Серия «Математика»
2020. Т. 32. С. 49—63

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

УДК 519.145
MSC 51E15, 15A04
DOI https://doi.org/10.26516/1997-7670.2020.32.49

Elementary Abelian 2-subgroups in an Autotopism Group of a Semifield Projective Plane *

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Abstract. We investigate the hypotheses on a solvability of the full collineation group for non-Desarguesian semifield projective plane of a finite order (the question 11.76 in Kourovka notebook). It is well-known that this hypotheses is reduced to the solvability of an autotopism group. We study the subgroups of even order in an autotopism group using the method of a spread set over a prime subfield. It is proved that, for an elementary abelian 2-subgroups in an autotopism group, we can choose the base of a linear space such that the matrix representation of the generating elements is convenient and uniform for odd and even order; it does not depend on the space dimension. As a corollary, we show the correlation between the order of a semifield plane and the order of an elementary abelian autotopism 2-subgroup. We obtain the infinite series of the semifield planes of odd order which admit no autotopism subgroup isomorphic to the Suzuki group $Sz(2^{2n+1})$. For the even order, we obtain the condition for the nucleus of a subplane which is fixed pointwise by the involutory autotopism. If we can choose such the nucleus as a basic field, then the linear autotopism group contains no subgroup isomorphic to the

* The author was supported by the Russian Foundation for Basic Research (Grant No. 19-01-00566 A.).

alternating group A_4. The main results can be used as technical for the further studies of the subgroups of even order in an autotopism group for a finite non-Desarguesian semifield plane. The obtained results are consistent with the examples of 3-primitive semifield planes of order 81, and also with two well-known non-isomorphic semifield planes of order 16.

Keywords: semifield plane, spread set, Baer involution, homology, autotopism.

1. Introduction

The coordinatization of points and lines in a finite projective plane enable us to study the geometric properties of a plane connected with the algebraic properties of its coordinatizing set. It is well-known that classical or Desarguesian projective plane is coordinatized by a field and a translation plane by a quasifield. A semifield ("quasitelo", according A.G. Kurosh) coordinatizes a translation plane with the property that its dual plane is also translation plane. Such a plane is called a semifield plane; it admits large groups of central collineations (or automorphisms).

There is a conjecture ([3], p. 178) concerning the solvability of a full collineation group of any non-Desarguesian semifield plane of a finite order (see also [11], Question 11.76, 1990). Presently this conjecture is confirmed for some classes of semifield planes, but there is no general approach to solve the problem. According [3] and Feit–Thompson theorem on a solvability of any group of odd order, it is necessary to study subgroups of even order in an autotopism group (collineations fixing a triangle).

In [5; 7] the author has shown that an involutory autotopism admits a matrix representation which is convenient for reasoning and calculations. The present paper develops an approach using linear spaces and spread sets to study autotopism subgroups of even order. The matrix representation is constructed for elementary abelian 2-subgroup of autotopisms which is generated by Baer involutions. This representation is uniform for semifield planes of even or odd order. The connection is determined for an order of a plane and 2-rank of an autotopism group. The main result is as follows.

Theorem 1. *Let π be a non-Desarguesian semifield plane of order p^N (p be prime) which autotopism group contains an elementary abelian 2-subgroup H of order 2^m and all its involutions are Baer,*

$$H = \langle\tau_1\rangle \times \langle\tau_2\rangle \times \cdots \times \langle\tau_m\rangle;$$

here τ_i are Baer involutions fixing pointwise different Baer subplanes π_i ($i = 1, 2, \ldots, m$). Then 2^m is a factor of N and a base of $2N$-dimensional linear space over $\mathbb{Z}_p$ can be chosen such that τ_i is determined by a block-diagonal *matrix with* 2^i *blocks of dimension* $(N/2^{i-1}) \times (N/2^{i-1})$ *as follows*

$$L = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix} \quad \text{if} \quad p > 2, \qquad L = \begin{pmatrix} E & E \\ 0 & E \end{pmatrix} \quad \text{if} \quad p = 2. \tag{1.1}$$

Note that blocks-submatrices by default have the same dimension everywhere in the paper. So, in (1.1) any block is of dimension $(N/2^i) \times (N/2^i)$.

Under additional conditions, the results are obtained on autotopism subgroup $H \simeq Sz(2^{2n+1})$ if $p > 2$ and $H \simeq A_4$ if $p = 2$ (Corollary 1 and Lemma 4).

2. Main definitions and preliminary discussion

State the main definitions, according to [3; 12].

A *semifield* is a set S with two binary algebraic operations $+$ and $*$ such that:

1) $\langle S, +\rangle$ is an abelian group with neutral element 0;

2) $\langle S \setminus \{0\}, *\rangle$ is a loop;

3) both distributivity laws hold, $a * (b + c) = a * b + a * c$, $(b + c) * a = b * a + c * a$ for any $a, b, c \in S$.

The weakening of two-sided distributivity to one-sided leads to a quasifield, left or right.

A semifield S contains the subsets N_r, N_m, N_l which are called *right, middle and left nuclei* respectively:

$$\begin{aligned} N_r &= \{n \in S \mid (a * b) * n = a * (b * n) \ \forall a, b \in S\}, \\ N_m &= \{n \in S \mid (a * n) * b = a * (n * b) \ \forall a, b \in S\}, \\ N_l &= \{n \in S \mid (n * a) * b = n * (a * b) \ \forall a, b \in S\}. \end{aligned} \tag{2.1}$$

Its intersection $N_0 = N_l \cap N_m \cap N_r$ is called the *nucleus* of a semifield, the set

$$Z = \{z \in N_0 \mid z * a = a * z \ \forall a \in S\}$$

is the *center* of a semifield. The nuclei and the center of a finite semifield are subfields; a semifield is a linear space over these subfields.

Consider a linear space W, n-dimensional over the finite field $GF(p^s)$, and a subset of linear transformations $R \subset GL_n(p^s) \cup \{0\}$ such that:

1) R consists of p^{ns} square $(n \times n)$-matrices over $GF(p^s)$;

2) R contains the zero matrix 0 and the identity matrix E;

3) for any $A, B \in R$, $A \neq B$, the difference $A - B$ is a nonsingular matrix.

The set R is called a *spread set* [3]. Consider a bijective mapping θ from W onto R and present the spread set as $R = \{\theta(y) \mid y \in W\}$. Determine the multiplication on W by the rule $x * y = x \cdot \theta(y)$ $(x, y \in W)$. Then $\langle W, +, *\rangle$ is a right quasifield of order p^{ns} [10; 12]. Moreover, if R is closed under addition then $\langle W, +, *\rangle$ is a semifield.

To construct and study finite semifields, the center Z is used usually as a basic field. But it may be more convenient to consider a linear space W and a spread set over a prime field $\mathbb{Z}_p$. In this case the mapping θ is presented using only linear functions; it greatly simplifies reasoning and calculations (also computer).

Further we consider a semifield W as n-dimensional linear space over $\mathbb{Z}_p$. Determine now a projective plane π of order p^n, using W:

1) the affine points of the plane π are the elements (x, y) $(x, y \in W)$ of the space $W \oplus W$;

2) the affine lines are the cosets to subgroups

$$V(\infty) = \{(0, y) \mid y \in W\}, \qquad V(m) = \{(x, x\theta(m)) \mid x \in W\} \quad (m \in W);$$

3) the set of all cosets to the subgroup $V(m)$ or $V(\infty)$ is the singular (or infinity) point (m) or (∞), respectively;

4) the set of all singular points is the singular (or infinity) line $[\infty]$;

5) the incidence is set-theoretical.

Such the projective plane π is a semifield plane, its full collineation group is $Aut\,\pi = T \rtimes G$. Here $T = \{\tau_{a,b} \mid a, b \in W\}$ is a translations group,

$$\tau_{a,b} : (x, y) \to (x + a, y + b), \quad x, y \in W,$$

G is a translation complement, it is a stabilizer of the point $(0, 0)$. The automorphisms from G are presented by linear transformations of $W \oplus W$:

$$\alpha : (x, y) \to (x, y) \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are $(n \times n)$-matrices over $\mathbb{Z}_p$. Note that the representation of G using linear transformations only is possible because we consider a spread set over a prime field. In other cases the collineations from G are represented by semilinear transformations.

The subgroup $\Lambda < G$ of collineations fixing a triangle with sides $[\infty]$, l_2, l_3 and vertices (∞), $(0, 0) \in l_2, l_3$, $P_3 \in [\infty]$, is called an *autotopism group*. Without loss of generality, we can assume that the linear autotopisms are determined by block-diagonal matrices,

$$\lambda : (x, y) \to (x, y) \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

here the matrices A and D satisfy the condition (for instance, see [6])

$$A^{-1}\theta(m)D \in R \quad \forall \theta(m) \in R. \tag{2.2}$$

The collineations fixing a closed configuration have special properties. It is well-known [3] that any involutory collineation is a central collineation or a Baer collineation.

A collineation of a projective plane is called *central* if it fixes a line pointwise (*the axis*) and a point linewise (*the center*). If the center is incident to the axis then a collineation is called *an elation*, and *a homology* in another case. The order of any elation is a factor of the order $|\pi|$ of a projective plane, and the order of any homology is a factor of $|\pi| - 1$.

The translation complement G of a non-Desarguesian semifield plane is $G = \Omega \rtimes \Lambda$, where Ω is the group of elations with the axis $[0]$ and the center (∞) (so-called *shears*). It is an elementary abelian group of order $|\pi|$. So, the investigation of a full collineation group of a semifield plane means a description of its autotopism group [3].

Any of nuclei (2.1) corresponds to the set of matrices [14]

$$\begin{aligned} R_l &= \{M \in GL_n(p) \cup \{0\} \mid MT = TM \ \forall T \in R\}, \\ R_m &= \{M \in R \mid MT \in R \ \forall T \in R\}, \\ R_r &= \{M \in R \mid TM \in R \ \forall T \in R\}; \end{aligned}$$

these sets are the subfields in $GL_n(p) \cup \{0\}$. Its intersection R_0 is called naturally the *nucleus of a plane*. The central collineations forms the cyclic subgroups in an autotopism group [14]:

1) $H_r \simeq N_r^* \simeq R_r^*$ is the group of homologies with the axis $[0, 0]$ and the center (∞);

2) $H_l \simeq N_l^* \simeq R_l^*$ is the group of homologies with the axis $[\infty]$ and the center $(0, 0)$;

3) $H_m \simeq N_m^* \simeq R_m^*$ is the group of homologies with the axis $[0]$ and the center (0).

The matrix representation of these groups is as follows:

$$H_r = \left\{ \begin{pmatrix} E & 0 \\ 0 & M \end{pmatrix} \,\middle|\, M \in R_r^* \right\}, \quad H_m = \left\{ \begin{pmatrix} M & 0 \\ 0 & E \end{pmatrix} \,\middle|\, M \in R_m^* \right\},$$

$$H_l = \left\{ \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \,\middle|\, M \in R_l^* \right\}.$$

A collineation of a projective plane π of order m is called *Baer collineation* if it fixes pointwise a subplane of order $\sqrt{|\pi|} = \sqrt{m}$ (*Baer subplane*).

Studying the autotopism subgroups of even order, we will focus on the subgroups generated by commuting involutions to obtain the important technical results. Naturally, we will distinguish cases of semifield planes of odd or even order due to the geometric meaning of the involutions.

Let π be a semifield plane of odd order. Then its autotopism group Λ always contains the normal elementary abelian subgroup H_0 of order 4 generated by homologies,

$$H_0 = \left\{ \varepsilon, h_1 = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}, h_2 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, h_3 = \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix} \right\}. \quad (2.3)$$

Each of these homologies is the unique involution in the cyclic group which is isomorphic to the multiplicative group of the nucleus (middle, right, left) of a coordinatizing semifield. Evidently, these homologies are not conjugated in Λ.

The case when the factor-group Λ/H_0 is of odd order is not of interest from the point of view of the solvability problem. So we assume further that Λ contains a Baer involution τ. If $|\pi| = 2^N$ then the autotopism group Λ does not contain the central collineations; for $p = 2$ we assume $\tau \in \Lambda$ too. We use the following results on the matrix representation of a Baer involution τ and a spread set of π which were obtained earlier in [5; 7].

Let π be a non-Desarguesian semifield plane of order p^N (p be prime). If its autotopism group Λ contains a Baer involution τ then $N = 2n$ is even and the base of $4n$-dimensional linear space over $\mathbb{Z}_p$ can be chosen such that

$$\tau = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \quad (2.4)$$

where the matrix $L \in GL_{2n}(p)$ is (1.1). The set of points of the Baer subplane π_τ which is fixed by τ consists of all vectors $(0, x, 0, y)$, where x, y are elements of n-dimensional linear space over the field $\mathbb{Z}_p$.

If $p > 2$ then the spread set R in $GL_{2n}(p) \cup \{0\}$ consists of the matrices

$$\theta(V, U) = \begin{pmatrix} m(U) & f(V) \\ V & U \end{pmatrix}. \quad (2.5)$$

Here $V \in Q$, $U \in K$, Q, K are some spread sets in $GL_n(p) \cup \{0\}$, K is the spread set of the Baer subplane π_τ, m, f are additive injective functions from K and Q to $GL_n(p) \cup \{0\}$, and $m(E) = E$.

If $p = 2$ then the spread set R in $GL_{2n}(2) \cup \{0\}$ consists of the matrices

$$\theta(V, U) = \begin{pmatrix} U + V + m(V) + w(V) & f(V) + m(U) \\ V & U + w(V) \end{pmatrix}, \quad (2.6)$$

where $U, V \in K$, K is the spread set of the Baer subplane π_τ in $GL_n(2) \cup \{0\}$. The additive functions m, f, w map K to the ring of $(n \times n)$-matrices over $\mathbb{Z}_2$, $m(E) = 0$, the function f is injective, the lower row of the matrix $w(V)$ consists of zeros for all $V \in K$.

Note that this result proved in [5] is a natural generalization of a matrix representation which was obtained in 1989 by N.L. Johnson et al. [1] for the case when a semifield plane of order 2^{2n} has a nucleus of order 2^n.

3. Results for semifield planes of odd order

Let π be a non-Desarguesian plane of odd order and τ be a Baer involution in the autotopism group Λ. We will study the involutions in Λ commuting with τ; it may be Baer involutions or homologies h_i (2.3).

Lemma 1. *Let π be a non-Desarguesian semifield plane of order p^N ($p > 2$ be prime), $\tau \in \Lambda$ be a Baer involution* (2.4)*. If $\sigma \neq \tau$ is a Baer involution in $C_\Lambda(\tau)$ then the restriction of σ onto the Baer subplane π_τ is a homology or a Baer involution. In the first case $\sigma = h_i\tau$ ($i = 1, 2, 3$); in the second case N is divisible by 4 and, with the appropriate base,*

$$\sigma = \begin{pmatrix} L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & L \end{pmatrix}. \tag{3.1}$$

Proof. Because σ commutes with τ,

$$\sigma = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_2 \end{pmatrix},$$

where $A_i, B_i \in GL_{N/2}(p)$, $A_i^2 = B_i^2 = E$ ($i = 1, 2$). Then $\sigma_\tau = \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}$ is an autotopism of the Baer subplane π_τ, identity or involutory homology, or Baer involution.

Let σ_τ be an identity mapping of π_τ, $A_2 = B_2 = E$. Then for any matrix $\theta(V, U)$ from the spread set R (2.5) the condition (2.2) holds:

$$\begin{pmatrix} A_1 & 0 \\ 0 & E \end{pmatrix} \theta(V, U) \begin{pmatrix} B_1 & 0 \\ 0 & E \end{pmatrix} \in R.$$

For instance, for $E \in R$ we have $B_1 = A_1$. Further, we can choose a new base of the linear space without changing τ such that the matrix A_1 leads to a Jordan normal form. Under the condition, σ is a Baer involution, so its characteristic roots are only -1 and 1, and in an equal amount. Moreover, its minimal polynomial is $\lambda^2 - 1$ and so the Jordan normal form of A_1 is necessarily $-E$, i.e. $\sigma = \tau$. We obtain the contradiction to the assumption of lemma.

Let σ_τ be the involutory homology with the axis $[0]$. Then $A_2 = -E$, $B_2 = E$ and $B_1 = -A_1$ from the condition (2.2). Converting A_1 to a Jordan normal form, we obtain diagonal matrix with diagonal elements ± 1. Show that $A_1 = \pm E$. Indeed, for $\theta(0, U)$ using (2.2) we have

$$\begin{pmatrix} A_1 & 0 \\ 0 & -E \end{pmatrix} \begin{pmatrix} m(U) & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} -A_1 & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} -A_1 m(U) A_1 & 0 \\ 0 & -U \end{pmatrix} \in R,$$

so $m(-U) = -A_1 m(U) A_1$, $A_1 m(U) = m(U) A_1$ for all $U \in K$. Assume that A_1 contains -1 and 1, then

$$A_1 = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix} \begin{matrix} k \\ N/2-k \end{matrix}$$

(here the number of rows is on the right). Divide the matrix $m(U)$ to the blocks of correspondent dimension and multiply:

$$m(U) = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \begin{matrix} k \\ N/2-k \end{matrix}$$

$$A_1 m(U) = \begin{pmatrix} -M_1 & -M_2 \\ M_3 & M_4 \end{pmatrix} = \begin{pmatrix} -M_1 & M_2 \\ -M_3 & M_4 \end{pmatrix} = m(U)A_1 \Rightarrow M_2 = M_3 = 0,$$

i.e. the matrices $m(U)$ are block-diagonal for all $U \in K$. It is impossible because the set $\{m(U) \mid U \in K\}$ is a spread set in $GL_{N/2}(p) \cup \{0\}$ and it consists of $p^{N/2}$ matrices determined uniquely, for instance, by lower row. If $0 < k < N/2$ then the number of variants is less. So, diagonal elements of the matrix A_1 are all -1 or all 1. We have the cases $\sigma = h_1$ or $\sigma = h_1\tau$.

If σ_τ is the involutory homology with the axis $[0,0]$ then the similar reasoning leads to $\sigma = h_2$ or $\sigma = h_2\tau$. If σ_τ is the involutory homology with the axis $[\infty]$ then $\sigma = h_3$ or $\sigma = h_3\tau$.

Consider now the case when σ_τ is a Baer involution. Then N is divisible by 4 and, with appropriate base, we have $A_2 = B_2 = L$. Converting A_1 and B_1 to a Jordan normal form, we deduce that corresponding matrices are both diagonal with diagonal elements ± 1. Moreover, from the condition (2.2) for $\theta(0,E) = E$ we have $A_1 = B_1$. Because the amount of (-1)-elements is equal to the amount of 1-elements, without loss of generality we assume $A_1 = B_1 = L$. The lemma is proved. □

Let π be a semifield plane of odd order p^N, $H < \Lambda$ is an elementary abelian 2-group of order 2^m without homologies. Then all involutions in H are Baer and the base of a linear space can be chosen such that all these involutions are represented by diagonal matrices. Enumerate the basic involutions in H: $\tau_1, \tau_2, \ldots, \tau_m$. Then the matrix τ_1 is formed by two diagonal matrices L of the dimension $(N/2 \times N/2)$, the matrix τ_2 is formed by four matrices L of the dimension $(N/4 \times N/4)$, and so on. The matrix τ_m is formed by 2^m matrices L of the dimension $(N/2^{m-1} \times N/2^{m-1})$; evidently, that $N/2^m \geq 1$. This reasoning proves the Theorem 1 in the case $p > 2$.

Corollary 1. *Let π be a non-Desarguesian semifield plane of order p^N ($p > 2$ be prime). If N is not divisible by 2^{2m+1} then an autotopism group of π does not contain the subgroups isomorphic to $Sz(2^{2n+1})$ for all $n \geq m$.*

Proof. A Sylow 2-subgroup in the Suzuki group $Sz(2^{2n+1})$ contains an elementary abelian subgroup of order 2^{2n+1}, its involutions are conjugated (for instance, see [15]). If H is such the autotopism subgroup then H does not contain homologies. Then the number 2^{2n+1} must be a factor of N. □

Corollary 2. *Let π be a non-Desarguesian semifield plane of order p^N ($p > 2$ be prime), where $N = 2^r \cdot s$, s is odd. If F is a subgroup of the autotopism group Λ, which is generated by homologies, then the 2-rank of the factor-group Λ/F is at most r.*

4. Results for semifield planes of even order

If $|\pi| = 2^N$ then any homology is of odd order, so an elementary abelian 2-subgroup in the autotopism group Λ contains only Baer involutions.

Lemma 2. *Let π be a non-Desarguesian semifield plane of order 2^N, $\tau \in \Lambda$ be a Baer involution* (2.4)*, where $L = \begin{pmatrix} E & E \\ 0 & E \end{pmatrix}$. If σ is a Baer involution in $C_\Lambda(\tau)$ and the Baer subplanes π_τ and π_σ are different then the restriction of σ onto π_τ is a Baer involution, N is divisible by 4, and σ is represented by* (3.1)*, with the appropriate base.*

Proof. If $\sigma \in C_\Lambda(\tau)$ is a Baer involution then

$$\sigma = \begin{pmatrix} A_1 & A_2 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & B_1 & B_2 \\ 0 & 0 & 0 & B_1 \end{pmatrix},$$

where $A_1^2 = B_1^2 = E$, $A_1 A_2 = A_2 A_1$, $B_1 B_2 = B_2 B_1$. The restriction of σ onto Baer subplane π_τ, $\sigma_\tau = \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}$ is not an identity or a homology. So, σ_τ is a Baer involution; then the bases of the subspaces $\{(0, x_2, 0, 0)\}$ and $\{(0, 0, 0, y_2)\}$ can be chosen such that $A_1 = B_1 = L$, $A_2 = B_2$, $A_2 L = L A_2$, then $A_2 = \begin{pmatrix} D_1 & D_2 \\ 0 & D_1 \end{pmatrix}$. Convert the base of $2N$-dimensional linear space using the transition matrix

$$T = \begin{pmatrix} E & C & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & E & C \\ 0 & 0 & 0 & E \end{pmatrix},$$

where $C = \begin{pmatrix} 0 & D_1 \\ D_1 & D_2 \end{pmatrix}$. Direct calculations show that in the new base the involution τ preserves its matrix representation, and the matrix $T\sigma T^{-1}$ become block-diagonal of the form (3.1).

This arguments complete the proof of Lemma 2 and, evidently, prove the Theorem 1 for the case $p = 2$. □

Remark 1. Without lost of generality, we can assume that in Theorem 1 a semifield plane π of order q^N is represented by a linear space over the nucleus $N_0 \supseteq GF(q)$ ($q = p^s$, p be prime), and Λ is a subgroup of linear autotopisms over $GF(q)$.

Consider the case when a Baer involution σ (in the notation of Lemma 2) acts on the Baer subplane π_τ as identity mapping.

Lemma 3. *Let π be a non-Desarguesian semifield plane of order 2^N, and $\tau, \sigma \in \Lambda$ are commuting Baer involutions fixing the same Baer subplane pointwise, $\pi_\tau = \pi_\sigma$. Then the nucleus K_0 of this subplane is of order ≥ 4, and, with appropriate base, τ is* (2.4),

$$\sigma = \begin{pmatrix} E & A & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & E & A \\ 0 & 0 & 0 & E \end{pmatrix}, \qquad A \in K_0^*. \tag{4.1}$$

Proof. It is enough to show that $A \in K_0^*$. Indeed, consider the condition (2.2) for σ and for the spread set (2.6):

$$\begin{pmatrix} E & A \\ 0 & E \end{pmatrix} \theta(V, U) \begin{pmatrix} E & A \\ 0 & E \end{pmatrix} \in R \quad \forall U, V \in K. \tag{4.2}$$

For $V = 0$ and arbitrary $U \in K$ from (4.2) we have

$$\begin{pmatrix} U & m(U) + AU + UA \\ 0 & U \end{pmatrix} = \begin{pmatrix} U & m(U) \\ 0 & U \end{pmatrix},$$

$UA = AU$, that is $A \in K_l^* = C_{GL_n(2)}(K)$, the matrix A is from the left nucleus of the subplane π_τ. For $U = 0$ and arbitrary $V \in K$:

$$\begin{pmatrix} E & A \\ 0 & E \end{pmatrix} \begin{pmatrix} V + m(V) + w(V) & f(V) \\ V & w(V) \end{pmatrix} \begin{pmatrix} E & A \\ 0 & E \end{pmatrix} =$$

$$= \theta(V, 0) + \begin{pmatrix} AV & VA + m(V)A + w(V)A + AVA + Aw(V) \\ 0 & VA \end{pmatrix} = \theta(V, VA),$$

it is true if $AV = VA \in K$ and

$$m(VA) = V(A + A^2) + m(V)A + w(V)A + Aw(V), \qquad \forall V \in K. \tag{4.3}$$

So, the matrix A is from the intersection of the right, middle and the left nuclei of the subplane π_τ, i.e. from the nucleus $K_0 = K_l \cap K_m \cap K_r$. □

If the nucleus K_0 of the subplane π_τ is a subfield (up to isomorphism) of the nucleus R_0 of the plane π then we can consider K_0 as a basic field and a subgroup Λ_0 of linear autotopisms over K_0. In this case $A = aE \in K_0^*$ is a scalar matrix, so the following lemma holds.

Lemma 4. *Let a nucleus R_0 of a plane π of even order 2^N contains a nucleus K_0 of a Baer subplane π_0. Then the subgroup Λ_0 of linear over K_0 autotopisms of π does not contain a subgroup H isomorphic to the alternating group A_4 with involutions fixing π_0 pointwise.*

Proof. Consider $H \simeq A_4$ as $\langle \tau, \sigma, \gamma \rangle$, where τ and σ are the commuting involutions, the collineation γ is of order 3 and $\gamma^{-1}\tau\gamma = \sigma$. According Lemma 3, the Baer involution τ is (2.4), σ is (4.1), and the condition (4.3) for the scalar matrix $A = aE \in K_0$ become

$$m(aV) + am(V) = V, \qquad \forall V \in K. \tag{4.4}$$

Because $\tau\gamma = \gamma\sigma$, we have

$$\gamma = \begin{pmatrix} B_1 & B_2 & 0 & 0 \\ 0 & aB_1 & 0 & 0 \\ 0 & 0 & C_1 & C_2 \\ 0 & 0 & 0 & aC_1 \end{pmatrix},$$

and $\sigma\gamma = \gamma\tau\sigma$ leads to $a^2 = a + 1$. Note, that $m(E) = 0$. Then the condition (4.4) for $V = E$ leads to $m(aE) = E$, and for $V = aE$ to

$$m(a^2E) + am(aE) = aE, \qquad m(aE) + m(E) + aE = aE, \qquad m(aE) = 0.$$

This contradiction proves the lemma. □

The question on a subgroup isomorphic to A_4 in the collineation group of a finite semifield plane has been raised for a long time. For instance, I.V. Busarkina (Sheveleva) used the absence of A_4 in an autotopism group to prove the solvability of a full collineation group for a p-primitive semifield plane with additional conditions for a spread set [13]. In the case of even order the author provided [4] examples of semifield planes admitting A_4 in the translation complement.

Lemma 5. *Let π be non-Desarguesian semifield plane of order 2^m with the right nucleus R_r of order 4^k, $k \geqslant 1$. Then π admits a subgroup of collineations isomorphic to the alternating group A_4.*

Proof. Let the right nucleus R_r of a plane π contains a subfield of order 4, and $M \in R_r^*$ be a primitive element of this subfield, $M^3 = E$. Consider two elations $\omega_1, \omega_2 \in \Omega$ with the axis $[0]$ and the center (∞), and the homology $\gamma \in H_r$ with the axis $[0,0]$ and the center (∞),

$$\omega_1 = \begin{pmatrix} E & E \\ 0 & E \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} E & M \\ 0 & E \end{pmatrix}, \qquad \gamma = \begin{pmatrix} E & 0 \\ 0 & M \end{pmatrix}.$$

Then we can prove by direct calculations that $|\omega_1| = |\omega_2| = 2$, $|\gamma| = 3$, $\gamma^{-1}\omega_1\gamma = \omega_2$, i.e. $\langle\omega_1, \omega_2\rangle \leftthreetimes \langle\gamma\rangle \simeq A_4$. □

5. Examples

Example 1. There are exactly two, up to isomorphism, non-Desarguesian semifield planes of order $16 = 2^4$ (for instance, see [5]). Their autotopism groups Λ are of order 18 and 108, the centralizier of a Baer involution in Λ equals to $\mathbb{Z}_2$ or $\mathbb{Z}_2 \leftthreetimes S_3$, respectively. This is consistent with the Lemmas 2 and 3.

Example 2. As was proved in [14], there are exactly 124 non-isomorphic semifield planes of order 256 with the left nucleus $R_l \simeq GF(16)$, which admit a Baer involution τ in the translation complement G_0. These planes are determined by spread sets of (2×2)-matrices over $GF(16)$, and $G/G_0 \simeq Aut\, R_l$. Because $|\pi| = 256 = 2^8$, an elementary abelian 2-subgroup in the autotopism group Λ is of order at most 8. According the main Theorem 1, the subgroup of linear autotopisms Λ_0 does not contain involutions commuting with τ.

The results obtained here are completely consistent with the computer calculations presented in [2]. The centralizier of a Baer involution τ in the linear autotopism group Λ_0 equals to

$$C_{\Lambda_0}(\tau) = H_l \times H_{rd} \times \langle\tau\rangle,$$

where $H_{rd} \simeq R_r^* \cap R_l^*$. All Baer involutions in Λ_0 are conjugated, the order Λ_0 is $2 \cdot 5^s \cdot 3^m$ $(s, m = 1, 2, 3)$ or $2 \cdot 5 \cdot 3^2 \cdot 17$, the group Λ_0 is solvable.

Example 3. There are exactly eight, up to isomorphism, semifield planes of order $81 = 3^4$ admitting a Baer involution (more detail, see [9]). For each of these 2-rank of the autotopism group Λ equals three, the group Λ is of order 2^m $(m = 8, \ldots, 11)$, it is solvable and it contains four or 100 (in a unique case) Baer involutions. Taking in account the involutory homologies, we see that the results are consistent with those proved above.

6. Conclusion

Evidently, that a convenient matrix representation of elementary abelian 2-subgroups in an autotopism group of a non-Desarguesian semifield projective plane allows us to investigate another even-order autotopism subgroups. In order to study the main problem of the solvability of a full collineation group, the author considers it possible to use the obtained (mainly technical) results to further study the series of planes that do not admit known non-abelian simple groups as automorphism groups.

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Received 24.12.2019

Элементарные абелевы 2-подгруппы в группе автотопизмов полуполевой проективной плоскости

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Аннотация. Изучается гипотеза разрешимости полной группы автоморфизмов недезарговой полуполевой проективной плоскости конечного порядка (вопрос 11.76 в Коуровской тетради). Как известно, эта гипотеза редуцируется к разрешимости группы автотопизмов. Изучая подгруппы четного порядка в группе автотопизмов, мы применяем метод с использованием регулярного множества над полем простого порядка. Показано, что для элементарной абелевой 2-подгруппы в группе автотопизмов выбор базиса линейного пространства позволяет построить матричное представление порождающих элементов, единообразное для полуполевых плоскостей четного и нечетного порядка и не зависящее от размерности пространства. В качестве следствия указано условие, связывающее порядок полуполевой плоскости и порядок элементарной абелевой 2-подгруппы автотопизмов. Выделена бесконечная серия полуполевых плоскостей нечетного порядка, не допускающих подгруппу автотопизмов, изоморфную группе Судзуки $Sz(2^{2n+1})$. В случае четного порядка плоскости получено условие на ядро подплоскости, поточечно фиксируемой автотопизмом порядка два. Выбор такого ядра в качестве основного поля приводит к отсутствию в группе линейных автотопизмов подгруппы, изоморфной знакопеременной группе A_4. Основные доказанные результаты являются техническими и необходимы для дальнейшего изучения подгрупп четного порядка в группе автотопизмов конечной недезарговой полуполевой плоскости. Результаты согласуются с приведенными в статье примерами 3-примитивных полуполевых плоскостей порядка 81, а также с хорошо известными двумя примерами неизоморфных полуполевых плоскостей порядка 16.

Ключевые слова: полуполевая плоскость, регулярное множество, бэровская инволюция, гомология, автотопизм.

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Поступила в редакцию 24.12.19