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Optimization of impulsive control systems with intermediate state constraints *

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Abstract. In this paper, we consider an optimal impulsive control problem with intermediate state constraints. The peculiarity of the problem consists in a non-standard way of specifying of intermediate constraints. So the constraints must be satisfied for at least one selection of a set-valued solution to the impulsive control system. We prove a theorem for the existence of an optimal control and propose the reduction procedure that transforms the initial optimal control problem with intermediate constraints into a hybrid problem with control parameters. This hybrid problem gives an equivalent description of the optimal impulsive control problem. We discuss a numerical algorithm based on a direct collocation method and give a schema to the corresponding numerical calculations for a test example.

Keywords: impulsive control, trajectory of bounded variation, intermediate state constraints, numerical method.

1. Introduction

We consider the optimal control problem (P_0):

$$\text{Minimize} \quad J_0 = l\big(x(b), V(b)\big)$$

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subject to the dynamics

$$\dot{x}(t) = f\big(t, x(t), u(t)\big) + G\big(t, x(t)\big)\, v(t), \quad x(a) = x_0, \tag{1.1}$$

$$\dot{V}(t) = \|v(t)\|_1, \qquad V(a) = 0, \tag{1.2}$$

$$\big(x(\theta_i), V(\theta_i)\big) \in A_i, \quad i = 1, \ldots, N, \tag{1.3}$$

$$u(t) \in U, \; v(t) \in K \text{ for a.e. } \; t \in T. \tag{1.4}$$

Here, $T = [a, b]$ is a fixed time interval from $\mathbb{R}$, U is a given compact subset of $\mathbb{R}^r$, K is a convex closed cone from $\mathbb{R}^m$, the constraint sets A_i, $i = \overline{1, N}$, are closed subsets of $\mathbb{R}^{n+1}$, $\theta = \{\theta_1, \ldots, \theta_N\}$ is a given vector of intermediate times such that $a < \theta_1 < \theta_2 < \cdots < \theta_{N-1} < \theta_N \le b$, $N < \infty$, $x_0 \in \mathbb{R}^n$ is a given vector of initial states. Symbol $\|\cdot\|_1$ stands for the Manhattan norm in $\mathbb{R}^m$, "a.e." denotes "almost everywhere with respect to the Lebesgue measure $\mathcal{L}$ on $\mathbb{R}$".

In Problem (P_0), functions $u(\cdot) \in L^\infty(T, \mathbb{R}^r)$ and $v(\cdot) \in L^\infty(T, \mathbb{R}^m)$ are controls whereas the corresponding function $V(\cdot)$ characterizes the total variation of $w(t) := \displaystyle\int_a^t v(\tau) d\tau$, namely, $V(t) = \displaystyle\sum_{j=1}^m \operatorname*{var}_{[a,t]} w_j(\cdot)$, $t \in T$. We say that $u(\cdot), v(\cdot)$ satisfying (1.4) are feasible controls if the intermediate conditions (1.3) hold for the correspondent solution to (1.1).

In general, Problem (P_0) is singular and does not have an optimal solution in the class of $\mathcal{L}$-measurable controls $u(\cdot)$, $v(\cdot)$ and absolutely continuous trajectory $x(\cdot)$. This is due to the fact that the right-hand side of Eq. (1.1) is pointwise unbounded and therefore one can take a sequence of controls converging in the sense of distributions to a generalized function (for example, the Dirac delta function) whereas the corresponding sequence of trajectories tends pointwise to a discontinuous function. For more detail we refer to [4; 7; 13]. The next example illustrates this remark.

Example 1. Let us consider the problem:

$$J_0(v) = \int_0^2 |v(t)| dt \to \inf, \tag{1.5}$$

$$\dot{x}(t) = x(t)v(t) + 4(1 - t), \quad x(0) = 0, \tag{1.6}$$

$$x(1/2) \in [0, 1], \qquad x(1) \in [3, 4], \quad x(2) = 0, \tag{1.7}$$

$$v(t) \in \mathbb{R}, \quad t \in T = [0, 2]. \tag{1.8}$$

We will show that the problem (1.5)–(1.8) does not have an optimal control from $L^\infty(T)$. First, we note that the inequality $J_0(v) > \ln(9/2)$ holds for an arbitrary feasible control $v(\cdot) \in L^\infty(T)$ due to the obvious estimates

$$\int_0^{\frac{1}{2}} |v(t)| dt > \ln\left(\frac{3}{2}\right), \quad \int_{\frac{1}{2}}^1 |v(t)| dt > \ln 2, \quad \int_1^2 |v(t)| dt > \ln\left(\frac{3}{2}\right)$$

following from the intermediate constraints (1.7). Moreover,

$$\inf_{v} J_0(v) = \ln(9/2).$$

Indeed, by taking the sequence of controls $\{v_k(\cdot)\}$ defined as follows

$$v_k(t) = \begin{cases} 0, & t \in \Big[0, \dfrac{1}{2} - \dfrac{1}{k}\Big), \\ \dfrac{-k^2 + 4k + 4 - 8k(1-t)}{2k + (-k^2 + 4k + 4)(t - 1/2)}, & t \in \Big[\dfrac{1}{2} - \dfrac{1}{k}, \dfrac{1}{2}\Big), \\ 0, & t \in \Big[\dfrac{1}{2}, 1 - \dfrac{1}{k}\Big), \\ \dfrac{3k^2 + 4 - 8(1-t)k}{6k + (3k^2 + 4)(t-1)}, & t \in \Big[1 - \dfrac{1}{k}, 1\Big), \\ \dfrac{-k^2 - 2 - 4(1-t)k}{3k - (k^2+2)(t-1)}, & t \in \Big[1, 1 + \dfrac{1}{k}\Big), \\ 0, & t \in \Big[1 + \dfrac{1}{k}, 2\Big] \end{cases}$$

for $k = 3, 4, \ldots$, we see that the corresponding trajectories $\{x_k(\cdot)\}$:

$$x_k(t) = \begin{cases} -2t^2 + 4t, & t \in \Big[0, \dfrac{1}{2} - \dfrac{1}{k}\Big), \\ \Big(2 + \dfrac{2}{k} - \dfrac{k}{2}\Big)\Big(t - \dfrac{1}{2}\Big) + 1, & t \in \Big[\dfrac{1}{2} - \dfrac{1}{k}, \dfrac{1}{2}\Big), \\ -2t^2 + 4t - \dfrac{1}{2}, & t \in \Big[\dfrac{1}{2}, 1 - \dfrac{1}{k}\Big), \\ \Big(\dfrac{3k}{2} + \dfrac{2}{k}\Big)(t-1) + 3, & t \in \Big[1 - \dfrac{1}{k}, 1\Big), \\ -\Big(k + \dfrac{2}{k}\Big)(t-1) + 3, & t \in \Big[1, 1 + \dfrac{1}{k}\Big), \\ -2t^2 + 4t, & t \in \Big[1 + \dfrac{1}{k}, 2\Big] \end{cases}$$

satisfy (1.7). Thereby the controls from $\{v_k(\cdot)\}$ are feasible. Since $J_0(v_k) \to \ln(9/2)$ when k tends to infinity, we conclude that $\{v_k(\cdot)\}$ is minimizing. The graphs of $v_k(\cdot)$ and $x_k(\cdot)$ are shown in Fig. 1.

Next, we consider functions $V_k(t) = \displaystyle\int_a^t |v_k(\tau)|d\tau$, $t \in T$, and measures $\mu_k = v_k(t)dt$, $k \geq 3$. We note that $V_k(t) = |\mu_k|\big([a,t]\big)$, $t \in T$, where $|\mu_k|$ denotes the total variation of measure μ_k.

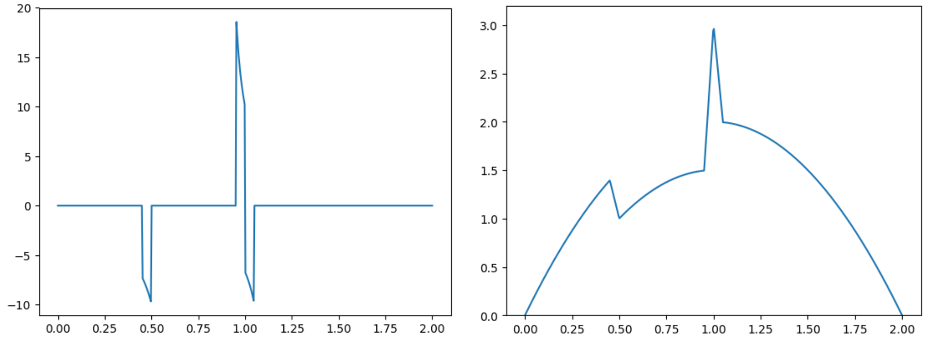

Figure 1. The minimizing sequence components v_k and x_k for $k = 20$

Let functions $\bar{x}(\cdot)$, $\bar{V}(\cdot)$ and measure $\bar{\mu}$ be defined by

$$\bar{x}(t) = \begin{cases} 4t - 2t^2, & t \in \big[0, 1/2\big), \\ 4t - 2t^2 - 1/2, & t \in \big[1/2, 1\big), \\ 3, & t = 1, \\ 4t - 2t^2, & t \in \big(1, 2\big], \end{cases} \qquad \bar{V}(t) = \begin{cases} 0, & t \in \big[0, 1/2\big), \\ \ln(3/2), & t \in \big[1/2, 1\big), \\ \ln(9/2), & t \in \big[1, 2\big] \end{cases}$$

$$\bar{\mu} = \ln(2/3)\,\delta\big(t - 1/2\big) + \ln(4/3)\,\delta\big(t - 1\big),$$

where $\delta(t-s)$ is the Dirac delta function concentrated at the point s. Then,

$$x_k(t) \to \bar{x}(t), \quad V_k(t) \to \bar{V}(t) \quad \text{for all } t \in T,$$

$$\mu_k \overset{*}{\to} \bar{\mu} \quad \text{(in the sense of distributions)}.$$

Thus, the problem (1.5)–(1.8) leads us to some its relaxation, in which the measure $\bar{\mu}$ and the discontinuous function $\bar{x}(\cdot)$ are a feasible impulsive control and a trajectory, respectively. We note that now $\bar{V}(\cdot) \neq |\bar{\mu}|\big([a, \cdot]\big)$, where $|\bar{\mu}|$ is the total variation of $\bar{\mu}$. Therefore, the concept of impulsive control must include additional components defining jumps of states x and V. The impulsive statement of the problem (1.5)–(1.8) and its numerical analysis are given in Section 4.

This paper continues the research started in [3]. We address the optimal impulsive control problem that is a relaxation extension of Problem (P_0). This paper is organized as follows. In Section 2, we describe an optimal impulsive control problem and prove the existence theorem for an optimal solution. In Section 3, we transform the optimal impulsive control problem into an equivalent hybrid problem. In Section 4, we discuss a numerical solution algorithm based on a direct collocation method and give a schema to the corresponding numerical calculations for a test case from Example 1.

2. Problem statement

Let $BV([a,b],\mathbb{R}^k)$ be the space of functions $y:[a,b]\to\mathbb{R}^k$ with bounded total variation on T. Let $y(a)=y(a^-)$, $y(b)=y(b^+)$.

We consider a tuple $\pi=\big(\mu,S,\{d_s,\omega_s(\cdot)\}_{s\in S}\big)$ such that: μ is a bounded Borel measure on T with values from K; the set S is an at most countable subset of the interval T and $S\supseteq S_{\rm d}(\mu):=\big\{s\in T\,\big|\,\mu(\{s\})\neq 0\big\}$; the non-negative numbers $\{d_s\}_{s\in S}$ and $\mathcal{L}$-measurable functions $\{\omega_s(\cdot)\}_{s\in S}$, where $\omega_s(\cdot):[0,d_s]\to co\ K_1$, $s\in S$, satisfy the following conditions

$$d_s\geq\big\|\mu\big(\{s\}\big)\big\|_1,\quad \int_0^{d_s}\omega_s(\tau)\,d\tau=\mu\big(\{s\}\big).$$

Here, $K_1:=\{v\in K\mid \|v\|_1=1\}$ and $co\,A$ is a convex hill of A.

We say that π is an impulsive control and consider the corresponding function $V=V[\pi]:\ T\to\mathbb{R}$ defined by the rule

$$V(t)=|\mu_{\rm c}|\big([a,t]\big)+\sum_{s\leq t,\ s\in S}d_s,\quad t\in(a,b],\quad V(a)=0,$$

where $\mu_{\rm c}$ denotes the continuous component in the Lebesgue decomposition of μ.

Let $M>0$ be given. We denote by $\mathcal{W}^M(T,K)$ the collection of all impulsive controls π for which $V[\pi](b)\leq M$.

Let us consider the impulsive control system

$$dx(t)=f\big(t,x(t),u(t)\big)\,dt+G\big(t,x(t)\big)\,\pi(dt),\quad x(a)=x_0. \tag{2.1}$$

The solution concept for (2.1) is understood in the sense of [1; 4–8; 13] (see also the references therein). For given an initial condition x_0 and controls $\big(u,\pi\big)$, where $u\in L^\infty(T,U)$, $\pi=\big(\mu,S,\{d_s,\omega_s(\cdot)\}_{s\in S}\big)\in\mathcal{W}^M(T,K)$, we introduce functions $x_{\rm r}(\cdot)$, $z^s(\cdot)$, $s\in S$:

$$\begin{aligned}x_{\rm r}(t)=x_0+\int_a^t f\big(\tau,x_{\rm r}(\tau),u(\tau)\big)\,d\tau+\int_a^t G\big(\tau,x_{\rm r}(\tau)\big)\,\mu_{\rm c}(d\tau)\\ +\sum_{s\in S,\ s\leq t}\big(z^s(d_s)-x_{\rm r}(s^-)\big),\quad t\in(a,b],\quad x_{\rm r}(a)=x_0,\end{aligned} \tag{2.2}$$

$$\frac{dz^s(\tau)}{d\tau}=G\big(s,z^s(\tau)\big)\,\omega_s(\tau),\quad z^s(0)=x_{\rm r}(s^-),\ \tau\in[0,d_s],\quad s\in S. \tag{2.3}$$

We note that $x_{\rm r}\in BV(T,\mathbb{R}^n)$; moreover, $x_{\rm r}(\cdot)$ is right continuous on $(a,b]$.

Then, we define a set-valued function $X_V:T\to{\rm comp}(\mathbb{R}^{n+1})$ as follows

(i) $X_V(t)=\big(x_{\rm r}(t),V(t)\big)$ for all $t\in T\setminus S$,

(ii) $X_V(s) = \{(z^s(\tau), V(s-)+\tau) \mid \tau \in [0,d_s]\}$ for all $t = s \in S$.
Here, comp(A) denotes the collection of all nonempty compact subset from A, and $V = V[\pi]$. We posit that

$$X_V(a-) = \big(x_{\mathrm{r}}(a), V(a)\big) = (x_0, 0), \quad X_V(b+) = \big(x_{\mathrm{r}}(b), V(b)\big).$$

We say that X_V is a solution to (2.1) corresponding to (u, π) and x_0. By an impulsive process we mean a tuple $\sigma = (X_V, u, \pi)$, where $u \in L^\infty(T, U)$, $\pi = \big(\mu, S, \{d_s, \omega_s(\cdot)\}_{s\in S}\big) \in \mathcal{W}^M(T, K)$, and X_V is the corresponding solution to (2.1).

We consider the optimal impulsive control problem (P):

$$\text{Minimize} \quad J = l\big(X_V(b+)\big)$$

over the impulsive processes $\sigma = (X_V, u, \pi)$ such that

$$X_V(\theta_i) \cap A_i \neq \emptyset, \quad i = 1, \ldots, N. \tag{2.4}$$

We say that $\sigma = (X_V, u, \pi)$ is feasible for Problem (P) provided that (2.4) holds. Let Σ be the collection of all feasible impulsive processes.

We note that every selection of X_V is a function from $BV(T, \mathbb{R}^{n+1})$. So we can also consider individual selections of X_V as solutions to (2.1). In this case, for a feasible process with path X_V, the intermediate constraints (2.4) must hold for at least one selection of X_V. Such an interpretation of intermediate constraints significantly distinguishes the problem considered in this paper from the problems in [8;9], where the constraints were imposed on one-sided values of trajectories at intermediate time points and thus the existence of an optimal solution could not be guaranteed. We show below that an optimal solution to Problem (P) exists under standard assumptions on the input functions.

In what follows, we accept the following assumptions:

(H_1) Function $l:\ \mathbb{R}^{n+1} \mapsto \mathbb{R}$ is continuous.

(H_2) Functions $f:\ T\times\mathbb{R}^n\times U \mapsto \mathbb{R}^n$, $G:\ T\times\mathbb{R}^n \mapsto \mathbb{R}^{n\times m}$ are continuous for all arguments and locally Lipschitz continuous in x. Moreover, there exist constants c_1, $c_2 > 0$ such that

$$\big\|f(t,x,u)\big\| \le c_1\left(1 + \|x\|\right), \quad \big\|G(t,x)\big\| \le c_2\left(1 + \|x\|\right)$$

for any $(t, x, u) \in T \times \mathbb{R}^n \times U$.

(H_3) The set $f(t,x,U) := \{f(t,x,u) \mid u \in U\}$ is convex for all $(t, x) \in T \times \mathbb{R}^n$.

The next lemma following from the result [8, Lemma 1] states the relation between problems (P) and (P_0).

Lemma 1. 1) *Let* $\{x_k(\cdot), V_k(\cdot), u_k(\cdot), v_k(\cdot)\}$ *be a sequence such that*

i) *for every* k *functions* $x_k(\cdot)$, $V_k(\cdot)$, $u_k(\cdot)$, *and* $v_k(\cdot)$ *satisfy* (1.1), (1.2), *and* (1.4),

ii) $\sup_k V_k(b) \le M$,

iii) $\lim_{k\to\infty} \big(x_k(\theta_i), V_k(\theta_i)\big) \in A_i, \quad i = 1, \ldots, N.$

Then, there exist an impulsive process $\sigma = \big(X_V, u, \pi\big) \in \Sigma$ *and its selection* $\big(x(t), V(t)\big) \in X_V(t)$, $t \in T$, *such that*

$$\big(x_k(t), V_k(t)\big) \to \big(x(t), V(t)\big) \quad \textit{for all } t \in T. \tag{2.5}$$

2) *For every* $\sigma = \big(X_V, u, \pi\big) \in \Sigma$ *and every selection* $\big(x(\cdot), V(\cdot)\big)$ *of* X_V *such that* $\big(x(a), V(a)\big) = (x_0, 0)$, *there exists a sequence of functions* $\big\{x_k(\cdot), V_k(\cdot), u_k(\cdot), v_k(\cdot)\big\}$ *such that the conditions i)–iii) and* (2.5) *hold.*

Theorem 1 (The existence of an optimal solution)**.** *Let* $\Sigma \ne \emptyset$. *Then, there exists an impulsive process* $\bar\sigma = \big(\bar X_V, \bar u, \bar\pi\big) \in \Sigma$ *such that*

$$J(\bar\sigma) = \min_{\sigma\in\Sigma} J(\sigma).$$

Proof. The proof of Theorem 1 is based on the space-time representation of impulsive processes (see, e.g., [5; 7; 8] and references therein). We consider the auxiliary optimal control problem (P_{a}):

$$\text{Minimize} \quad \tilde J = l\big(\tilde y(s_b), \tilde m(s_b)\big)$$

subject to the dynamics

$$\tilde\eta'(s) = \tilde\omega_0(s), \quad \tilde\eta(0) = a, \quad \tilde\eta(s_b) = b, \tag{2.6}$$

$$\tilde y'(s) = f\big(\tilde\eta(s), \tilde y(s), \tilde\nu(s)\big)\,\tilde\omega_0(s) + G\big(\tilde\eta(s), \tilde y(s)\big)\,\tilde\omega(s), \quad \tilde y(0) = x_0, \tag{2.7}$$

$$\tilde m'(s) = 1 - \tilde\omega_0(s), \qquad \tilde m(0) = 0, \tag{2.8}$$

$$\big(\tilde\eta(s_i), \tilde y(s_i), \tilde m(s_i)\big) \in \{\theta_i\} \times A_i, \quad i = 1, \ldots, N, \tag{2.9}$$

$$s_b \in [b - a, b - a + M], \tag{2.10}$$

$$\tilde\nu(s) \in U, \quad \big(\tilde\omega_0(s), \tilde\omega(s)\big) \in co\,\tilde K_1 \text{ for a.e. } s \in [0, s_b]. \tag{2.11}$$

In this problem, the trajectories are absolutely continuous functions $\tilde\eta(\cdot)$, $\tilde y(\cdot)$, and $\tilde m(\cdot)$; the controls are $\tilde\nu(\cdot) \in L^\infty\big([0, s_b], \mathbb{R}^r\big)$ and $\big(\tilde\omega_0(\cdot), \tilde\omega(\cdot)\big) \in L^\infty\big([0, s_b], \mathbb{R}^{m+1}\big)$; $\rho = (s_1, \ldots, s_N, s_b)$ is a vector of non-fixed times such that $0 < s_1 < \cdots < s_N \le s_b$; the control constraint set $\tilde K_1 := \{(\tilde\omega_0, \tilde\omega) \in [0,1] \times K \mid \tilde\omega_0 + ||\tilde\omega||_1 = 1\}$. Let $s_0 := 0$, $s_{N+1} := s_b$.

We denote by σ_{a} a feasible process to Problem (P_{a}); i.e.,

$$\sigma_{\mathrm{a}} = \big(\rho, \tilde\eta, \tilde y, \tilde\nu, \tilde\omega_0, \tilde\omega\big),$$

where the components satisfy (2.6)–(2.11). Let Σ_{a} be the collection of all feasible processes σ_{a}. Under the assumptions (H_1)–(H_3) the sets Σ and Σ_{a} are related to each other (for more details, see [8]) and

$$\min_{\sigma\in\Sigma} J(\sigma) = \min_{\sigma_{\mathrm{a}}\in\Sigma_{\mathrm{a}}} \tilde J(\sigma_{\mathrm{a}}).$$

Since $\Sigma \neq \emptyset$, the existence of an optimal solution to Problem (P) immediately follows from the uniform boundedness and equicontinuity of the set of solutions $\big(\rho, \tilde{\eta}, \tilde{y}\big)$ to the system (2.6)–(2.8) (the proof of these properties follows closely the lines of the proof of similar results from [5; 13])).

Theorem 1 is thus proven. □

3. Hybrid problem

In this section, we transform Problem (P_{a}) into a hybrid problem, named (P_{h}). To this end, we perform a standard time change for Problem (P_{a}). In order not to complicate the notation, we denote variables for the transformed problem in the same way as for (P_{a}).

Let $\tau_0 := 0$, $\tau_{N+1} := 1$. We fix arbitrary time moments $\tau_i \in [0,1]$, $i = \overline{1,N}$, such that

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_{N-1} < \tau_N \leq \tau_{N+1} = 1.$$

Let $\zeta := (\tau_1, \dots, \tau_N)$.

In the hybrid problem, the controls are functions $\nu(\cdot)$, $\omega_0(\cdot)$, $\omega(\cdot)$ and a vector of control parameters $g = (g_1, g_2, \dots, g_{N+1})$, where $g_i \geq 0$, $i = \overline{1, N+1}$. We note that by definition $g_i > 0$ for every $i \in \{1, \dots, N\}$ and the identity $g_{N+1} = 0$ implies $\tau_N = 1$ (i.e., $\theta_N = b$). Let

$$Z(g) := \operatorname{co}\Big\{(\omega_0, \omega) \in \mathbb{R}_+ \times K \;\;\Big|\;\; \omega_0 + \sum_{j=1}^{m} |\omega_j| = g\Big\}.$$

Introduce some notations: for $i \in \{1, \dots, N+1\}$

$$\alpha_i := \tau_i - \tau_{i-1}, \quad m(\tau_i) := \sum_{j=1}^{i} \alpha_j g_j - (\theta_i - a), \quad q_i := \big(y(\tau_i), m(\tau_i)\big).$$

Then, Problem (P_{h}) is described by the following relations:

$$l\big(q_{N+1}\big) \to \min,$$

$$\eta'(\tau) = \omega_0(\tau), \quad \eta(0) = a, \quad \eta(1) = b, \tag{3.1}$$

$$y'(\tau) = f\big(\eta(\tau), y(\tau), \nu(\tau)\big)\, \omega_0(\tau) + G\big(\eta(\tau), y(\tau)\big)\, \omega(\tau), \quad y(0) = x_0, \tag{3.2}$$

$$\eta(\tau_i) = \theta_i, \qquad q_i \in A_i, \quad i = 1, \dots, N, \tag{3.3}$$

$$\nu(\tau) \in U \text{ for a.e. } \tau \in [0,1], \tag{3.4}$$

$$\big(\omega_0(\tau),\omega(\tau)\big)\in Z(g_i) \text{ for a.e. } \tau\in[\tau_{i-1},\tau_i],\quad i=\overline{1,N+1}, \tag{3.5}$$

$$\alpha_i g_i \ge \theta_i-\theta_{i-1},\quad i=\overline{1,N+1},\quad \sum_{j=1}^{N+1}\alpha_j g_j \le b-a+M. \tag{3.6}$$

Let us comment on the relation between problems (P_{a}) and (P_{h}) and show how to define σ_{a} by using σ_{h}.

For given $\sigma_{\mathrm{h}}=\big(\zeta,g;\eta,y,\nu,\omega_0,\omega\big)$ satisfying (3.1)–(3.6), the corresponding $\sigma_{\mathrm{a}}=\big(\rho;\tilde\eta,\tilde y,\tilde\nu,\tilde\omega_0,\tilde\omega\big)$ is defined by the following rule:

1) the vector of intermediate time moments $\rho=(s_1,\dots,s_N,s_b)$:

$$s_0=0,\quad s_i=g_i(\tau_i-\tau_{i-1})+s_{i-1},\quad i=\overline{1,N},\quad s_b=g_{N+1}(1-\tau_N)+s_N.$$

2) controls $\tilde\omega_0(\cdot)$, $\tilde\omega(\cdot)$, $\tilde\nu(\cdot)$ and trajectories $\tilde\eta(\cdot)$, $\tilde y(\cdot)$ are defined as follows

$$\tilde\omega_0(s)=\frac{1}{g_i}\omega_0\big(\tau^i(s)\big),\quad \tilde\omega(s)=\frac{1}{g_i}\omega\big(\tau^i(s)\big),\quad \tilde\nu(s)=\nu\big(\tau^i(s)\big),$$

$$\tilde\eta(s)=\eta\big(\tau^i(s)\big),\quad \tilde y(s)=y\big(\tau^i(s)\big),$$

where $\tau^i(s)=\tau_{i-1}+(s-s_{i-1})/g_i$, $s\in[s_{i-1},s_i]$, $i=\overline{1,N}$. And the same holds for $i=N+1$ if $g_{N+1}\ne 0$.

4. Numerical solution of (P_{h})

For solving (P_{h}) it is convenient to use direct collocation methods since they can easily handle intermediate constraints. Various variational methods (see, e.g., [10; 11]) can also be applied to (P_{h}); but these approaches are quite involved and lead to heavy computations.

Recall that in direct collocation methods we transform the initial optimal control problem to a finite dimensional optimization problem. The latter depends on the choice of the numerical scheme used to solve our dynamic system.

Consider Example 1 above and write down the corresponding optimal impulsive control problem:

$$J=V(2)\to\min, \tag{4.1}$$

$$dx(t)=x(t)\pi(dt)+4(1-t),\quad x(0)=0, \tag{4.2}$$

$$x(1/2)\in[0,1],\quad x(1)\in[3,4],\quad x(2)=0, \tag{4.3}$$

$$\pi\in\mathcal{W}\big(T,\mathbb{R}\big). \tag{4.4}$$

Here, $T=[0,2]$, $\theta_1=1/2$, $\theta_2=1$, $A_1=[0,1]$, $A_2=[3,4]$, $A_3=\{0\}$. The impulsive control is a triple $\pi=\big(\mu,S,\{d_s,\omega_s(\cdot)\}_{s\in S}\big)$, where $\mu\in C^*(T,\mathbb{R})$,

$S \subset [0,2]$ is an at most countable set containing all atoms of μ. In addition, π contains, for any $s \in S$, a nonnegative number d_s and a measurable function $\omega_s : [0, d_s] \to [-1,1]$ such that

$$d_s \geq |\mu(\{s\})|, \quad \int_0^{d_s} \omega_s(\tau)d\tau = \mu(\{s\}).$$

In this case $V(\cdot)$ is given by

$$V(t) = |\mu_{\rm c}|([a,t]) + \sum_{s\leq t,\, s\in S} d_s, \quad t \in (0,2], \quad V(0)=0,$$

and the solution to (4.2) that corresponds to π is the tuple $\big(x_{\rm r}(\cdot), \{z^s(\cdot)\}_{s\in S}\big)$ such that

$$x_{\rm r}(t) = 4t - 2t^2 + \int_0^t x_{\rm r}(\tau)\,\mu_{\rm c}(d\tau) + \sum_{s\in S,\ s\leq t} \big(z^s(d_s) - x_{\rm r}(s^-)\big), \tag{4.5}$$

$$\frac{dz^s(\tau)}{d\tau} = z^s(\tau)\,\omega_s(\tau), \quad z^s(0) = x_{\rm r}(s^-), \quad \tau \in [0, d_s], \quad s \in S. \tag{4.6}$$

Let $X : T \to \mathrm{comp}(\mathbb{R})$ be a set-valued map defined by

$$X(t) = \begin{cases} x_{\rm r}(t), & t \in T \setminus S \\ \big\{z^s(\tau) \mid \tau \in [0, d_s]\big\}, & t = s \in S. \end{cases}$$

We say that a process $\sigma = \big(X, \pi\big)$ is feasible in (4.1)–(4.4) if the following intermediate constraints hold instead of (4.3):

$$X(\theta_i) \cap A_i \neq \emptyset, \quad i = \overline{1,3}. \tag{4.7}$$

Note that selections of X can be also thought of as solutions to (4.2). With this interpretation the intermediate constraint (4.3) must hold for some selection of a feasible process X, i.e., there exists $x(\cdot) \in BV(T,\mathbb{R})$ such that $x(t) \in X(t)$ for $t \in T$ and $x(\theta_i) \in A_i,\ i = \overline{1,3}$.

By transforming (4.1)–(4.4), we obtain the auxiliary problem

$$\hat{J} = \sum_{i=1}^{3} \alpha_i g_i - (b-a) \to \min,$$
$$\eta'(\tau) = u(\tau), \quad \eta(0) = a, \quad \eta(\tau_1) = \theta_1, \quad \eta(\tau_2) = \theta_2, \quad \eta(1) = b,$$
$$y'(\tau) = 4(1-\eta(\tau))u(\tau) + y(\tau)v(\tau), \quad \tau \in [0,1],$$
$$y(0) = x_0, \quad y(\tau_1) \in [0,1], \quad y(\tau_2) \in [3,4], \quad y(1) = 0,$$
$$u(\tau) \geq 0, \quad \begin{cases} u(\tau) + |v(\tau)| \leq g_1, & \tau \in [0, \tau_1], \\ u(\tau) + |v(\tau)| \leq g_2, & \tau \in [\tau_1, \tau_2], \\ u(\tau) + |v(\tau)| \leq g_3, & \tau \in [\tau_2, 1], \end{cases}$$
$$\alpha_i g_i \geq \theta_i - \theta_{i-1}, \quad i = 1,2,3, \quad \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 \leq b - a + M.$$

Here, $x_0 = 0$, $a = 0$, $b = 2$, $\theta_0 = a$, $\theta_1 = 1/2$, $\theta_2 = 1$, $\theta_3 = b$, $\tau_0 = 0$, $\tau_3 = 1$. The intermediate time moments τ_1, $\tau_2 \in [0, 1]$ and nonnegative parameters α_i satisfy the identities $\alpha_i = \tau_i - \tau_{i-1}$, $i = 1, 2, 3$.

Let us choose $\tau_1 = 1/4$, $\tau_2 = 1/2$, $M = 3$. Then $\alpha_1 = 1/4$, $\alpha_2 = 1/4$, $\alpha_3 = 1/2$. Now we define a partition $\{t^k : k = 0, \ldots, N_3\}$ on T such that $t^0 = 0$, $t^{N_1} = \tau_1$, $t^{N_2} = \tau_2$, $t^{N_3} = \tau_3$, $t_{k+1} - t_k = h$ for all k. Heun's method (a two-stage Runge–Kutta scheme) applied to our dynamical system yields the following optimization problem (P_{discr}):

$$\frac{g_1}{4} + \frac{g_2}{4} + \frac{g_3}{2} - 2 \to \min,$$
$$(\hat{\eta}_{k+1} - \eta_k)h^{-1} = u_k,$$
$$2(\eta_{k+1} - \eta_k)h^{-1} = u_k + u_{k+1},$$
$$(\hat{y}_{k+1} - y_k)h^{-1} = 4(1 - \eta_k)u_k + y_k v_k,$$
$$2(y_{k+1} - y_k)h^{-1} = 4(1 - \eta_k)u_k + y_k v_k + 4(1 - \hat{\eta}_{k+1})u_{k+1} + \hat{y}_{k+1} v_{k+1},$$
$$k = 0, \ldots, N_3 - 1,$$
$$\eta_0 = 0, \quad \eta_{N_1} = 1/2, \quad \eta_{N_2} = 1, \quad \eta_{N_3} = 2,$$
$$y_0 = 0, \quad y_{N_1} \in [0, 1], \quad y_{N_2} \in [3, 4], \quad y_{N_3} = 0,$$
$$u_k \geq 0, \quad k = 0, \ldots, N_3,$$
$$u_k - g_1 \leq v_k \leq g_1 - u_k, \quad k = 0, \ldots, N_1,$$
$$u_k - g_2 \leq v_k \leq g_2 - u_k, \quad k = N_1 + 1, \ldots, N_2,$$
$$u_k - g_3 \leq v_k \leq g_3 - u_k, \quad k = N_2 + 1, \ldots, N_3,$$
$$g_i \geq 2 \quad i = 1, 2, 3, \quad \frac{g_1}{4} + \frac{g_2}{4} + \frac{g_3}{2} \leq 5.$$

Here the decision variables are η_k, $\hat{\eta}_k$, y_k, $\hat{y}_k$, u_k, v_k, g_i.

Problem (P_{discr}) is solved by using COIN-OR Ipopt, a nonlinear solver implementing an interior point line search filter method. The analytical solution of (P_{h}) and a numerical solution of (P_{discr}) for $N_3 = 100$ are presented in Fig. 2. Note that if N_3 grows (i.e., the grid step decreases) the numerical and analytical solutions become visually indistinguishable, though the control function still jumps around $t = 0.35$ and $t = 0.5$. Further optimization, which uses the classical Euler scheme, allows to discard these perturbations.

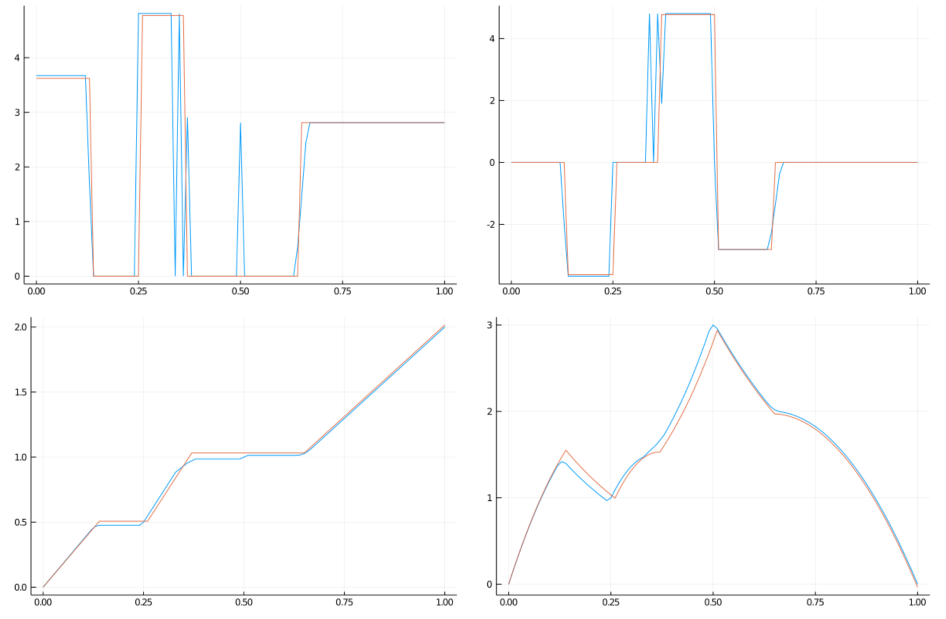

Figure 2. The analytical solution of (4.1)–(4.4) and its numerical solution for $N_3 = 100$. Top: controls u and v, bottom: trajectories η and y.

5. Conclusion

In this paper, the intermediate state constraints are interpreted in a special way which provides the existence of an optimal control. The problem of state constraints for impulsive processes has been thoroughly studied by B.M. Miller (see references in [1;5;7]). It is clear that a relaxation extension of singular optimal control problems admits various interpretations of state constraints and therefore leads to different statements of optimal impulsive control problems. In the most studied case, the intermediate constraints are imposed on one-sided values of feasible trajectories from $BV(T, \mathbb{R}^n)$. For this case, necessary and sufficient optimality conditions were obtained both in a form of Maximum principle and in a variational form based on using of functions of the Lyapunov type (see the literature review in [8]). In this paper, we consider a new case when a trajectory is represented by a set-valued function X_V whereas the constraint at an intermediate point θ_i means that $X_V(\theta_i) \cap A_i \neq \emptyset$. Such constraints may arise in different applications. The reduction procedure proposed in this paper allows to replace the initial optimal impulsive control problem with intermediate constraints with a hybrid problem and then with a finite-dimensional optimization problem. It is shown that in some cases the hybrid problem can be effectively solved by direct collocation methods.

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Оптимизация импульсных управляемых систем с промежуточными фазовыми ограничениями

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Аннотация. Рассматривается задача оптимального импульсного управления с промежуточными фазовыми ограничениями. Особенность задачи состоит в нестандартном способе задания промежуточных ограничений, которые должны выполняться хотя бы для одного селектора многозначного решения импульсной управляемой системы. Доказана теорема существования оптимального решения, и представлена гибридная задача с управляющими параметрами, которая дает эквивалентное описание задачи оптимального импульсного управления. Обсуждается вариант численного решения гибридной задачи, основанный на прямом коллокационном методе, и представлена схема соответствующих численных расчетов для тестового примера.

Ключевые слова: импульсное управление, траектории ограниченной вариации, промежуточные фазовые ограничения, численный метод.

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