ${}_SA$ be an S-act, $a, b \in A$, $\theta_1, \theta_2 \in Con({}_SA)$. Then $a\,(\theta_1 \vee \theta_2)\,b$ if and only if there are the elements $x_0, x_1, \ldots, x_{2n}$ in A such that $a = x_0$, $x_{2n} = b$, $x_{2k}\,\theta_1\,x_{2k+1}$ and $x_{2k+1}\,\theta_2\,x_{2k+2}$ for all $k \in \{0, 1, \ldots, n-1\}$.*

A lattice $(L, \wedge, \vee)$ is called *modular* if $(a \vee b) \wedge c = a \vee (b \wedge c)$ for all $a, b, c \in L$ with $a \leqslant c$.

Theorem 2. *[9] A lattice L is modular iff the conditions $a \leqslant b$, and $a \vee c = b \vee c$, $a \wedge c = b \wedge c$ for some $c \in L$ imply $a = b$ for all $a, b \in L$.*

A congruence θ on an S-act ${}_SA$ is called *perforating* if there are S-acts ${}_SB$, ${}_SC$ and elements $b_1, b_2 \in B$, $c_1, c_2 \in C$ such that

$$ {}_SA = {}_S B \sqcup_S C, (b_1, b_2) \notin \theta, (c_1, c_2) \notin \theta, (b_1, c_1) \in \theta, (b_2, c_2) \in \theta. $$

Theorem 3. *[8] A lattice $Con({}_SA)$ is modular if and only if the following conditions are true:*

(1) the S-act ${}_SA$ contains no more than three connected components;

(2) the latices of congruences on all connected components of an S-act ${}_SA$ are modular;

(3) there are no perforating congruences on an S-act ${}_SA$.

A lattice $(L, \wedge, \vee)$ is called *distributive* if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all $a, b, c \in L$. It is clear that a distributive lattice is modular.

Let $\leq$ be a total ordering on S and 1 be a minimal element in S. Then $(S; \cdot)$ is a commutative monoid relative to the operation $a \cdot b = \max\{a, b\}$ for $a, b \in S$, at that 1 is identity of monoid S. This monoid is called a *linearly ordered monoid.* If $(S; \leq)$ is well-ordered set then linearly ordered monoid $(S; \cdot)$ is called a *well-ordered monoid.*

Proposition 1. *[10] Let S be a well-ordered monoid. Then the lattice on any cyclic S-act is distributive.*

3. S-acts over a well-ordered monoid with modular congruence lattices

Lemma 1. *Let S be a linearly ordered monoid, ${}_SA$ be an S-act, $a, b \in A$, $Sa = Sb$. Then $a = b$.*

Proof. Let the conditions of the Lemma hold. Since $Sa = Sb$, we have $a = sb$ and $b = ta$ for some $s, t \in S$. Suppose that $s \leq t$. Then $a = sb = sta = ta = b$. □

Lemma 2. *Let S be a linearly ordered monoid, ${}_SA$ be an S-act, $a, b, c \in A$, $Sc \subseteq Sb \subseteq Sa$, θ be a congruence on an S-act ${}_SA$ and $a\theta c$. Then $a\theta b\theta c$.*

Proof. Since $Sc \subseteq Sb \subseteq Sa$, we have $b = ta$ and $c = sb = sta$ for some $s, t \in S$. Since $a\theta c$, we have $b = ta\theta tc = tsta = tsa = c$. Hence $b\theta c$. Therefore, $a\theta c\theta b$, that is $a\theta b$. □

Lemma 3. *Let S be a well-ordered monoid, ${}_SA$ be a connected S-act. Then the following conditions are true:*

(1) for any a, $b \in A$ there is a minimal element $m \in S$ such that $Sa \cap Sb = Sma$ and $ma = mb$;

(2) there are $a_i \in A$ $(i \in I)$ such that $A = \bigcup_{i \in I} Sa_i$ and $a_i \notin Sa_j$ for any different elements $i, j \in I$.

Proof. Let us prove (1). The existence of minimal element $m \in S$ such that $ma = mb$ follows from the connectivity of ${}_SA$ and the well-ordering of the monoid S. Let us check for equality $Sa \cap Sb = Sma$. Obviously $Sma \subseteq Sa \cap Sb$. Let $c \in Sa \cap Sb$. Then $c = s_0a = s_1b$ for some s_0, $s_1 \in S$. This means that $c = sa = sb$, where $s = \max\{s_0, s_1\}$. In particular, $s \geq m$. Therefore, $c = sa = sma \in Sma$.

To prove (2), let $A = \{a_\alpha \mid \alpha \in \kappa\}$, where κ is some ordinal. Then we assume $I = \{\beta \in \kappa \mid$ *there are no* $\gamma \in \kappa$ *such that* $Sa_\beta \subset Sa_\gamma\}$. □

Theorem 4. *Let S be a well-ordered monoid. The lattice $Con({}_SA)$ is modular if and only if the following conditions are true:*

(1) an S-act ${}_SA$ contains no more than three connected components;

(2) if $a_1, a_2 \in A$, $s \in S$, $s \neq 1$ and $Sa_1 \cap Sa_2 = Ssa_1 \cap Ssa_2$ then $sa_1 = sa_2$, or $sa_1 = ra_1$, or $sa_2 = ra_2$ for some $r \in S$, $r < s$;

(3) if $a_1, a_2, a_3 \in A$ and $s \in S$ such that $s \neq 1$, $a_i \notin Sa_j$, $sa_i = sa_j$ and $Sa_i \cap Sa_j = Ssa_1$ for any different $i, j \in \{1, 2, 3\}$, then $sa_i = ra_i$ for some $r \in S$, $r < s$, and $i \in \{1, 2, 3\}$.

Proof. Necessity. Let $Con({}_SA)$ be a modular lattice. By Theorem 3 we have (1).

Let us prove (2). Suppose for contradiction that there are elements $a_1, a_2 \in A$ and $s \in S$ such that $s \neq 1$, $Sa_1 \cap Sa_2 = Ssa_1 \cap Ssa_2$, $sa_1 \neq sa_2$ and $sa_i \neq ra_i$ for all $r < s$ and for all $i \in \{1, 2\}$. Let us define the equivalence relations θ_1, θ_2, η on the set A as follows:

$(u, v) \in \theta_1 \Leftrightarrow u, v \in Ssa_1 \cup Ssa_2$, or $u, v \in Sa_1 \setminus Ssa_1$, or $u, v \in Sa_2 \setminus Ssa_2$, or $u = v$;

$(u, v) \in \theta_2 \Leftrightarrow (u, v) \in \theta_1$ or $u, v \in (Sa_1 \setminus Ssa_1) \cup (Sa_2 \setminus Ssa_2)$;

$(u, v) \in \eta \Leftrightarrow u, v \in \{ta_1 \mid t \leq s\}$, or $u, v \in \{ta_2 \mid t \leq s\}$, or $u = v$.

We show that the relation θ_1 is a congruence on ${}_SA$. Let $u, v \in Sa_1 \cup Sa_2$, $t \in S$. If $t \geq s$ then $tu = tsu \in Ssa_1 \cup Ssa_2$ and $tv = tsv \in Ssa_1 \cup Ssa_2$, that is $(tu, tv) \in \theta_1$. So, we assume that $t < s$. If $u, v \in Ssa_1 \cup Ssa_2$ then $tu, tv \in Ssa_1 \cup Ssa_2$ and $(tu, tv) \in \theta_1$. Let $u, v \in Sa_1 \setminus Ssa_1$ and $tu \in Ssa_1$. Then $tu = stu = su$ and $u = la_1$ for some $l \in S$. As $u \notin Ssa_1$ then $l < s$. Hence $tla_1 = tu = su = sla_1 = sa_1$. Because of the inequality $tl < s$ we get the contradiction. So $tu \notin Ssa_1$. Similarly, it is shown that $tv \notin Ssa_1$. Thus, $tu, tv \in (Sa_1 \setminus Ssa_1)$, i.e. $(tu, tv) \in \theta_1$. Therefore, the relation θ_1 is a congruence on the S-act ${}_SA$.

Similarly, it is proved that the relation θ_2 is a congruence on the S-act ${}_SA$.

We show that the relation η is a congruence on ${}_SA$. Let $(u,v) \in \eta$, $r \in S$. We can assume that $u = t_1a_1$ and $v = t_2a_1$ for some $t_1 \leq s$, $t_2 \leq s$. Since

$$rt_1 \leq s \Leftrightarrow r \leq s \Leftrightarrow rt_2 \leq s$$

then $(ru, rv) \in \eta$. So η is a congruence on the S-act ${}_SA$.

It is clear that $\theta_1 \subseteq \theta_2$. Now we show that $\theta_1 \subset \theta_2$. If $a_1 \in Ssa_1$, then $Sa_1 = Ssa_1$ and by Lemma 1 we have $1 \cdot a_1 = a_1 = sa_1$, this contradicts the assumption. Therefore, $a_1 \notin Ssa_1$. If $a_1 \in Sa_2$ then $a_1 \in Sa_1 \cap Sa_2 = Ssa_1 \cap Ssa_2 \subseteq Ssa_1$, contradiction. So, $a_1 \notin Ssa_1 \cup Ssa_2$ and $a_1 \in Sa_1 \setminus Ssa_1$. Similarly, $a_2 \notin Ssa_1 \cup Ssa_2$ and $a_2 \in Sa_2 \setminus Ssa_2$. Hence it is proved that $(a_1, a_2) \in \theta_2$. Note that $(Sa_1 \setminus Ssa_1) \cap (Sa_2 \setminus Ssa_2) = \emptyset$. Indeed,

$$(Sa_1 \setminus Ssa_1) \cap (Sa_2 \setminus Ssa_2) = (Sa_1 \cap Sa_2) \setminus (Ssa_1 \cup Ssa_2) =$$

$$= (Ssa_1 \cap Ssa_2) \setminus (Ssa_1 \cup Ssa_2) = \emptyset.$$

So, $(a_1, a_2) \notin \theta_1$. Thus, $(a_1, a_2) \in \theta_2 \setminus \theta_1$.

To prove the equality $\theta_1 \wedge \eta = \theta_2 \wedge \eta$ it is enough to check the inclusion $\theta_2 \wedge \eta \subseteq \theta_1$. Indeed, let $(u,v) \in \theta_2 \wedge \eta$. Since $(u,v) \in \eta$ then we can assume, for example, that $u = t_1a_1$, $v = t_2a_1$ for some $t_1, t_2 \leq s$. Moreover, since $(u,v) \in \theta_2$ we can assume $u, v \in (Sa_1 \setminus Ssa_1) \cup (Sa_2 \setminus Ssa_2)$ (otherwise $(u,v) \in \theta_1$, as claimed). If $u \in Sa_2 \setminus Ssa_2$ then $u \in Sa_1 \cap Sa_2 = Ssa_1 \cap Ssa_2 \subseteq Ssa_2$, contradiction. So $u \in Sa_1 \setminus Ssa_1$ and, similarly, $v \in Sa_1 \setminus Ssa_1$. Hence, $(u,v) \in \theta_1$.

Let us prove $\theta_1 \vee \eta = \theta_2 \vee \eta = \rho(Sa_1 \cup Sa_2)$, where $\rho(Sa_1 \cup Sa_2)$ is the Rees congruence. Clearly, $\theta_1 \subseteq \theta_2 \subseteq \rho(Sa_1 \cup Sa_2)$ and $\eta \subseteq \rho(Sa_1 \cup Sa_2)$, i.e. $\theta_1 \vee \eta \subseteq \theta_2 \vee \eta \subseteq \rho(Sa_1 \cup Sa_2)$. Note that $a_1 \eta sa_1 \theta_1 sa_2 \eta a_2$. Then $(a_1, a_2) \in \theta_1 \vee \eta$. Let $t \in S$. Since $ta_1 \eta a_1$ (if $t \leq s$) or $ta_1 \theta_1 sa_1$ (if $t \geq s$) then $(ta_1, a_1) \in \theta_1 \vee \eta$. Analogously, $(ta_2, a_2) \in \theta_1 \vee \eta$. So for all $u, v \in Sa_1 \cup Sa_2$ we have $(u,v) \in \theta_1 \vee \eta$, i.e. $\rho(Sa_1 \cup Sa_2) \subseteq \theta_1 \vee \eta \subseteq \theta_2 \vee \eta$.

Thus, $\theta_1 \subset \theta_2$, $\theta_1 \wedge \eta = \theta_2 \wedge \eta$ and $\theta_1 \vee \eta = \theta_2 \vee \eta$. By Theorem 2 it contradicts the modularity of the lattice $Con({}_SA)$.

To prove the condition (3) we suppose to the contrary that there are $a_1, a_2, a_3 \in A$ and $s \in S$ such that $s \neq 1$, $Sa_i \cap Sa_j = Ssa_1$, $a_i \notin Sa_j$, $sa_i = sa_j$ for all different $i, j \in \{1, 2, 3\}$ and $sa_i \neq ra_i$ for all $r < s$ and for all $i \in \{1, 2, 3\}$. Let us define the equivalence relations θ_1, θ_2, η on the set A as follows:

$(u,v) \in \theta_1 \Leftrightarrow u, v \in (Sa_1 \cup Sa_2) \setminus Ssa_1$, or $u, v \in Ssa_1$, or $u, v \in Sa_3 \setminus Ssa_1$, or $u = v$;

$(u,v) \in \theta_2 \Leftrightarrow (u,v) \in \theta_1$ or $u, v \in Sa_3$;

$(u,v) \in \eta \Leftrightarrow u, v \in (Sa_2 \cup Sa_3) \setminus Ssa_1$, or $u, v \in Sa_1$, or $u = v$.

We show that θ_1 is a congruence on ${}_SA$. Let $(u,v) \in \theta_1$, $r \in S$. The proof

is divided into three cases.
Case 1: $u, v \in Ssa_1$. Then it is obviously that $ru, rv \in Ssa_1$ and $(ru, rv) \in \theta_1$ for any $r \in S$.
Case 2: $u, v \in (Sa_1 \cup Sa_2) \setminus Ssa_1$. If $r \geq s$ then $ru = rsu \in Ssa_1$ and $rv = rsv \in Ssa_1$, that is $ru, rv \in Ssa_1$ and $(ru, rv) \in \theta_1$. Suppose that $r < s$ and, for example, $u \in Sa_1 \setminus Ssa_1$. Then $u = ta_1$ for some $t < s$, i.e. $ru = rta_1$, where $rt < s$. By assumption, $sa_1 \neq rta_1$. By Lemma 1 $Ssa_1 \subset Srta_1 = Sru$, that is $ru \notin Ssa_1$. So $ru \in (Sa_1 \cup Sa_2) \setminus Ssa_1$. Similarly, $rv \in (Sa_1 \cup Sa_2) \setminus Ssa_1$, that is $(ru, rv) \in \theta_1$.
Case 3: $u, v \in Sa_3 \setminus Ssa_1$. This case is considered similarly to case 2.
Thus, it is proved that θ_1 is the congruence on ${}_SA$. Similarly checked that θ_2, η are the congruences on ${}_SA$.

It is clear that $\theta_1 \subseteq \theta_2$. Let us show that $\theta_1 \subset \theta_2$. Since $sa_1 = sa_3$ then $sa_1, a_3 \in Sa_3$, i.e. $(sa_1, a_3) \in \theta_2$. By assumption, $a_3 \notin Sa_1$, in particular, $a_3 \notin Ssa_1$. So, $(sa_1, a_3) \notin \theta_1$. Thus, $(sa_1, a_3) \in \theta_2 \setminus \theta_1$.

To prove the equality $\theta_1 \wedge \eta = \theta_2 \wedge \eta$ it is enough to check the inclusion $\theta_2 \wedge \eta \subseteq \theta_1$. Indeed, let $(u, v) \in \theta_2 \wedge \eta$. Since $(u, v) \in \theta_2$ then we can assume that $u, v \in Sa_3$ (otherwise $(u, v) \in \theta_1$, as claimed). Since $(u, v) \in \eta$ than we have either $u, v \in Sa_2 \setminus Ssa_1$ (then $(u, v) \in \theta_1$), or $u, v \in Sa_3 \setminus Ssa_1$ (then $(u, v) \in \theta_1$ too), or $u, v \in Sa_1$. In the last case we have $u, v \in Sa_1 \cap Sa_3 = Ssa_1$, i.e. $(u, v) \in \theta_1$.

Let us prove $\theta_1 \vee \eta = \theta_2 \vee \eta = \rho(Sa_1 \cup Sa_2 \cup Sa_3)$, where $\rho(Sa_1 \cup Sa_2 \cup Sa_3)$ is the Rees congruence. Clearly, $\theta_1 \subseteq \theta_2 \subseteq \rho(Sa_1 \cup Sa_2 \cup Sa_3)$ and $\eta \subseteq \rho(Sa_1 \cup Sa_2 \cup Sa_3)$, i.e. $\theta_1 \vee \eta \subseteq \theta_2 \vee \eta \subseteq \rho(Sa_1 \cup Sa_2 \cup Sa_3)$. Note that $sa_1 \theta_1 a_1 \theta_1 a_2 \eta a_3$ for all $s \in S$. Then $sa_1 \theta_1 ta_2 \eta ta_3$ for all $s, t \in S$, i.e. $\rho(Sa_1 \cup Sa_2 \cup Sa_3) \subseteq \theta_1 \vee \eta \subseteq \theta_2 \vee \eta$.

Thus, $\theta_1 \subset \theta_2$, $\theta_1 \wedge \eta = \theta_2 \wedge \eta$ and $\theta_1 \vee \eta = \theta_2 \vee \eta$. By Theorem 2 it contradicts the modularity of the lattice $Con({}_SA)$.

Sufficiency. Let us prove few statements first.

Lemma 4. *Let the condition (2) of the theorem is true, ${}_SA$ be a connected S-act and a, $b \in A$. Then the following statements are true:*

(1) if $\theta \in Con({}_SA)$, $a\theta b$ and $m \in S$ is the minimum element with the conditions $Sa \cap Sb = Sma$ and $ma = mb$ then $sa\theta b$ and $a\theta sb$ for all $s < m$;

(2) if $\theta \in Con({}_SA)$, $a\theta b$ and $Sa \cap Sb \subset Sc \subseteq Sa$ then $a\theta c\theta b$;

(3) if $\theta_1, \ldots, \theta_n \in Con({}_SA)$, $\theta = \theta_1 \circ \ldots \circ \theta_n$, $a\theta b$ and $m \in S$ is the minimum element with the conditions $Sa \cap Sb = Sma$ and $ma = mb$ then $sa\theta b$ and $a\theta sb$ for all $s < m$.

Proof. Since ${}_SA$ is a connected S-act then $Sa \cap Sb \neq \emptyset$.

Let us prove (1). If $m = 1$ then $a = b$ and (1) is satisfied. Let $m > 1$. If $(mb, b) \in \theta$ and $s < m$ then $Smb \subseteq Ssb \subseteq Sb$ implies $b\theta mb$, i.e. by Lemma 2 we have $a\theta b\theta sb$ and $sa\theta sb\theta b$. Suppose that $(mb, b) \notin \theta$. Then $(mb, a) \notin \theta$. We assume $k_1 = \min\{r \in S \mid (rb, a) \notin \theta\}$, $k_2 = \min\{r \in S \mid (rb, b) \notin \theta\}$.

If $k_2 < k_1$ then by the definition of k_1 we have $b\theta a\theta k_2 b$ which contradicts the definition of k_2. If $k_1 < k_2$ then by the definition of k_2 we have $a\theta b\theta k_1 b$ which contradicts the definition of k_1. So $k_1 = k_2 = k$ and $1 < k \leq m$. Let $x \in Sa \cap Sb$. Then $x = sa = sb$ for some $s \in S$. It means that $sb \in Sa$ and by the definition of m we have $k \leq m \leq s$. Hence, $kx = ksa = sa = x$ and $kx = ksb = sb = x$, i.e. $x \in Ska \cap Skb$. Thus, $Sa \cap Sb = Ska \cap Skb$. By the condition (2) of the theorem we have $la = ka$ for some $l < k$, or $lb = kb$ for some $l < k$, or $ka = kb$. In the first case, by the definition of $k_2 = k$ we have $lb\theta b$; the condition $a\theta b$ implies $kb\theta ka = la\theta lb\theta b$, that contradicts the definition of $k_2 = k$. In the second case, $(lb, b) = (kb, b) \notin \theta$, that contradicts the definition of $k_2 = k$. And in the third case, since $k \leq m$, from the definition of element m we have $k = m$; from the definition of element k we have $sa\theta sb\theta b\theta a$ for all $s < m = k$.

Let us prove (2). By Lemma 3 there exists a minimal element $m \in S$ such that $Sa \cap Sb = Sma$ and $ma = mb$. Since $Sma \subset Sc \subseteq Sa$ then $c = sa$ for some $s < m$. So by (1) we have $c = sa\theta b$.

Let us prove (3). Suppose $s < m$. By induction on n we will prove $sa\theta b$ ($a\theta sb$ is proved similarly). If $n = 1$ then (3) is done by (1). Suppose $n > 1$, and for $n-1$ (3) is done, and $\eta = \theta_2 \circ \ldots \circ \theta_n$. Then $a\theta_1 c\eta b$ for some $c \in A$. Since ${}_SA$ is a connected S-act then $Sa \cap Sc \neq \emptyset$ and $Sc \cap Sb \neq \emptyset$. By Lemma 3 there exists minimal element $r \in S$ such that $Sa \cap Sc = Sra$ and $ra = rc$. If $s < r$ then by (1) we have $sa\theta_1 c$, i.e. $sa\theta_1 c\eta b$ and $sa\theta b$. If $s \geq r$ then $m > r$. By Lemma 3 there exists a minimal element $k \in S$ such that $Sc \cap Sb = Skb$ and $kb = kc$. If $s < k$ then from the induction hypothesis we obtain $sc\eta b$, i.e. $sa\theta_1 sc\eta b$ and $sa\theta b$. Suppose that $s \geq k$. Then $m > k$. If $r \geq k$ then the equation $kb = kc$ implies $rb = rc = ra$. So, by the choice of the element m, we have $r \geq m$. Contradiction. If $k > r$ then the equation $ra = rc$ implies $ka = kc = kb$. So, by the choice of the element m we have $k \geq m$. Contradiction. □

Lemma 5. *Let the conditions (2) and (3) of the theorem are true, ${}_SA$ be a connected S-act and $c, c_1, c_2, c_3 \in A$ are such that $Sc_i \cap Sc_j = Sc$ for all i, j, $1 \leq i < j \leq 3$. Then the following statements are true:*

(1) if $\theta, \eta \in Con({}_SA)$ and $c_1\theta c_2\eta c_3$ then $c_1\theta c$ or $c\eta c_3$;

(2) if $\theta_1, \ldots, \theta_n, \eta_1, \ldots, \eta_m \in Con({}_SA)$, $\theta = \theta_1 \circ \ldots \circ \theta_n$, $\eta = \eta_1 \circ \ldots \circ \eta_m$ and $c_1\theta c_2\eta c_3$ then $c_1\theta c$ or $c\eta c_3$.

Proof. Let $\theta_1, \ldots, \theta_n, \eta_1, \ldots, \eta_m \in Con({}_SA)$, $\theta = \theta_1 \circ \ldots \circ \theta_n$, $\eta = \eta_1 \circ \ldots \circ \eta_m$ and $c_1\theta c_2\eta c_3$. If $c_i \in Sc_j$ for some different $i, j \in \{1, 2, 3\}$ then $Sc_i \cap Sc_j = Sc_i = Sc$ and by Lemma 1 we have $c = c_i$, that is $c_1\theta c$ or $c\eta c_3$. Suppose that $c_i \notin Sc_j$ for all different $i, j \in \{1, 2, 3\}$. By Lemma 3 for all i, j, $i < j$, there exists a minimal element $s_{ij} \in S$ such that $Sc_i \cap Sc_j = Ss_{ij}c_i$ and $s_{ij}c_i = s_{ij}c_j$. Since $Sc_i \cap Sc_j = Sc$ then by Lemma 1 we have $c = s_{ij}c_i$ where $i < j$. If $s_{12} < s_{23}$ then by Lemma 4(3) we have $c = s_{12}c_2\eta c_3$, i.e. $c\eta c_3$.

Similarly, if $s_{23} < s_{12}$ then $c\theta c_1$. So, we can assume that $s_{12} = s_{23} = s$. Therefore, $sc_1 = sc_2 = sc_3 = c$ and $Sc_i \cap Sc_j = Ssc_1$ for all different $i, j \in \{1, 2, 3\}$.

Let us prove (1). By condition (3) of the Theorem there exists $r < s$ such that $sc_i = rc_i = c$ for some $i \in \{1, 2, 3\}$. By Lemma 4 (1) we have $rc_1\theta c_2$, $c_1\theta rc_2$, $rc_2\eta c_3$ and $c_2\eta rc_3$. If $rc_1 = c$ then $c_1\theta rc_2\theta rc_1 = c$. Similarly, if $rc_3 = c$ then $c\eta c_3$. If $rc_2 = c$ then $c_1\theta rc_2 = c$ and $c = rc_2\eta c_3$.

By induction on $m + n$ let us prove (2). If $m = n = 1$ then the required statement follows from (1). Let $m + n > 2$. Without loss of generality we can assume that $n > 1$. Let $\xi = \theta_1 \circ \ldots \circ \theta_{n-1}$. Then $c_1\xi x\theta_n c_2$ for some $x \in A$. Let $k = \min\{s' \in S \mid s'c_2 = s'x\}$. Then by Lemma 3 $Skc_2 = Sc_2 \cap Sx$. In view of well-ordering of the monoid S there is one of three cases.

Case 1: $s = k$. In this case $Sc_2 \cap Sx = Sc$. Let $t = \min\{s' \in S \mid s'c_3 = s'x\}$. Then by Lemma 3 $Stc_3 = Sc_3 \cap Sx$. Again in view of well-ordering of the monoid S we have $s \leqslant t$ or $s > t$. In the first case, $Sc = Ssc_3 \supseteq Stc_3 = Sc_3 \cap Sx \supseteq Sc$, so $Sc = Stc_3$, that is by Lemma 1 we have $c = tc_3$; thus, $Sc_2 \cap Sx = Sc_3 \cap Sx = Sc_2 \cap Sc_3 = Sc$; from the induction hypothesis applied to relation $x\theta_n c_2\eta c_3$, we obtain $c_1\xi x\theta_n c$ or $c\eta c_3$, i.e. $c_1\theta c$ or $c\eta c_3$. Let $s > t$. Since $x\theta_n c_2$, $Sx \cap Sc_2 = Ssc_2$, $sc_2 = sx$ and $t < s$, then by Lemma 4 (1) we have $tx\theta_n c_2$. Let $l = \min\{s' \in S \mid s'c_1 = s'x\}$. Then by Lemma 3 $Slc_1 = Sc_1 \cap Sx$ and $lc_1 = lx$. Since $sc_1 = sc_2 = sx$, it follows that $Sx \cap Sc_1 \supseteq Ssc_1$ and $l \leq s$. If $t \geq l$ then $tc_1 = tlc_1 = tlx = tlc_3 = tc_3$, i.e. $tc_1 = tc_3$ and $t \geq s$, contradiction. Let $t < l$. By Lemma 4 (3) we have $c_1\xi tx\theta_n c_2$. Note that $Sc = Sc_1 \cap Sc_2 \supseteq Slx \cap Stx = Slx \supseteq Ssx = Ssc_2 = Sc$. Hence $Slx = Sc$. By Lemma 1 we have $lx = c$. So $Sc_1 \cap Stx = Slx = Sc$. Therefore, $Sc_i \cap Stx = Sc_1 \cap Sc_2 = Sc$ for all $i \in \{1, 2\}$. By the induction hypothesis, we obtain $c_1\xi c$ or $c\theta_n c_2$. If $c\theta_n c_2$ then $c_1\xi x\theta_n c_2\theta_n c$, i.e. $c_1\xi x\theta_n c$ and $c_1\theta c$. Thus, $c_1\theta c$ or $c\eta c_3$.

Case 2: $s > k$. In this case $Sc = Ssc_2 \subseteq Skc_2$. Then $Sc_i \cap Skc_2 = Sc_i \cap Sc_2 \cap Skc_2 = Sc \cap Skc_2 = Sc = Ssc_i$ for all $i \in \{1, 3\}$. It is clear that $sc_1 = sc_3 = skc_2$. Since $s > k$ then by Lemma 4 (3) we have $c_1\xi kc_2$ and $kc_2\eta c_3$, i.e. $c_1\xi kc_2\eta c_3$. By the induction hypothesis, we have $c_1\xi c$ or $c\eta c_3$, i.e. $c_1\theta c$ or $c\eta c_3$.

Case 3: $s < k$. Since $Sx \cap Sc_2 = Skc_2$, $kc_2 = kx$ and $s < k$, then by Lemma 4 (1) we have $x\theta_n sc_2 = c$. Hence $c_1\xi x\theta_n c$, i.e. $c_1\theta c$. □

Let S be a well-ordered monoid and ${}_SA$ satisfies the conditions (1)–(3) of Theorem. If ${}_SA$ is a cyclic S-act then by Proposition 1 $Con({}_SA)$ is a distributive lattice, therefore, $Con({}_SA)$ is a modular lattice.

Let ${}_SA = \bigcup_{i \in I} {}_SSa_i$ be a connected S-act and $|I| > 1$. By Lemma 3 (2) we can assume that $a_i \notin Sa_j$ $(i \neq j)$.

For $\theta \in Con({}_SA)$, we denote $(\theta \restriction Sa_i) \cup 0_A \in Con({}_SA)$ by θ^i. Note that $(\theta_1 \wedge \theta_2)^i = \theta_1^i \wedge \theta_2^i$ for all $\theta_1, \theta_2 \in Con({}_SA)$ and $i \in I$.

Suppose that $\theta_1, \theta_2, \eta \in Con({}_SA)$ such that $\theta_1 \subseteq \theta_2, \theta_1 \wedge \eta = \theta_2 \wedge \eta, \theta_1 \vee \eta = \theta_2 \vee \eta$. By Theorem 2 it is enough to prove $\theta_1 = \theta_2$.

Let $i \in I$. Then $\theta_1^i \subseteq \theta_2^i$, $\theta_1^i \wedge \eta^i = \theta_2^i \wedge \eta^i$.

We divide the proof into several steps.

Step I. We show that $\eta^i \vee \theta_1^i = \eta^i \vee \theta_2^i$.

Clearly, $\eta^i \vee \theta_1^i \subseteq \eta^i \vee \theta_2^i$. Let $ta_i(\theta_2^i \vee \eta^i)ra_i$ and $ta_i \neq ra_i$. Then $ta_i(\theta_2 \vee \eta)ra_i$. Hence $ta_i(\theta_1 \vee \eta)ra_i$. By Lemma 1 we have $Sra_i \neq Sta_i$. Let, for example, $Sra_i \subset Sta_i$. By Theorem 1 there exist $n \in \omega$, $k_0, \ldots, k_{2n} \in I$ ($k_0 = k_{2n} = i$) and $d_j \in Sa_{k_j}$ $(0 \leq j \leq 2n)$ such that $ta_i = d_0$, $ra_i = d_{2n}$, $d_{2j}\theta_1 d_{2j+1}\eta d_{2j+2}$ $(0 \leq j < n)$. Since ${}_SA$ is a connected S-act then $Sa_i \cap Sd_k \neq \emptyset$ $(0 \leq k \leq 2n)$. Let $Sb_k = Sa_i \cap Sd_k$ $(0 \leq k \leq 2n)$.

By induction on n we prove $(*)$: if $d_0(\theta_2^i \vee \eta^i)d_{2n}$, $d_{2j}\theta_1 d_{2j+1}\eta d_{2j+2}$ for all j, $0 \leq j < n$, and $Sd_{2n} \subseteq Sd_0$, then $d_0(\theta_1^i \vee \eta^i)d_{2n}$.

Let $n = 1$. Then $Sd_0 \subseteq Sb_1$, or $Sb_1 \subset Sd_2$, or $Sd_2 \subseteq Sb_1 \subset Sd_0$. In the first case, we have $Sd_0 \subseteq Sd_1$; since $d_1\eta d_2$ and $Sd_2 \subseteq Sd_0 \subseteq Sd_1$, by Lemma 2 we have $d_0\eta^i d_2$, i.e. $d_0(\theta_1^i \vee \eta^i)d_2$. In the second case, since $d_0\theta_1 d_1$ and $Sd_0 \cap Sd_1 = Sb_1 \subset Sd_2 \subseteq Sd_0$, by Lemma 4 (2) we have $d_0\theta_1 d_2$, i.e. $d_0(\theta_1^i \vee \eta^i)d_2$. Let $Sd_2 \subseteq Sb_1 \subset Sd_0$, i.e. the first and second cases are wrong. Since $d_1\eta d_2$ then by Lemma 2 we have $b_1\eta^i d_2$ and $b_1\eta d_1$. Since $d_0\theta_1 d_1$ then by Lemma 4 (2) we have $c\theta_1 d_0$ and $c\theta_1 d_1$ for all $c \in A$ such that $Sb_1 \subset Sc \subseteq Sd_0$. Since $d_0(\theta_2^i \vee \eta^i)d_2$ then by Lemma 2 we have $d_0(\theta_2^i \vee \eta^i)b_1$. Theorem 1 and Lemma 2 imply the existence of element $c \in A$ such that $Sb_1 \subset Sc \subseteq Sd_0$ and $c\eta b_1$ or $c\theta_2 b_1$. If $c\eta b_1$ then $d_0\theta_1 c\eta b_1\eta d_2$, i.e. $d_0(\theta_1^i \vee \eta^i)d_2$. If $c\theta_2 b_1$ then in view of $\theta_1 \subseteq \theta_2$ we have $b_1\theta_2 c\theta_2 d_1$; since $b_1\eta d_1$ then $b_1(\theta_2 \wedge \eta)d_1$; so the equality $\theta_1 \wedge \eta = \theta_2 \wedge \eta$ implies $b_1(\theta_1 \wedge \eta)d_1$, in particular, $b_1\theta_1 d_1$; hence $d_0\theta_1 d_1\theta_1 b_1\eta d_2$, $d_0\theta_1 b_1\eta d_2$ and $d_0(\theta_1^i \vee \eta^i)d_2$. Thus, for $n = 1$ $(*)$ is proved.

Let $n > 1$. Consider two cases.

Case 1: $Sd_0 \subseteq Sb_j$ for some $j \geq 2$. Since $Sd_{2n} = Sb_{2n} \subset Sd_0$ then we can choose j such that $Sb_{j+1} \subset Sd_0 \subseteq Sb_j$ and $j \geq 2$. Since $d_j\zeta d_{j+1}$, where $\zeta \in \{\theta_1, \eta\}$, and $Sd_j \cap Sd_{j+1} = Sd_0 \cap Sd_{j+1} = Sb_{j+1} \subset Sd_0 \subseteq Sd_j$, then by Lemma 4 (2) we have $d_0\zeta d_{j+1}$. Let j be even number. Then $\zeta = \theta_1$ and $d_0\theta_1 d_{j+1}\eta d_{j+2}$. Besides that, $d_{2k}\theta_1 d_{2k+1}\eta d_{2k+2}$ for all k, $j+2 \leq 2k < 2n$. From the induction hypothesis we obtain $d_0(\theta_1^i \vee \eta^i)d_{2n}$, as claimed. Let j be odd number. Then $j \geq 3$, $\zeta = \eta$ and $d_0\theta_1 d_0\eta d_{j+1}$. Besides that, $d_{2k}\theta_1 d_{2k+1}\eta d_{2k+2}$ for all k, $j+1 \leq 2k < 2n$. From the induction hypothesis we obtain $d_0(\theta_1^i \vee \eta^i)d_{2n}$, as claimed.

Case 2: $Sb_{j+1} \subset Sd_{2n}$ for some $j \leq 2n-3$. Since $Sd_{2n} \subset Sd_0 = Sb_0$ then we can choose j such that $Sb_{j+1} \subset Sd_{2n} \subseteq Sb_j$ and $j \leq 2n-3$. Since $d_j\zeta d_{j+1}$, where $\zeta \in \{\theta_1, \eta\}$, and $Sd_j \cap Sd_{j+1} = Sd_0 \cap Sd_{j+1} = Sb_{j+1} \subset Sd_{2n} \subseteq Sd_j$, then by Lemma 2 we have $d_{2n}\zeta d_j$. From the reasoning given in case 1, it follows $d_0(\theta_1^i \vee \eta^i)d_{2n}$, as claimed.

Hence we can assume that

$$Sb_j \subset Sd_0 \ for \ all \ j \geq 2 \ and \ Sd_{2n} \subseteq Sb_{j+1} \ for \ all \ j \leq 2n-3.$$

Let $n > 2$. Again consider two cases.

Case 1: $Sd_{2n} \subseteq Sb_{j+1} \subset Sb_j \subset Sd_0$ for some j, $2 \leq j \leq 2n-3$. Since $d_j \zeta d_{j+1}$, where $\zeta \in \{\theta_1, \eta\}$, and $Sd_j \cap Sd_{j+1} = Sd_0 \cap Sd_{j+1} = Sb_{j+1} \subset Sb_j \subseteq Sd_j$, then by Lemma 4 we have $d_{j+1} \zeta b_j$. By Theorem 1 $d_0(\theta_2^i \vee \eta^i)d_{2n}$ implies $d_0(\theta_2^i \vee \eta^i)b_j$, $b_j(\theta_2^i \vee \eta^i)d_{2n}$. Let j be even number. Then $j \leq 2n-4$, $\zeta = \theta_1$ and $d_j \theta_1 b_j \theta_1 d_{j+1}$. Since $j \leq 2n-4$, $d_j \theta_1 b_j$, $d_0(\theta_2^i \vee \eta^i)b_j$ and $d_{2k}\theta_1 d_{2k+1} \eta d_{2k+2}$ for all k, $0 \leq 2k < 2n$, then from the induction hypothesis we obtain $d_0(\theta_1^i \vee \eta^i)b_j$. Since $j \geq 2$, $b_j \theta_1 d_{j+1}$, $b_j(\theta_2^i \vee \eta^i)d_{2n}$ and $d_{2k}\theta_1 d_{2k+1} \eta d_{2k+2}$ for all k, $0 \leq 2k < 2n$, then from the induction hypothesis we obtain $b_j(\theta_1^i \vee \eta^i)d_{2n}$. Hence $d_0(\theta_1^i \vee \eta^i)d_{2n}$. Let j be odd number. Then $j \geq 3$, $\zeta = \eta$ and $d_j \eta b_j \eta d_{j+1}$. Since $j \leq 2n-3$, $d_j \eta b_j$, $d_0(\theta_2^i \vee \eta^i)b_j$ and $d_{2k}\theta_1 d_{2k+1} \eta d_{2k+2}$ for all k, $0 \leq 2k < 2n$, then from the induction hypothesis we obtain $d_0(\theta_1^i \vee \eta^i)b_j$. Since $j \geq 3$, $b_j \eta d_{j+1}$, $b_j(\theta_2^i \vee \eta^i)d_{2n}$ and $d_{2k}\theta_1 d_{2k+1} \eta d_{2k+2}$ for all k, $0 \leq 2k < 2n$, then from the induction hypothesis we obtain $b_j(\theta_1^i \vee \eta^i)d_{2n}$. Hence $d_0(\theta_1^i \vee \eta^i)d_{2n}$.

Case 2: $Sd_{2n} \subseteq Sb_j \subset Sb_{j+1} \subset Sd_0$ for some j, $2 \leq j \leq 2n-3$. From the reasoning given above when considering case 1, it follows $d_0(\theta_1^i \vee \eta^i)d_{2n}$.

Thus, we can assume that for $n > 2$

$$Sb_{2n-1} \subset Sd_0, \ Sd_{2n} \subseteq Sb_1 \ and \ Sd_{2n} \subseteq Sb_2 = Sb_3 = \ldots = Sb_{2n-2} \subset Sd_0.$$

Clearly, it is true for $n = 2$.

By Lemma 1 $b_i = b_j = b$ for all i, j, $2 \leq i, j \leq 2n-2$, i.e. $Sd_{2n} \subseteq Sb \subset Sd_0$.

We show that $b(\theta_1^i \vee \eta^i)d_{2n}$. If $Sb \subseteq Sb_{2n-1}$ then in view of $d_{2n-1}\eta d_{2n}$ by Lemma 2 we obtain $b\eta d_{2n}$. If $Sb_{2n-1} \subset Sb$ then in view of $d_{2n-2}\theta_1 d_{2n-1}$ and $Sd_{2n-2} \cap Sd_{2n-1} = Sd_0 \cap Sd_{2n-1} = Sb_{2n-1} \subset Sb \subseteq Sd_{2n-1}$, by Lemma 4 we obtain $b\theta_1 d_{2n-1}$, i.e. $b(\theta_1 \circ \eta)d_{2n}$. Since $b(\theta_2^i \vee \eta^i)d_{2n}$ then by induction basis we have $b(\theta_1^i \vee \eta^i)d_{2n}$.

Hence to prove $d_0(\theta_1^i \vee \eta^i)d_{2n}$, it is enough to prove $d_0(\theta_1^i \vee \eta^i)b$.

If $Sd_0 \cap Sd_1 = Sb_1 \subset Sb \subseteq Sd_0$ then in view of $d_0\theta_1 d_1$ by Lemma 4 (1) we have $d_0\theta_1 b$, i.e. $d_0(\theta_1^i \vee \eta^i)b$. So we will assume that $Sb \subseteq Sb_1$. Since $d_0(\theta_2^i \vee \eta^i)d_{2n}$, it is clear that $d_0(\theta_2^i \vee \eta^i)b$. The proof is divided into three cases.

Case 1: $Sd_j \cap Sd_{j+1} = Sb$ for some $j \in \{2, \ldots, 2n-3\}$. Let $d_{j-1}\zeta d_j \zeta' d_{j+1}$, where ζ, ζ' are different elements $\{\theta_1, \eta\}$. Since $Sd_0 \cap Sd_j = Sd_0 \cap Sd_{j+1} = Sd_j \cap Sd_{j+1} = Sb$ then by Lemma 5 (2) we obtain $d_0\theta_1 \circ \ldots \circ \zeta b$ or $b\zeta' d_{j+1}$, i.e. $d_0\theta_1 \circ \ldots \circ d_j \zeta' d_{j+1} \zeta' b$. Since $j \leq 2n-3$, then in both cases from the induction hypothesis we obtain $d_0(\theta_1^i \vee \eta^i)b$. So, in this case $*$ is proved.

Case 2: $Sd_j \cap Sd_{j+1} \neq Sb$ for all $j \in \{2, \ldots, 2n-3\}$ and $Sd_i \cap Sd_k = Sb$ for some different elements $i, k \in \{2, \ldots, 2n-2\}$. Hence $Sd_i \cap Sd_{2n-2} = Sb$

for some $i \in \{2, \ldots, 2n-3\}$. Let j be a minimum element such that $2 \le j \le 2n-3$ and $Sd_j \cap Sd_{2n-2} = Sb$. Then $Sd_{j+1} \cap Sd_{2n-2} \supset Sb$ and $Sd_j \cap Sd_{j+1} = Sb$, that contradicts our assumption.

Case 3: $Sb \subset Sd_i \cap Sd_k$ for all different elements $j, k \in \{2, \ldots, 2n-2\}$. By Lemma 3 (1) we have $\bigcap\limits_{2 \le i \le 2n-2} Sd_i = Sd$ for some $d \in A$. So, $Sb \subset Sd$. We have three sub-cases: $Sb = Sb_1$ and $Sb = Sd_1 \cap Sd$, or $Sb = Sb_1$ and $Sb \subset Sd_1 \cap Sd = Sc$, or $Sb \subset Sb_1$. In the thirst case we have $Sd_1 \cap Sd_0 = Sd_1 \cap Sd = Sd_0 \cap Sd$, since $d_0 \theta_1 d_1 \eta d_2$ and $Sd_1 \cap Sd_2 = Sb \subset Sd \subseteq Sd_2$ then by Lemma 4 (2) we have $d_1 \eta d$, and then by Lemma 5 (1) we have $b\theta_1 d_0$, i.e. $d_0(\theta_1^i \vee \eta^i)b$, or $b\eta d$; if $b\eta d$ then by induction basis $d_0\theta_1 d_1 \eta d \eta b$ implies $d_0(\theta_1^i \vee \eta^i)b$. In the second case, since $d_0\theta_1 d_1$ and $Sd_0 \cap Sd_1 = Sb \subset Sc \subseteq Sd_1$, then by Lemma 4 (2) we have $d_0\theta_1 c$; if $Sc \subseteq Sd_{2n-1}$ then by Lemma 2 we have $d_{2n-1}\eta d_{2n}$ implies $c\eta b$, i.e. $d_0\theta_1 c\eta b$; by induction basis we have $d_0(\theta_1^i \vee \eta^i)b$. If $Sc \cap Sd_{2n-1} \subset Sc$ then by Lemma 4 (2) $d_{2n-2}\theta_1 d_{2n-1}$ and $Sd_{2n-2} \cap Sd_{2n-1} \subseteq Sd_{2n-2} \cap Sc \cap Sd_{2n-1} = Sc \cap Sd_{2n-1} \subset Sc \subseteq Sd_{2n-2}$ imply $c\theta_1 d_{2n-1}$; so $d_0\theta_1 c\theta_1 d_{2n-1}\eta d_{2n}$, hence by induction basis we have $d_0(\theta_1^i \vee \eta^i)d_{2n}$ and by Lemma 2 we have $d_0(\theta_1^i \vee \eta^i)b$. Let $Sb \subset Sb_1$. Note that by Lemma 4 (2) $d_1\eta d_2$ implies $u\eta d_1$ and $v\eta d_2$ for all u, v such that $Sb \subset Su \subseteq Sb_1 \subseteq Sd_1$ and $Sb \subset Sv \subseteq Sd \subseteq Sd_2$, in particular, $b_1\eta d_1$ and $d\eta d_2$. Let $Sb \subset Sb_{2n-1}$. Clearly, $Sb_1 \cap Sb_{2n-1} = Su$ where $u = b_1$ or $u = b_{2n-1}$. By Lemma 2 $d_{2n-1}\eta d_{2n}$ and $Sd_{2n} \subseteq Sb \subset Su \subseteq Sd_{2n-1}$ imply $u\eta b$. As noted above $u\eta d_1$. Hence $d_0\theta_1 d_1\eta u\eta b$, so by induction basis we have $d_0(\theta_1^i \vee \eta^i)b$. Let $Sb = Sb_{2n-1} \subset Sd \cap Sd_{2n-1}$. By Lemma 3 (1) we have $Sd \cap Sd_{2n-1} = Sv$ for some $v \in A$. By Lemma 2 $d_{2n-1}\eta d_{2n}$ implies $v\eta b$. As noted above $v\eta d$, i.e. $d_0\theta_1 d_1\eta d\eta v\eta b$. By induction basis we have $d_0(\theta_1^i \vee \eta^i)b$. Let $Sb_{2n-1} \subset Sb$ or $Sb_{2n-1} = Sb = Sd \cap Sd_{2n-1}$. In the first case, by Lemma 4 (2) $d_{2n-2}\theta_1 d_{2n-1}$ implies $d\theta_1 b$. Let us prove $d_0(\theta_1^i \vee \eta^i)b$ or $d\theta_1 b$ in the second case. By Lemma 4 (2) $d_{2n-2}\theta_1 d_{2n-1}$ and $Sb \subset Sd \subseteq Sd_{2n-2}$ imply $d\theta_1 d_{2n-1}$. By Lemma 5 (1) $d_1\eta d\theta_1 d_{2n-1}$ and $Sd_1 \cap Sd = Sd_1 \cap Sd_{2n-1} = Sd \cap Sd_{2n-1} = Sb$ imply either $d_1\eta b$, i.e. $d_0\theta_1 d_1\eta b$ and by induction basis $d_0(\theta_1^i \vee \eta^i)b$, or $b\theta_1 d_{2n-1}$, i.e. $b\theta_1 d$. It remains to prove that $b\theta_1 d$ implies $d_0(\theta_1^i \vee \eta^i)b$. Let $b\theta_1 d$. Since $d_0(\theta_2^i \vee \eta^i)b$ then there exists $u \in A$ such that $Sb \subset Su \subseteq Sb_1$ and $u\eta b$ or $u\theta_2 b$. As noted above we have $u\eta d_1$, i.e. $d_0\theta_1 d_1\eta u$ and by induction basis we have $d_0(\theta_1^i \vee \eta^i)u$. If $u\eta b$ then $d_0(\theta_1^i \vee \eta^i)u\eta b$, i.e. $d_0(\theta_1^i \vee \eta^i)b$. Let $u\theta_2 b$. Since $b\theta_1 d$ then $b\theta_2 d$. So, $u\theta_2 b\theta_2 d$. Since $u\eta d$ then $u(\theta_2 \wedge \eta)d$. Since $\theta_2 \wedge \eta = \theta_1 \wedge \eta$ then $u\theta_1 d$. So, $u\theta_1 d\theta_1 b$ and $d_0(\theta_1^i \vee \eta^i)u\theta_1 b$, i.e. $d_0(\theta_1^i \vee \eta^i)b$.

Therefore, we proof $d_0(\theta_1^i \vee \eta^i)d_{2n}$. So, $(*)$ is proved. Thus, $\theta_1^i \vee \eta^i = \theta_2^i \vee \eta^i$ and step I is finished.

Since by Proposition 1 the lattice $Con({}_S Sa_i)$ is distributive then it is modular and by Theorem 2 we have $\theta_1^i = \theta_2^i$.

Step II. We show that $\theta_1 = \theta_2$.

We suppose to the contrary that $\theta_1 \subset \theta_2$. Then there are $d_0, d_1 \in A$ and different $k_0, k_1 \in I$ such that $d_0 \in Sa_{k_0}$, $d_1 \in Sa_{k_1}$ and $(d_0, d_1) \in \theta_2 \setminus \theta_1$. Then $d_0 \notin Sd_1$ and $d_1 \notin Sd_0$. By Lemma 3 (1) there exists a minimum element $s \in S$ such that $Ssd_0 = Sd_0 \cap Sd_1$ and $sd_0 = sd_1 = b$. Note that $sd_0 \neq rd_0$ and $sd_1 \neq rd_1$ for all $r \in S$, $r < s$. Indeed, suppose for example that $sd_0 = rd_0$ for some $r \in S$, $r < s$; by Lemma 4 (2) we have $d_0\theta_2 rd_0\theta_2 d_1$; since $\theta_1^{k_0} = \theta_2^{k_0}$ and $\theta_1^{k_1} = \theta_2^{k_1}$ then $d_0\theta_1 rd_0\theta_1 d_1$, i.e. $d_0\theta_1 d_1$, contradiction.

Since $d_0\theta_2 d_1$ then $d_0(\theta_2 \vee \eta)d_1$. Since $\theta_1 \vee \eta = \theta_2 \vee \eta$ then $d_0(\theta_1 \vee \eta)d_1$. By Theorem 1 $d_1\eta d_2\theta_1 d_3 \ldots d_{2n-1}\eta d_{2n} = d_0$ for some $d_2, \ldots d_{2n-1} \in A$ and $n \geq 1$. Note that $n > 1$ (otherwise $d_0(\theta_2 \wedge \eta)d_1$, so $d_0(\theta_1 \wedge \eta)d_1$ and $d_0\theta_1 d_1$, contradiction).

There is one of two cases.

Case 1: $Sd_k \cap Sd_0 \subseteq Sb$ and $Sd_k \cap Sd_1 \subseteq Sb$ for some k, $1 < k < 2n$. Let i be a minimum number such that $Sd_i \cap Sd_1 \subseteq Sb$ and $1 < i < 2n$, j be a maximum number such that $Sd_j \cap Sd_0 \subseteq Sb$ and $1 < j < 2n$. Consider only the case $i > 2$, $j < 2n-2$ as the most difficult. By Lemma 3 (1) we have $Sr_1d_1 = Sd_{i-1} \cap Sd_1 \supset Sb$, $Sr_0d_0 = Sd_{j+1} \cap Sd_0 \supset Sb$ for some $r_0, r_1 \in S$. Note that $r_1 < s$, $r_0 < s$. By Lemma 4 (2) $d_1\theta_2 r_1d_1\theta_2 r_0d_0\theta_2 d_0$. Clearly, $Sd_i \cap Sd_{i-1} \subseteq Sb$.

Suppose that

$$Sd_i \cap Sd_1 = Sb \textit{ and } Sd_i \cap Sd_0 \supset Sb. \tag{1}$$

Let $Sd_i \cap Sd_0 = St_0d_0$ and $d_{i-1}\xi d_i$, where $\xi \in \{\theta_1, \eta\}$. Since $Sd_{i-1} \cap Sd_i \subseteq Sb \subset St_0d_0 \subseteq Sd_i$ and $Sd_{i-1} \cap Sd_i \subseteq Sb \subset Sr_1d_1 \subseteq Sd_{i-1}$, then in view of $d_{i-1}\xi d_i$ by Lemma 4 (2) we have $t_0d_0\xi r_1d_1$. Since $d_0\theta_2 d_1$, $Sd_0 \cap Sd_1 = Sb \subset St_0d_0 \subseteq Sd_0$ and $Sd_0 \cap Sd_1 = Sb \subset Sr_1d_1 \subseteq Sd_1$, then by Lemma 4 (2) we have $d_0\theta_2 t_0d_0\theta_2 r_1d_1\theta_2 d_1$. Since $\theta_1^{k_0} = \theta_2^{k_0}$ and $\theta_1^{k_1} = \theta_2^{k_1}$, then $d_0\theta_1 t_0d_0$ and $d_1\theta_1 r_1d_1$. If $\xi = \theta_1$ then $d_0\theta_1 t_0d_0\theta_1 r_1d_1\theta_1$, i.e. $d_0\theta_1 d_1$, contradiction. If $\xi = \eta$ then $t_0d_0(\theta_2 \wedge \eta)r_1d_1$; since $\theta_1 \wedge \eta = \theta_2 \wedge \eta$ then $t_0d_0(\theta_1 \wedge \eta)r_1d_1$ and $d_0\theta_1 t_0d_0\theta_1 r_1d_1\theta_1 d_1$, i.e. $d_0\theta_1 d_1$, contradiction.

Let (1) not be done. We show that

$$r_1d_1\eta b \vee r_1d_1\theta_1 b. \tag{2}$$

Let $Sd_i \cap Sd_1 \subset Sb$. Since $d_i\xi d_{i-1}$, where $\xi \in \{\theta_1, \eta\}$, then in view of $Sd_{i-1} \cap Sd_i \subset Sb \subseteq Sd_{i-1}$ and $Sd_{i-1} \cap Sd_i \subset Sr_1d_1 \subseteq Sd_{i-1}$ by Lemma 4 (2) we have $b\xi d_{i-1}$ and $r_1d_1\xi d_{i-1}$, i.e. $b\xi r_1d_1$. Let $Sd_i \cap Sd_1 = Sb$. Because (1) is wrong then $Sd_i \cap Sd_0 = Sb$. Since $d_1\theta_2 d_0$ and $Sd_0 \cap Sd_1 = Sb \subset Sr_1d_1 \subseteq Sd_1$, by Lemma 4 (2) we have $r_1d_1\theta_2 d_0$. Since $d_i\xi d_{i-1}$, where $\xi \in \{\theta_1, \eta\}$, and $Sd_i \cap Sd_{i-1} \subset Sr_1d_1 \subseteq Sd_{i-1}$, then by Lemma 4 (2) we have $r_1d_1\xi d_i$. By Lemma 5 (1) we have $d_0\theta_2 b$, or $d_i\xi b$. If $d_0\theta_2 b$ then $b\theta_2 d_0\theta_2 r_1d_1$ and in view $\theta_1^{k_1} = \theta_2^{k_1}$ we have $b\theta_1 r_1d_1$. If $d_i\xi b$ then $b\xi d_i\xi r_1d_1$.

Thus, (2) is proven. Just like (2), it is proved $r_0d_0\eta b \vee r_0d_0\theta_1 b$.

If $r_1d_1\theta_1 b$ or $r_0d_0\theta_1 b$ then $r_1d_1\theta_2 b$ or $r_0d_0\theta_2 b$; hence $d_1\theta_2 b\theta_2 d_0$; in view $\theta_1^{k_0} = \theta_2^{k_0}$ and $\theta_1^{k_1} = \theta_2^{k_1}$ we have $d_1\theta_1 d_0$, contradiction. Let $r_1d_1\eta b\eta r_0d_0$.

Then $d_0\theta_2 d_i$, $Sd_0 \cap Sd_i \subset Sr_0d_0 \subseteq Sd_0$ and $Sd_0 \cap Sd_i \subset Sr_1d_1 \subseteq Sd_1$ by Lemma 4 (2) imply $r_1d_1\theta_2 r_0d_0$. Then $r_1d_1(\theta_2 \wedge \eta)r_0d_0$, i.e. in view $\theta_2 \wedge \eta = \theta_1 \wedge \eta$ we have $d_1\theta_1 r_1d_1\theta_1 r_0d_0\theta_1 d_0$, contradiction.

Case 2: $Sd_k \cap Sd_0 \supset Sb$ or $Sd_k \cap Sd_1 \supset Sb$ for all k, $1 < k < 2n$. Then we have $Sd_k \cap Sd_1 \supset Sb$ for all k, $1 < k < 2n$, or $Sd_k \cap Sd_0 \supset Sb$ for all k, $1 < k < 2n$, or $Sd_{k_1} \cap Sd_0 \supset Sb$ and $Sd_{k_2} \cap Sd_1 \supset Sb$ for some k_1, k_2, $1 < k_1, k_2 < 2n$. In the first case by Lemma 3 (1) we have $Su_1 = Sd_{2n-1} \cap Sd_1 \supset Sb$ for some $u_1 \in A$. Since $d_{2n-1}\eta d_0$ and $d_0\theta_2 d_1$ then by Lemma 4 (2) we have $d_0(\eta \wedge \theta_2)u_1$ and $d_1\theta_2 u_1$. Since $\theta_1^{k_1} = \theta_2^{k_1}$ then $d_1\theta_1 u_1$ Since $\theta_1 \wedge \eta = \theta_2 \wedge \eta$ then $d_0(\theta_1 \wedge \eta)u_1$, in particular, $d_0\theta_1 u_1$. Therefore, $d_0\theta_1 u_1\theta_1 d_1$, i.e. $d_0\theta_1 d_1$, contradiction. The second case is similar to case a. Let $Sd_{k_1} \cap Sd_0 \supset Sb$ and $Sd_{k_2} \cap Sd_1 \supset Sb$ for some k_1, k_2, $1 < k_1, k_2 < 2n$. Since $Sd_k \cap Sd_0 \supset Sb$ or $Sd_k \cap Sd_1 \supset Sb$ for all k, $1 < k < 2n$ then $Sd_k \cap Sd_1 \supset Sb$ and $Sd_{k+1} \cap Sd_0 \supset Sb$ for some k, $1 < k < 2n-1$, or $Sd_k \cap Sd_0 \supset Sb$ and $Sd_{k+1} \cap Sd_1 \supset Sb$ for some k, $1 < k < 2n-1$. Let $Su_1 = Sd_k \cap Sd_1 \supset Sb$ and $Su_0 = Sd_{k+1} \cap Sd_0 \supset Sb$ for some k, $1 < k < 2n-1$ (an other sub-case is considered similarly). Then $Sd_k \cap Sd_{k+1} \subset Su_1$ and $Sd_k \cap Sd_{k+1} \subset Su_0$. Since $d_k\xi d_{k+1}$, where $\xi \in \{\theta_1, \eta\}$, $Sd_k \cap Sd_{k+1} \subset Su_0 \subseteq Sd_{k+1}$ and $Sd_k \cap Sd_{k+1} \subset Su_1 \subseteq Sd_k$, then by Lemma 4 (2) we have $u_0\xi d_{k+1}$ and $u_1\xi d_k$, therefore, $u_0\xi u_1$. Since $d_0\theta_2 d_1$, $Sd_0 \cap Sd_1 = Sb \subset Su_0 \subset Sd_0$ and $Sd_0 \cap Sd_1 = Sb \subset Su_1 \subset Sd_1$, then by Lemma 4 (2) we have $u_0\theta_2 d_0$, $u_1\theta_2 d_1$, hence $u_0\theta_2 u_1$, i.e. $u_0(\theta_2 \wedge \eta)u_1$ or $u_0\theta_1 u_1$. Since $\theta_1 \wedge \eta = \theta_2 \wedge \eta$ then $u_0\theta_1 u_1$. Since $\theta_1^{k_0} = \theta_2^{k_0}$ and $\theta_1^{k_1} = \theta_2^{k_1}$ then $u_0\theta_1 d_u$ and $u_1\theta_1 d_1$. Therefore, $d_0\theta_1 u_0\theta_1 u_1\theta_1 d_1$, i.e. $d_0\theta_1 d_1$, contradiction.

Thus, the lattice $Con({}_SA)$ is modular.

Step III. Let ${}_SA$ not be connected S-act. As we have proved, the lattices of congruences on connected components of ${}_SA$ are modular. Let us prove that the lattice $Con({}_SA)$ is modular. By Theorem 3 it is enough to prove that there are no perforating congruences on ${}_SA$. Suppose that ${}_SB \sqcup {}_SC \subseteq {}_SA$, θ is the perforating congruences on ${}_SA$, $b_1, b_2 \in B$, $c_1, c_2 \in C$, $(b_1, c_1) \in \theta$, $(b_2, c_2) \in \theta$, $(b_1, b_2) \notin \theta$, $(c_1, c_2) \notin \theta$. By point 1 of Theorem we can assume that ${}_SB$ is connected S-act.

We show that $sb_i\theta b_i$ for all $i \in \{1, 2\}$ and $s \in S$. Note that $Sb_i \cap Sc_i = Ssb_i \cap Ssc_i = \emptyset$ for all $s \in S$. Suppose that there is $l \in S$ such that $(b_i, lb_i) \notin \theta$. Let $t = \min\{l \mid (b_i, lb_i) \notin \theta\}$. Then $t > 1$. By point 2 of Theorem we have $tb_i = rb_i$ or $tc_i = rc_i$ for some $r \in S$, $r < t$. In the first case, we have $tb_i = rb_i\theta b_i$. Contradiction. In the second case, we have $tb_i\theta tc_i = rc_i\theta rb_i$, contradiction.

Since ${}_SB$ is connected S-act then $r_1b_1 = r_2b_2$ for some $r_1, r_2 \in S$. As proved above $b_1\theta r_1b_1 = r_2b_2\theta b_2$, contradiction. Therefore, there are no perforating congruences on ${}_SA$ and by Theorem 3 the lattice $Con({}_SA)$ is modular. □

4. Conclusion

In this paper, we obtain a description of S-acts over a well-ordered monoid with modular congruence lattice. In [10], S-acts over linearly ordered monoids with linearly ordered congruence lattices and S-acts over a well-ordered monoid with distributive congruence lattices are characterized. The questions of describing S-acts over linearly ordered monoids with distributive congruence lattices and with modular congruence lattice remain open.

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S-полигоны над вполне упорядоченным моноидом с модулярной решеткой конгруэнций

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Аннотация. Исследование относится к структурной теории полигонов, подразумевающей описание полигонов над теми или иными классами моноидов или обладающих теми или иными свойствами, например удовлетворяющих какому-либо требованию, предъявляемому к решётке конгруэнций. Конгруэнции универсальной алгебры — это ядра гомоморфизмов этой алгебры в другие. Знание всех конгруэнций означает знание всех гомоморфных образов алгебры. Левый S-полигон над моноидом S — это множество A, на котором моноид S действует слева, причем единица этого моноида действует тождественно. Рассматриваются полигоны над линейно упорядоченными и над вполне упорядоченными моноидами, где под линейно упорядоченным моноидом S понимается линейно упорядоченное множество с минимальным элементом и с бинарной операцией max, относительно которой S является, очевидно, коммутативным моноидом; под вполне упорядоченным моноидом S понимается вполне упорядоченное множество с бинарной операцией max, относительно которой S также является коммутативным моноидом. Статья является продолжением авторского исследования с М. С. Казаком, где приводится описание S-полигонов над линейно упорядоченными моноидами с линейной решеткой конгруэнций и S-полигонов над вполне упорядоченными моноидами с дистрибутивной решеткой конгруэнций. Описываются S-полигоны над вполне упорядоченными моноидами, решетки конгруэнций которых модулярны.

Ключевые слова: полигон над моноидом, решетка конгруэнций алгебры, модулярная решетка.

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