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Studying Semigroups Using the Properties of Their Prime m-Ideals

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Abstract. In this article, we present the idea of m-ideals, prime m-ideals and their associated types for a positive integer m in a semigroup. We present different chrarcterizations of semigroups through m-ideals. We demonstrate that the ordinary ideals, and their relevent types differ from the m-ideals and their assocated types by presenting concrete examples on the maximal, irreducible and strongly irreducible m-ideals. We conclude from the study that the introduction of the m-ideal will explore new fields of studies in semigroups and their applications.

Keywords: completely prime m-ideals, strongly prime m-ideals, maximal m-ideals, irreducible m-ideals, strongly Irreducible m-ideals.

1. Introduction

Semigroups are the fundamental blocks of almost all algebraic structures. Semigroups are characterized in several ways by using the properties and types of their ideals. The concept of prime ideals in algebraic structures originated as a generalization of the concept of prime numbers. Prime ideals are as important in algebraic structures as the prime numbers in the field of arithmetic [5].

Ideals are generalized in different ways. One way to generalize ideals is through positive integers. This was initiated by Lajos [9]. Chinram et al., characterized the quasi ideals through two positive integers in semirings [3]. *Ansari et. al.,* characterized the quasi ideals in semigroups through the two positive integers [2]. Ansari et. al., also characterized the non-associated structures ([6–8; 21]) in his article referred as [1]. *Eqbal et. al.,* gave the concept of $(m,\ n)$ semirings [4]. *Mahboob et. al.,* defined some types of ideals like (m,n)-hyperideals in ordered semihypergroups using two positive integers m and n [10]. *Pibaljommee et. al.,* presented (m,n)-bi-quasi hyperideals in semihyperrings [18].

The other way to generalize ideals is through a single positive integer m. *Munir et. al.,* generalized the bi ideals in the semiring through a positive integer m and called them m-bi ideals [15]. The author again presented the concept of the m-bi ideals in the semigroups [11]. *Nakkhasen et. al.,* gave the concept of the m-bi-hyperideals in semihyperrings through a positive integer m [16]. *Munir et al.,* presented the idea of m-quasi ideals in semirings and other related concepts like m-regular and m-weakly regular semirings in [12]. In the field of fuzzification, Munir et. al., in his article [14], characterized the semigroups through introducing the concept of prime fuzzy m-bi ideals.

Generalization of ideals through two non-negative integers (m,n) and one positive integer m are two different ways to study the different properties of semigroups. These generalizations naturally motivated us to present the idea of m-ideals in semigroups. Theory of m-ideals is useful to explore new results associated with the subsets, e.g., subsemigroups of semigroups. These study the subsets of semigroups on the lines of the study of the semigroups themselves. In this way, the m-ideals will help in selecting the usable samples of larger finite semigroups being used in different scientific fields like automata theory. As an extension of this work, the researchers can investigate the properties of ideals for every subset of the semigroup.

In order to explore the nature and structures of m-ideals along with their related types in semigroups, we have divided the contents of this paper into five sections. Section 1 discusses the introduction and essential motivation of our research work. Section 2 presents the preliminary concepts from the literature which will be used to build up the theory of the m-ideals. Section 3 deals with the definition and the properties of m-ideals and prime m-ideals. Section 4 studies the maximal m-ideals and Section 5 deals with the irreducible and strongly irreducible m-ideals, and their basic properties.

2. Preliminaries

In this section, we call the essential definitions from the literature of semigroups which will be used in building the concepts of m-ideals.

Definition 2.1. *A non-void set S satisfying the closure law and the associative law under a given binary operation $\cdot$ from $S \times S$ to S is known as a semigroup.*

Definition 2.2. *The product of two non-void subsets A, and B of a semigroup S is defined by $AB = \{ab : a \in A, b \in B\}$.*

For a subset A of a semigroup S, and a positive integer m, we have, $A^m = AAA...A$(m-times) [15]. Considering multiplication of subsets, a subsemigroup of a semigroup S is a nonempty subset A with the property that $A^2 \subseteq A$, where $A^2 = AA \subseteq A$. Since $A^3 = AAA \subseteq A^2 \subseteq A$, i.e., $A^3 \subseteq A^2$, and $A^3 \subseteq A$. In this way, we reason that $A^l \subseteq A^m$ for any two positive integers l and m, to the extent that $l \geq m$. Subsequently, $A^m \subseteq A$, for all positive integers m, however the converse does not follow.

Definition 2.3. *Let A be a subsemigroup of a semigroup S. If the proposition $AS \subseteq A$ holds, then A is known as a right ideal of A. If $SA \subseteq A$ holds, then A is known as a left ideal of S. In case, if A is both a right and a left ideal of S, A is known as a two-sided ideal or simply an ideal of S.*

If A is a two-sided ideal of S, then $SAS = (SA)S \subseteq AS \subseteq A$. Conversely, if $SAS \subseteq A$ and S has a left identity e, then $AS = eAS \subseteq SAS \subseteq A$, so that A is a right ideal of S. Comparably, if S has a right identity and $SAS \subseteq A$, then A is a left ideal of S. Consequently, if S has a two-sided identity and $SAS \subseteq A$, then A is a two-sided ideal of S. So, without identity, $SAS \subseteq A$ may not infer either $SA \subseteq A$ or $AS \subseteq A$.

Definition 2.4. *Let A be a subsemigroup of a semigroup S. If the proposition $A^m SA^n \subseteq A$ holds, for any two non-negative integers m and n, then A is called an (m, n)-ideal of S [9].*

3. m-Ideals and Prime m-Ideals

In this section, we first present the ideas m-ideals, then the ideas of prime m-ideals and their associated are presented.

3.1. m-Ideals

Definition 3.2. *Let S be a semigroup, a subsemigroup A of S is called an m-left(m-right) ideal of S if $S^m A \subseteq A(AS^m \subseteq A)$, for any positive integer m. In the case that A is both an m-left and m-right ideals of S, then A is called an m-ideal of S. For this situation, we have $S^m AS^m \subseteq A$.*

Each left/right ideal of S is 1-left/1-right ideal of S. More explicitly, every m-left/m-right ideal of S is n-left/n-right ideal of S for all $m \geq n$, however, the opposite does not follow. This follows from the foregoing results in the Preliminary Section 2, and is demonstrated in the following Example 3.3.

Intersection of m-left and m-right ideal of S is m-quasi ideal of S, as $AS^m \subseteq A$, and $S^m A \subseteq A$ infer $AS^m \cap S^m A \subseteq A$. For a more detailed investigation of m-bi ideals in semigroups, m-bi ideals in semirings and m-quasi ideals in semirings, the References [11], [15], and [12] can be respectively followed.

Example 3.3. *Let $S = \{1,2,3,4,5,6\}$ be the semigroup with the binary operation defined on its elements in the Table 1 [5]. It is evident that*

Table 1

$\cdot$	1	2	3	4	5	6
1	1	1	1	4	4	4
2	1	1	1	4	4	4
3	1	1	2	4	4	5
4	4	4	4	1	1	1
5	4	4	4	1	1	1
6	4	4	5	1	1	2

$S^2 = \{1,2,4,5\}$. The subset $A = \{1,2,3,4\}$ of S is a subsemigroup of S. In addition, $S^2 A = \{1,4\} \subseteq A$. That is, A is 2-left ideal of A. Similarly, A is 2-right ideal of A. Eventually, A is 2-ideal of S. Nonetheless, A is not a left ideal or a right ideal of S, on the reasons that

$$AS = \{1,2,3,4\}\{1,2,3,4,5,6\} = \{1,2,4,5\} \nsubseteq A.$$

Here, $AS = SA$. Consequently, A is not an ideal of S.

Theorem 3.1. *The product of an m-left ideal and n-left ideal of S is a* $\max(m,n)$*-left ideal of S, m and n are any two positive integers.*

Proof. Let L_1 be m-left and L_2 be n-left ideals of S. Consider

$$S^{\max(m,n)} L_1 L_2 \subseteq S^m L_1 L_2 \subseteq L_1 L_2,$$

so $L_1 L_2$ is $\max(m,n)$-left of S. □

Corollary 3.4. *The product of two m-left ideals of S is an m-left ideal of S.*

Proof. Straightforward. □

Also, we can express the following theorems.

Theorem 3.2. *The product of an m-right ideal and n-right ideal of S is a* $\max(m, n)$*-right ideal of S.*

Proof. Straightforward. □

Corollary 3.5. *The product of two m-right ideals of S is a m-right ideal of S.*

Proof. Straightforward. □

Theorem 3.3. *The product of an m-ideal and n-ideal of S is a* $\max(m, n)$*-ideal of S.*

Proof. Since the product is both m-left ideal and m-right ideal, so this product is $\max(m, n)$-ideal. □

Corollary 3.6. *The product of m-two-sided ideals of a semigroup S is an m-two-sided ideal of S.*

Proof. Straightforward. □

Remark 3.7. *Any finite collection of m-left(m-right, m-two-sided) ideals of a semigroup S, taken in any order, is also an m-left (m-right, m-two-sided) ideal of S. This result likewise holds for the distinct positive whole numbers.*

Theorem 3.4. *If L is an m-left ideal and R an m-right ideal of semigroup S, then LR is a m-ideal of S.*

Proof. Since $S^m LR \subseteq LR$, and $LRS^m \subseteq LR$, so LR is a m-ideal of S. □

Remark 3.8. *In the above case, the product RL needs neither be an m-left nor an m-right ideal of S.*

Theorem 3.5. *For some natural number n the intersection of any collection of m_1, m_2, $\cdots$, m_n -left(-right, -two-sided) ideals of a semigroup S is an s-left(-right, -two-sided) ideal of S, where $s = \max(m_1, m_2, ..., m_n)$.*

Proof. Clear. □

Remark 3.9. *The intersection of an m-left ideal L and an m-right ideal R, $L \cap R$ is not an m-ideal.*

Definition 3.10. *The principal m-left ideal generated by an element $a \in S$ is the m-left ideal $S^m a$. If the left identity $e \in S$, then $a = ea \in S^m a$. If S does not have a left identity, then we may have $a \notin S^m a$. For instance, the principal ideal generated by* 3 *in the multiplicative semigroup of multiples of* 3 *does not possess* 3.

The principal m-right ideal generated by a is characterized to be aS^m. If S possesses a right identity, then $a \in aS$; otherwise, a may or may not be in aS^m.

The principal two-sided m-ideal generated by a is characterized to be $S^m a S^m$, which has a if S has a two-sided identity.

3.11. Prime m-Ideals

In this part of the section, we characterize the prime m-left, m-right and m-ideals in semigroups.

Definition 3.12. *An m-left ideal P of a semigroup S is said to be prime m-left ideal if $AB \subseteq P$ infers that $A \subseteq P$ or $B \subseteq P$ for any two m-left ideals A, B of S.*

Similarly, we can define the prime m-right ideal and prime m ideals of S.

Definition 3.13. *An m-left(m-right, m-two-sided) ideal P of a semigroup S is said to be completely prime if $a, b \in S$ and $ab \in P$ infer that either $a \in P$ or $b \in P$.*

Along these lines, we can define the completely prime m-right ideal and m ideals of S.

Remark 3.14. *Completely primeness implies primeness; the opposite follows if the semigroup is commutative [17].*

Theorem 3.6. *The union of an arbitrary collection of completely prime m-right ideals of a semigroup S is a completely prime m-right ideal of S.*

Proof. Let $\{P_i : i \in I\}$ be a collection of completely prime m-right ideals of S, and let $a, b \in S$ and $ab \in \underset{i\in I}{\cup} P_i$. This gives $ab \in P_i$, for some $i \in I$. But, P_i is completely prime, so $a \in P_i$ or $b \in P_i$, for some $i \in I$. Therefore, either $a \in \underset{i\in I}{\cup} P_i$ or $b \in \underset{i\in I}{\cup} P_i$. This brings that $\underset{i\in I}{\cup} P_i$ is completely prime. In a similar way, we can prove this theorem for m-left and m-two-sided completely prime ideals. □

The union of two or more ideals may be completely prime, despite the fact that none of them is completely prime.

The following example shows that the product of two or more completely prime m-ideals may not be a completely prime m-ideals.

Example 3.15. *Consider the set $S = \{1, 2, 3, 4, 5\}$ together with the multiplication defined on its elements as is given in Table 2, [5]. $S^2 = \{1, 2, 4, 5\}$. 2-right ideals $\{1, 3, 4, 5\}$ and $\{4, 5\}$ are completely prime, while their product $\{1, 4, 5\}$ is not completely prime.*

In the following example, the product of completely prime m-ideals may turn out to be completely prime.

Example 3.16. *Consider the set $S = \{1, 2, 3, 4, 5\}$, with multiplication defined on its elements as depicted in Table 3, [5]. Then, $S^2 = \{1, 2, 3, 4\}$. The product $\{3, 4\}$ of the completely prime 2-right ideals $\{2, 3, 4\}$ and $\{4\}$ is also completely prime 2-right ideals.*

·	1	2	3	4	5
1	1	1	1	1	5
2	2	2	2	2	5
3	1	1	1	1	5
4	4	4	4	4	5
5	5	5	5	5	5

Table 2

·	1	2	3	4	5
1	1	1	3	3	3
2	2	2	3	3	3
3	3	3	3	3	3
4	4	4	4	4	4
5	4	4	4	4	4

Table 3

Generally, the intersection of completely prime m-right/m-left ideals or m-ideals of a semigroup S is not completely prime. See the example given below.

Example 3.17. *Let $S = \{1, 2, 3, 4, 5\}$ alongwith Table 4 as given in Reference [5]. The intersection of completely prime 2-right ideals $\{1, 2\}$ and $\{1, 3\}$, namely, $\{1\}$ is not completely prime 2-right ideal of S because $3.2 = 1 \in \{1\}$, but neither 2 nor 3 is in $\{1\}$.*

·	1	2	3	4	5
1	1	1	1	1	5
2	2	2	2	2	5
3	1	1	3	3	5
4	2	2	4	4	5
5	5	5	5	5	5

Table 4

·	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	2	2	2
3	0	2	2	2

Table 5

In the proceeding part, we build up the theory of strongly prime m-ideals and semiprime m-ideals and their relationship with the prime m-ideals.

Definition 3.18. *A m-ideal P of a semigroup S is known as a strongly prime m-ideal if the proposition $P_1P_2 \cap P_2P_1 \subseteq P$ infers either $P_1 \subseteq P$ or $P_2 \subseteq P$ for any two m-ideals P_1 and P_2 of S.*

Definition 3.19. *A m-ideal P of a semigroup S is known as semiprime m-ideal if the statement $P_1^2 \subseteq P$ brings $P_1 \subseteq P$ for any m-ideal P_1 of S.*

The property of strongly primeness implies primeness; however, the opposite is not true. Primeness implies semiprimeness, the converse does not hold.

Example 3.20. *The semigroup S itself is always a totally prime, strongly prime, prime, a semiprime m-ideal of S.*

In addition to the semigroup S itself, S can have other these kinds of ideals. This is demonstrated in the following example.

Example 3.21. *Consider the semigroup $S = \{0, 1, 2, 3\}$ with the binary operation $\cdot$ given in the Table 5. Taking $m = 2$, we get $S^2 = \{0, 1, 2\}$. We see that*

1) The semigroup S being the 2-left ideal and 2-right ideal is 2-ideal. This ideal is completely prime, strongly prime, prime and semiprime 2-ideal of S.

2) The set $\{0\}$ being the 2-left and 2-right ideal is additionally 2-ideal. This ideal is compeltely prime, prime and semiprime 2-ideal of S. However, $\{0\}$ is not strongly prime 2-right ideal(off course 2-ideal) of S on the grounds that $\{0,1\}\{0,2\}\cap\{0,2\}\{0,1\} = \{0,1\}\cap\{0,2\} = \{0\}$, however none of $\{0,1\}$ and $\{0,2\}$ is contained in $\{0\}$.

3) $\{0,1\}$ is 2-right ideal as $\{0,1\}S^2 = \{0,1\}\{0,1,2\} = \{0,1\} \subseteq \{0,1\}$ $\Rightarrow \{0,1\}S^2 \subseteq \{0,1\}$. $\{0,1\}$ is not 2-left ideal of S as $S^2\{0,1\}$ $= \{0,1,2\}\{0,1\} = \{0,1,2\} \nsubseteq \{0,1\} \Rightarrow S^2\{0,1\} \nsubseteq \{0,1\}$. $\{0,1\}$ is totally prime, prime, semiprime and strongly prime 2-right ideal.

4) $\{0,2\}$ is 2-right ideal as $\{0,2\}S^2 = \{0,2\}\{0,1,2\} = \{0,2\} \subseteq \{0,2\}$ $\Rightarrow \{0,2\}S^2 \subseteq \{0,2\}$. $\{0,2\}$ is not 2-left ideal of S as $S^2\{0,2\}$ $= \{0,1,2\}\{0,2\} = \{0,1,2\} \nsubseteq \{0,2\} \Rightarrow S^2\{0,2\} \nsubseteq \{0,2\}$. $\{0,2\}$ is totally prime, prime, semiprime and strongly prime 2-right ideal of S. This is to be noticed that $\{0,2\}$ is not totally prime right ideal of S in light of the fact that $3 \times 3 = 2$, and $3 \notin \{0,2\}$.

5) $\{0,1,2\}$ is 2-ideal as $S^2\{0,1,2\}S^2 = \{0,1,2\} \Rightarrow S^2\{0,1,2\}S^2 \subseteq \{0,1,2\}$. $\{0,1,2\}$ is totally prime, prime, semiprime and strongly prime 2-right ideal of S; yet not a totally prime right ideal in light of the fact that $3 \times 3 = 2$, and $3 \notin \{0,1,2\}$. More remarks on the ideal viz., $\{0,1,2\}$ will be given in Section 3, Remark 4.3.

Example 3.22. *Consider a semigroups S having at least two elements with the property that $yx = x \ \forall\ x, y \in S$; called right zero semigroup. Since for an arbitrary element $x \in S$, $xx = x$. So, $S^2 = S$. Subsequently, $S^m = S$ for any whole number $m \geq 1$. Let $P \subseteq S$, then $S^mP = SP = P$. This* *implies that P is m-left ideal of S. Thus, each subset of S is an m-left ideal of S.*

In this semigroup, each m-left becomes a prime m-left ideal; off course a semiprime m-left ideal. This is on the grounds that for m-ideals M_1, M_2, we have $M_1M_2 = M_2$. Additionally, if M is any m-left ideal of S with the condition that $|S - M| \geq 2$, then for any two different elements $a, b \in S-M$, $(M\cup\{a\})(M\cup\{b\})\cap(M\cup\{b\})(M\cup\{a\}) = (M\cup\{a\})\cap(M\cup\{b\}) = M$, but neither $(M \cup \{a\})$ nor $(B \cup \{b\})$ is a subset of M. This implies M is not strongly prime m-left ideal.

Remarks 3.23. *1) An arbitrary subset of a left zero semigroup, having at least two elements, is its m-right ideal. Every m-right is a prime(semiprime) m-right ideal, but not a strongly prime m-right ideal.*

2) A subset of a zero semigroup, with at least two elements, is its m-ideal. Every m-ideals is a prime as well as semiprime m-ideal, but not a strongly prime m-ideal.

Example 3.24. *We consider the Kronecker delta semigroup,S, defined by following relation:*

$$xy = \begin{cases} x & if \quad x = y, \\ 0 & otherwise. \end{cases}$$

Additionally, S is assumed to possess atleast three elements including zero. Since $xx = x, \quad \forall \quad x \in S$, $S^m = S$. Let R be any right ideal of $S(m = 1)$, then $S^mR = SR = R$. This makes R an m-right ideal of S for all m. Additionally, if $R_1^2 \subseteq R$, then since $R_1^2 = R_1$ for any right ideals R_1 and R of S, so $R_1 \subseteq R$. This infers that all right ideals of S are semiprime m-right ideals of S. If P is an m-right ideal of S with the condition that $|S - P| > 2$, then P is not a prime m-ideal of S because for any two distinct elements $a, b \in S - P$, $(P \cup \{a\})(P \cup \{b\}) = (P \cup \{a\}) \cap (P \cup \{b\}) = P$, however, neither $(P \cup \{a\})$ nor $(P \cup \{b\})$ is contained in P. This result shows that every semiprime m-right ideal is not prime. In particular, $\{0\}$ is a semiprime m-ideal of S which is not a prime m-ideal.

4. Maximal m-Ideals

Alongside prime, strongly prime and semiprime ideals, maximal ideals are vital to be considered for characterizing semigroups in a benefitting way. The accompanying lines present the ideas of the maximal m-ideals in semigroups.

Definition 4.1. *An m-ideal M of a semigroup S, different from S^m, is said to be a maximal m-ideal in S if there does not exist another m-ideal, M_1 in S, such that $M \subset M_1 \subset S^m$ [19].*

Similarly, we can interpret the maximality idea for the m-left and m-right ideals of S.

Definition 4.2. *An m-ideal $K \neq \{0\}$(If $0 \in S$) of a semigroup S is termed as a minimal m-ideal of S if $\nexists$ any other proper m-ideal, K_1 in S such that $K_1 \subset K \subset S$ [19].*

Analogously, the minimality idea can be extended for the m-left and m-right ideals. Recalling Example 3.21 from Section 3, we see that $M = \{0, 1\}$ and $N = \{0, 2\}$ are the maximal m-right ideals of $S = \{0, 1, 2, 3\}$. The m-right ideals $K = \{0, 1\}$ and $J = \{0, 2\}$ are the minimal m-right ideals of S.

Theorem 4.1. *If $S^m = (S^m)^2$ for a semigroup S, then every maximal m-ideal M of S is a prime m-ideal of S, where m is a positive integer.*

Proof. Let M be a maximal m-ideal of semigroup S. To show that M is prime, let $P_1P_2 \subset M$ for any two m-ideals P_1 and P_2 of S. Suppose on contradiction that neither P_1 nor P_2 is contained in M. Since $P_1 \nsubseteq M$ and M is maximal, we have $P_1 \cup M = S^m$, subsequently $P \subset P_1$, where $P = S^m - M$ is the complement of M with respect to S^m. Comparably, we get $P \subset P_2$. Along these lines, we get

$$P^2 \subset P_1P_2 \tag{4.1}$$

Since $S^m = (S^m)^2$, $S^m = (M \cup P)^2 = M^2 \cup MP \cup PM \cup P^2 \subset M \cup P^2$. That is, $S^m \subset M \cup P^2$. This gives $S^m \cap P \subset (M \cup P^2) \cap P = (M \cap P) \cup (P^2 \cap P) = (M \cap (S^m - M)) \cup (P^2 \cap P) = \emptyset \cup (P^2 \cap P) = (P^2 \cap P)$, consequently, we get

$$P \subset P^2 \tag{4.2}$$

From (4.1) and (4.2), using transitive property of *set inclusion*, we get $P \subset P_1P_2$, which means $S^m - M \subset P_1P_2 \subset M$, that is $S^m - M \subset M$, a contradiction. This completes the proof of the theorem. □

Remarks 4.3. *1) If $S = S^2$, then $S^m = S^{2m}$, but the converse does not follow.*

2) If $S \neq S^2$, then Theorem 4.1 does not follow. This is clear from Example 3.21 that the maximal 2-right ideal $\{0, 2\}$ is not prime on the basis that $\{0, 2, 3\}\{0, 2, 3\} \subseteq \{0, 2\}$, however, $\{0, 2, 3\} \nsubseteq \{0, 2\}$, so $\{0, 2\}$ is not prime.

The following theorems deal with the sets of maximal m-ideals, their intersections and their complement sets in the semigroups.

Theorem 4.2. *[19]. Let $\{M_\alpha : \alpha \in \Omega\}$ be the family of various maximal m-ideals of S. Assume $|\Omega| \geq 2$ and indicate $P_\alpha = S^m - M_\alpha$ and $M = \bigcap\limits_{\alpha\in\Omega} M_\alpha$, we have,*

1) $P_\alpha \cap P_\beta = \emptyset$ for $\alpha \neq \beta$.

2) $S^m = (\bigcup\limits_{\alpha\in\Omega} P_\alpha) \cup M$.

3) For each $\nu \neq \alpha$, we have $P_\alpha \subset M_\nu$.

4) If J is an m-ideal of S and $J \cap P_\alpha \neq \emptyset$, then $P_\alpha \subset J$.

5) For $\alpha \neq \beta$, we have

$$P_\alpha^m P_\beta P_\alpha^m \subset M,$$

that is M is not empty.

Proof. The case $|\Omega| = 1$ is obvious.

1) We have $M_\alpha \cup M_\beta = S^m$ for $\alpha \neq \beta$. In this way, $P_\alpha \cap P_\beta = (S^m - M_\alpha) \cap (S^m - M_\beta) = S^m - (M_\alpha \cup M_\beta) = \emptyset$.

2) Since $M = \bigcap\limits_{\alpha\in\Omega} M_\alpha = \bigcap\limits_{\alpha\in\Omega} (S^m - P_\alpha) = S^m - \bigcup\limits_{\alpha\in\Omega} P_\alpha$. In this way, $S^m = (\bigcup\limits_{\alpha\in\Omega} P_\alpha) \cup M$.

3) For $\nu \neq \alpha$, we have $P_\alpha = S^m \cap P_\alpha = (M_\nu \cup P_\nu) \cap P_\alpha = M_\nu \cap P_\alpha$. In this way, $P_\alpha \subset M_\nu$.

4) Since $J \cap P_\alpha \neq \emptyset$ and J is a m-ideal of S, whereas M_α is the maximal m-ideal, therefore the set $M_\alpha \cup J$ is an m-ideal of S greater than M_α. Thus, $M_\alpha \cup J = S^m$. Since $M_\alpha \cap P_\alpha = \emptyset$, we have $P_\alpha \cap M_\alpha \cup J = P_\alpha \cap S^m$, i.e., $P_\alpha \cap (M_\alpha \cup J) = P_\alpha \cap S^m$, which gives that $(P_\alpha \cap M_\alpha) \cup (P_\alpha \cap J) = P_\alpha$, and, $\emptyset \cup (P_\alpha \cap J) = P_\alpha$, i.e., $(P_\alpha \cap J) = P_\alpha$, which gives that $P_\alpha \subset J$.

5) Suppose on contradiction that $\exists\ u_\alpha, u_\delta \in P_\alpha$ and $u_\beta \in P_\beta$ such that $u_\alpha u_\beta u_\delta = u_\gamma$ and $u_\gamma \notin M$. Utilizing Part(2), we can discover P_γ such that $u_\gamma \in P_\gamma$. On the other hand, $P_\gamma \neq P_\alpha$. Then, $P_\alpha \subset S - P_\gamma = M_\gamma$. That is, $P_\alpha \subset S_\gamma$ and correspondingly, $P_\delta \subset S_\gamma$. This gives, $P_\alpha^m P_\beta P_\alpha^m \subset S^m M_\gamma S^m \subset M_\gamma S^m \subset M_\gamma$, thus, $u_\gamma \in S_\gamma$, which is a contradiction to $u_\gamma \in P_\gamma = M \backslash S_\gamma$. Assume now, $P_\gamma = P_\beta$. Then, $P_\beta \subset S - P_\gamma = M_\gamma$ and $P_\alpha^m P_\beta P_\alpha^m \subset S^m M_\alpha S^m \subset M_\alpha$, subsequently $u_\gamma \in M_\alpha = S - P_\alpha$, which is a contradiction to $u_\gamma \in P_\gamma$. In this way, $P_\alpha^m P_\beta P_\alpha^m \subset M$, and $M \neq \emptyset$.

□

Theorem 4.3. *Let S be a semigroup containing maximal m-ideals and let M be the intersection of all maximal m-ideals of S. Then, each prime m-ideal of S containing M and different from S^m is a maximal m-ideal of S.*

Proof. Let N be a prime m-ideal of S containing M and $N \neq S^m$. Then, Theorem 4.2: Part(4), $N = S^m - (\underset{\nu\in\Omega}{\cup} P_\nu) = \underset{\nu\in\Omega}{\cap}(S^m - P_\nu) = \underset{\nu\in\Omega}{\cap} M_\nu$, where $\Omega \neq \emptyset$. If $|\Omega| = 1$, we have $N = M_\nu$, for example, N is a maximal m-ideal of S and the theorem is proved. We will show that $|\Omega| \geq 2$ is not possible. Assume on opposite that $|\Omega| \geq 2$. Let $\beta \in \Omega$ and indicate $H = \underset{\nu\in\Omega,\nu\neq\beta}{\cup} P_\nu$. Then, we have $N = H \cap M_\beta$. Since both H and M_β are m-ideals, their product is also m-ideal, thus $HM_\beta \subset H \cap M_\beta = N$. Since N is prime m-ideal, so either $H \subset N$ or $M_\beta \subset N$. We talk about these two prospects independently:

1) When $H \subset N$. Since $N \subset H$ as well, so $N = H$. Further $H = N = H \cap M_\beta$ suggests $H \subseteq M_\beta$, by Theorem 4.2: Part(3), we have $P_\beta \subseteq \underset{\nu\in\Omega,\nu\neq\beta}{\cup} P_\nu = H$. Consequently, $P_\beta \subset S_\beta$, a contradiction to $P_\beta \cap M_\beta = \emptyset$.

2) When $M_\beta \subset N$. Since additionally, $N \subset M_\beta$, so $N = M_\beta$. Now, $N = M_\beta = H \cap M_\beta$ would infer $M_\beta \subset H$. Since M_β is maximal and H is an proper subset of S, so $H = M_\beta$. The relation $P_\beta \subset H = M_\beta$ gives another contradiction.

These two cases complete the proof of the theorem. □

Theorem 4.4. *S is a semigroup containing at least one maximal m-ideal. A prime m-ideal N different from S^m is a maximal m-ideal of S $\Longleftrightarrow$ $M \subset N$, where $M = \underset{\alpha\in\Omega}{\cap} M_\alpha$, M_α are maximal m-ideals of S.*

Proof. If N is a maximal m-ideal, then $M \subset N$. On the other hand, if $M \subset N$, then by Theorem 4.3, N is a maximal m-ideal of S. □

5. Irreducible and Strongly Irreducible m-ideals

Definition 5.1. *A m-ideal I of a semigroup S is known as an irreducible (strongly irreducible) m-ideal if the proposition $I_1 \cap I_2 = I$ ($I_1 \cap I_2 \subseteq I$) infers either $I_1 = I$ or $I_2 = I$ (either $I_1 \subseteq I$ or $I_2 \subseteq I$), for m-ideals I_1 and I_2 of S.*

A strongly irreducible m-ideal is irreducible; however, the opposite does not hold. See the accompanying example.

Example 5.2. *We have $S = \{1, 2, 3, 4, 5, 6, 7\}$; a semigroup with the binary operation $\cdot$ given in Table 6.*

Table 6

$\cdot$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	1
3	1	2	3	4	2	2	1
4	1	2	2	2	3	4	1
5	1	2	5	6	2	2	1
6	1	2	2	2	5	6	1
7	1	1	1	1	1	1	1

If we take $m = 2$, $S^2 = \{1, 2, 3, 4, 5, 6\}$. We observe that

1) $\{1\}$ *is 2-right and 2-left ideal of S, so is 2-ideal of S. $\{1\}$is both irreducible and strongly irreducible.*
2) $\{1, 2\}$ *being 2-right and 2-left ideal of S is 2-ideal of S.*
3) $\{1, 2, 3, 4\}$ *is a 2-right ideal, but not 2-left ideal. $\{1, 2, 3, 4\}$ is an irreducible 2-right ideal.*
4) $\{1, 2, 5, 6\}$ *is a 2-right ideal, but not 2-left ideal. $\{1, 2, 5, 6\}$ is an irreducible 2-right ideal.*
5) *S is 2-right ideal and 2-left ideal, so is 2-ideal of S. S is irreducible.*

The condition when a semiprime m-ideal is a prime m-ideal in a semigroup is elaborated in the following proposition.

Proposition 5.3. *A strongly irreducible semiprime m-ideal of a semigroup is a strongly prime m-ideal.*

Proof. Let P be an irreducible semiprime m-ideal of semigroup S. If P_1 and P_2 are two m-ideals of S with the additional assumption that

$$P_1P_2 \cap P_2P_1 \subseteq P. \tag{5.1}$$

Then, since $P_1 \cap P_2$ being intersection of m-ideals is an m-ideal, so after simplification, we get,

$$(P_1 \cap P_2)^2 \subseteq P_1P_2 \cap P_2P_1. \tag{5.2}$$

Consolidating (5.1) and (5.2) through the application of transitive property of inclusion of sets, we have, $(P_1 \cap P_2)^2 \subseteq P$. This implies $P_1 \cap P_2 \subseteq P$, as P is a semiprime. Additionally, since P is strongly irreducible m-ideal of S, so $P_1 \subseteq P$ or $P_2 \subseteq P$; resulting P into a strongly prime m-ideal of S. □

Proposition 5.4. *For any m-ideal P of a semigroup S, such that $c \in S$ and $c \notin P$, $\exists$ an irreducible m- ideal I such that $P \subseteq I$ and $c \notin I$.*

Proof. Take $\mathcal{P} = \{P : \; P$ is an m- ideal of S so that $c \in S$ and $c \notin P\}$. Then $\mathcal{P} \neq \emptyset$, because $P \in \mathcal{P}$. $\mathcal{P}$ is clearly a partially ordered set under the binary operation of *inclusion* of m-ideals in $\mathcal{P}$. If $\mathcal{S}$ is any totally ordered subset of $\mathcal{P}$, then $T = \underset{T_\alpha \in \mathcal{S}, \alpha \in \wedge}{\cup} S_\alpha$ is an m-ideal of S containing P. So we can find a maximal m- ideal, J, in $\mathcal{P}$. To show that J is irreducible, we suppose $J = J_1 \cap J_2$ for two m- ideals J_1 and J_2 of S. If, on contrary, both J_1 and J_2 contain J properly, then $c \in J_1$ and $c \in J_2$. Hence $c \in J_1 \cap J_2 = J$; which contradicts the hypothesis that $c \notin J$. Thus $J = J_1$ or $J = J_2$; implying that J is an irreducible m-ideal. □

Theorem 5.1. *The following propositions are equivalent [20]:*

1) *The set $\mathcal{R}$ of all m-right ideals of a semigroup S is totally ordered under the inclusion of sets,*
2) *Every m-right ideal of S is strongly irreducible m-right ideal,*
3) *Every m-right ideal of S is irreducible m-right ideal.*

Proof. $(1 \Rightarrow 2)$: Let R is an m-ideal of S, then for any two m-right ideals R_1, R_2 of S, $R_1 \cap R_2 \subseteq R$ follows. Since $\mathcal{R}$ is *totally ordered* under *set inclusion*, either $R_1 \subseteq R_2$ or $R_2 \subseteq R_1$. This gives, either $R_1 \cap R_2 = R_1$ or $R_1 \cap R_2 = P_2$. Eventually from the hypothesis, $R_1 \cap R_2 \subseteq R$, we infer either $R_1 \subseteq R$ or $R_2 \subseteq R$; making R a strongly irreducible m-right ideal of S.

$(2 \Rightarrow 3)$: This follows immediately from the fact that the strongly irreducible m-right ideals of S are its irreducible m-right ideal.

$(3 \Rightarrow 1)$: Assume that $R_1 \cap R_2 = R_1 \cap R_2$ holds for two m-right ideals R_1 and R_2 of S. Since each m-right ideal of S is irreducible m-right ideal, $R_1 = R_1 \cap R_2$ or $R_2 = R_1 \cap R_2$, which further implies $R_1 \subseteq R_2$ or $R_2 \subseteq R_1$. Therefore, R_1 and R_2 are comparable. That is, $\mathcal{R}$ is totally ordered under inclusion of sets. □

6. Conclusion

The idea of m-ideal in the semigroups was introduced. The kinds of the prime, completely prime, semiprime and strongly prime m-ideals were introduced for the classification of these ideals. It is hoped that studies of larger algebraic structures can also be carried out more fruitfully by characterizing them through m-ideals. The applications of the m-ideals in the finite and the infinite semigroups theory is quite obvious from the foregoing examples in the text.

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Изучение полугрупп с помощью свойств простых m -идеалов

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Аннотация. Вводятся понятия m -идеала, простого m -идеала и связанных с ними понятий для положительного целого числа m в полугруппе. Рассматриваются различные характеристики полугрупп через m -идеалы. Демонстрируется, что классическое понятие идеала и связанные с ним понятия отличаются от понятия m

-идеала и связанных с ним понятий на конкретных примерах максимальных, неприводимых и сильно неприводимых m-идеалов. Делается вывод, что введение понятия m -идеала откроет новые области исследований полугрупп и их приложений.

Ключевые слова: вполне простые m -идеалы, строго простые m -идеалы, максимальные m -идеалы, неприводимые m -идеалы, сильно неприводимые m -идеалы.

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