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## On Covering Bounded Sets by Collections of Circles of Various Radii

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**Abstract.** This paper is devoted to the problem of constructing an optimal covering of a two-dimensional figure by the union of circles. The radii of the circles, generally speaking, are different. Each of them is equal to the product of some positive coefficient and the parameter  $r$  common to all circles, which is the objective function to be minimized. We carried out an analytical study of the problem and obtained expressions that allow us to describe the generalized Dirichlet zones for the considered case. We propose an iterative procedure correcting the coordinates of the circles' centers that form the covering, which is based on finding the Chebyshev centers of the generalized Dirichlet zones. This procedure does not impair the properties of the covering. A computational algorithm is proposed and implemented. It includes the multistart method to generate the initial positions of points and the iterative procedure. We carried out a computational experiment to find optimal coverings by sets of circles at various coefficients that determine the radius of each of them. Two and three different types of circles are used. Both convex and non-convex polygons are taken as the covered sets. The analysis of the calculation results was carried out, which allowed us to draw conclusions about the properties of the constructed coverings.

**Keywords:** optimization, circle covering problem, generalized Dirichlet zone, Chebyshev center, iterative algorithm, computational experiment.

### 1. Introduction

The problem of the optimal covering constructing of a bounded set on the plane is one of the main challenges of computational geometry [8]. Often it

is considered in the traditional formulation: it is necessary to cover a given set with a certain number of equal circles [9]. And even in such a relatively pure form, it is NP-hard. In recent years, non-classical versions of this problem have been considered. Coverage elements can be different, as well as be circles in some non-Euclidean metric. Such statements arise in connection with the tasks of infrastructure logistics [2;6] when one needs to take into account special constraints. For example, service areas of various logistics centers can have different radii, or a service zone can be heterogeneous. Besides, some tasks need to find reserve or multiple coverings [7].

This article is devoted to constructing the optimal covering of a bounded set by circles of different radii. Assume that the radii are proportional to the variable  $r$ , and its minimization is the objective function of the problem.

It was well-known Hungarian mathematician G. Fejes Tóth [10] who hypothesized the lower boundary of the covering density. The hypothesis was proved only after 27 years [11], and this gave an impulse to a more active study of this problem. In [4], the authors suggest a sufficient condition for the covering to be “solid”. The article [3] presents simple constructive estimates of the upper and lower boundaries of the covering density.

Analytical methods for covering and packaging problems usually have a limited range of applicability. Therefore, the primary research tool is a numerical experiment. Among a significant number of such publications, we point out the paper [1] proposed a successful algorithm of branch-boundaries, which allows one to check whether a polygon is covered by a given set of circles.

In this paper, we continue a long cycle of articles devoted to optimal circle covering problem (CCP). Earlier, we studied CCP [7], including multiple and reserve coverings in non-Euclidean metrics, the research methodology is based on the construction of  $n$ -networks [5]. In this article, we consider the new problem of covering a flat set with different circles, for which  $n$ -networks, generally speaking, are not applicable. To solve it, we propose a computational algorithm and prove theorems on its properties. A computational experiment is carried out for the cases of two and three different types of covering circles. It shows the efficiency of the proposed approach, and also makes it possible to conclude the coverings’ properties.

## 2. Formulation

Assume we are given a compact set  $M \subset \mathbb{R}^2$  and a set of  $n \in \mathbb{N}$  positive numbers  $\alpha_i, i = \overline{1, n}$ . We address to optimal circle covering problem (CCP) in the following formulation. It is required to find the optimal covering of the set  $M$  by the union of  $n$  circles  $O(\mathbf{s}_i, \alpha_i r), i = \overline{1, n}$ , whose centers form the array  $S = \{\mathbf{s}_i\}_{i=1}^n$ , and the radii are proportional to the numbers  $\alpha_i, i = \overline{1, n}$ . The objective function is  $r \rightarrow \min$ . In this formulation,

the problem can have various interpretations in geometry, approximation theory, and control theory.

**Definition 1.** A covering  $\Xi_n$  of a compact set  $M \subset X$  by  $n$  circles with radii  $r_i, i = \overline{1, n}$  is a union  $O(\mathbf{x}_1, r_1) \cup O(\mathbf{x}_2, r_2) \cup \dots \cup O(\mathbf{x}_n, r_n)$ , if

$$M \subseteq \bigcup_{i=\overline{1, n}} O(\mathbf{x}_i, r_i).$$

**Definition 2.** A covering  $\Xi_n$  is an optimal covering of  $M$ , if  $r$  is minimal.

The problem of finding optimal covering comes to determine a set  $S$  of  $n$  points for which

$$R_M(S) = \max_{\mathbf{x} \in M} \min_{i=\overline{1, n}} \varphi^{(i)}(\mathbf{x}) \quad (2.1)$$

is minimal. Here

$$\varphi^{(i)}(\mathbf{x}) \triangleq \frac{\|\mathbf{x} - \mathbf{s}_i\|}{\alpha_i}, i = \overline{1, n}. \quad (2.2)$$

$R_M(S)$  means such minimal  $r$ , for which  $M$  belongs to a union of circles  $\Xi_n$ .

The problem is a generalization of the problem of finding the best Chebyshev  $n$ -network of the set.

### 3. Solution method

#### 3.1. DIVIDING THE SET $M$ INTO ZONES

In the article, we develop the previously used procedures for constructing coverings by sets of congruent circles. Their basis includes two steps: the construction of the partition of the set  $M$  into zones of influence of points  $\mathbf{s}_i \in S$  (centers of the covering circles) and the shift of points in order to minimize the radius of the circle in which this zone can be inscribed. However, since we consider unequal circles having different radii proportional to the numbers  $\alpha_i, i = \overline{1, n}$ , the structure of the zones will be different.

**Definition 3.** The domain of the dominance of point  $\mathbf{s}_i$  over point  $\mathbf{s}_j$  is called the set

$$D^{(i,j)}(S) \triangleq \left\{ \mathbf{x} \in \mathbb{R}^2 : \varphi^{(i)}(\mathbf{x}) \leq \varphi^{(j)}(\mathbf{x}) \right\}.$$

For the convenience, we assume that  $D^{(i,i)}(S) = \mathbb{R}^2$ .

**Theorem 1** (On the structure of the dominance domain). *Let  $\mathbf{s}_i, \mathbf{s}_j$  be different points from  $S$ . Then the following statements hold.*

1) If  $\alpha_i < \alpha_j$ , then  $D^{(i,j)}(S)$  is a circle

$$D^{(i,j)}(S) = O(\mathbf{v}, r^*(\alpha_i, \alpha_j, \mathbf{s}_i, \mathbf{s}_j)), \quad (3.1)$$

with a center in

$$\mathbf{v} = \mathbf{s}_i + \frac{\alpha_i^2}{\alpha_j^2 - \alpha_i^2}(\mathbf{s}_i - \mathbf{s}_j) \quad (3.2)$$

having a radius

$$r^*(\alpha_i, \alpha_j, \mathbf{s}_i, \mathbf{s}_j) = \frac{\alpha_i \alpha_j}{|\alpha_j^2 - \alpha_i^2|} \|\mathbf{s}_i - \mathbf{s}_j\|. \quad (3.3)$$

2) If  $\alpha_i = \alpha_j$ , then  $D^{(i,j)}(S)$  is a half-plane

$$D^{(i,j)}(S) = \{\mathbf{x} \in \mathbb{R}^2: \|\mathbf{x} - \mathbf{s}_i\| \leq \|\mathbf{x} - \mathbf{s}_j\|\}. \quad (3.4)$$

3) If  $\alpha_i > \alpha_j$ , then  $D^{(i,j)}(S)$  is an unbounded set

$$D^{(i,j)}(S) = \{\mathbf{x} \in \mathbb{R}^2: \|\mathbf{x} - \mathbf{w}\| \geq r^*(\alpha_i, \alpha_j, \mathbf{s}_i, \mathbf{s}_j)\}, \quad (3.5)$$

$$\mathbf{w} = \mathbf{s}_j + \frac{\alpha_i^2}{\alpha_i^2 - \alpha_j^2}(\mathbf{s}_j - \mathbf{s}_i). \quad (3.6)$$

*Proof.* Let us begin with case 1). Without loss of generality, we assume that the points  $\mathbf{s}_i$  and  $\mathbf{s}_j$  have coordinates  $(0, 0)$  and  $(0, d)$ ,  $d > 0$ , respectively. Consider the geometric set  $X = \{\mathbf{x}\} = \{(x, y)\}$  of points which obey

$$\varphi^{(i)}(\mathbf{x}) = \varphi^{(j)}(\mathbf{x}). \quad (3.7)$$

From formula (2.2) and the assumption about the location of network points it follows, that

$$\varphi^{(i)}(\mathbf{x}) = \varphi^{(i)}(x, y) = \sqrt{x^2 + y^2}/\alpha_i, \quad (3.8)$$

$$\varphi^{(j)}(\mathbf{x}) = \varphi^{(j)}(x, y) = \sqrt{(x-d)^2 + y^2}/\alpha_j. \quad (3.9)$$

Substituting the values (3.8) and (3.9) into the equality (3.7), we obtain the equality

$$\sqrt{x^2 + y^2}/\alpha_i = \sqrt{(x-d)^2 + y^2}/\alpha_j,$$

that can be reduced to the form of the canonical equation of a circle

$$\left(x + \frac{d\alpha_i^2}{\alpha_j^2 - \alpha_i^2}\right)^2 + y^2 = \left(\frac{d\alpha_i\alpha_j}{\alpha_j^2 - \alpha_i^2}\right)^2. \quad (3.10)$$

Now we prove that the circle defined by equation (3.10) coincides with the boundary of the set (3.1). Assumptions about the choice of the coordinate system means that  $\|\mathbf{s}_i - \mathbf{s}_j\| = d$ , and according to formula (3.2)

$$\mathbf{v} = \mathbf{s}_i + \frac{\alpha_i^2}{\alpha_j^2 - \alpha_i^2}(\mathbf{s}_i - \mathbf{s}_j) = \left( -\frac{d\alpha_i^2}{\alpha_j^2 - \alpha_i^2}, 0 \right).$$

At the same time, (3.3) takes the form

$$r^*(\alpha_i, \alpha_j, \mathbf{s}_i, \mathbf{s}_j) = \frac{\alpha_i \alpha_j}{|\alpha_j^2 - \alpha_i^2|} d.$$

Thus, the boundary of the disk  $O(\mathbf{v}, r^*(\alpha_i, \alpha_j, \mathbf{s}_i, \mathbf{s}_j))$  coincides with the circle (3.10). This means that the set  $D^{(i,j)}(S)$  coincides with the part of the plane that is bounded by this circle  $\partial O(\mathbf{v}, r^*(\alpha_i, \alpha_j, \mathbf{s}_i, \mathbf{s}_j))$  and contains the point  $\mathbf{s}_i$ , i.e (3.1).

Let us turn to case 2). It is elementary, since in this case the difference between the functions  $\varphi^{(i)}(\mathbf{x})$  and  $\varphi^{(j)}(\mathbf{x})$  coincides with the difference between the Euclidean distances from  $\mathbf{x}$  to the points  $\mathbf{s}_i$  and  $\mathbf{s}_j$  multiplied by a positive number  $\alpha_i^{-1}$ . Therefore, the boundary of the dominance domains coincides with the middle perpendicular to the segment  $[\mathbf{s}_i, \mathbf{s}_j]$ , and the set  $D^{(i,j)}(S)$  is the half-plane that contains point  $\mathbf{s}_i$ . Formula (3.4) is proved.

Let's consider case 3). It is similar to case 1) if we interchange the points  $\mathbf{s}_i$  and  $\mathbf{s}_j$ . Therefore, we can similarly prove that the geometrical location of the points for which (3.7) holds is a circle of radius (3.3) centered at the point (3.6). However, since in case 3) the point  $\mathbf{s}_i$  is located outside the disk bounded by this circle, then  $D^{(i,j)}(S)$  does not coincide with the circle, but with its complement to the plane  $\mathbb{R}^2$ , i.e. (3.5). □

**Definition 4.** *The generalized Dirichlet zone of the point  $\mathbf{s}_i$  in the set  $M$  for given numbers  $\alpha_i, i = \overline{1, n}$  is called the set*

$$D^{(i)}(S, M) \triangleq \left\{ \mathbf{m} \in M : \varphi^{(i)}(\mathbf{m}) = \min_{j=\overline{1, n}} \varphi^{(j)}(\mathbf{m}) \right\}. \quad (3.11)$$

The domains  $D^{(i)}(S, M)$  are a generalization of the Dirichlet zones, which were introduced for the equal circles covering problem. Dirichlet zones are the geometrical places of points located no farther from one of the elements of the  $\mathbf{s}_i$   $n$ -network  $S_n$  than from others. Moreover, the generalized Dirichlet zones have a much more complex shape. In particular, their boundary may contain circular arcs. In addition, they can be non-convex and even multiply connected. From formula (3.11) it follows that the generalized Dirichlet zones can be found as the intersection of the domains

of dominance of the point with the  $M$

$$D^{(i)}(S, M) = M \cap \bigcap_{j=\overline{1, n}} D^{(i, j)}(S). \quad (3.12)$$

The boundaries  $D^{(i)}(S, M)$  can contain both segments and arcs of circles of different radius. However, it is further convenient to pass to their approximations, for example, by polygons.

### 3.2. FINDING NEW CENTERS

**Definition 5.** *The Chebyshev center of a closed bounded set  $M \in \mathbb{R}^2$  is the point  $\mathbf{c}(M)$  satisfying the equality*

$$h(M, \{\mathbf{c}(M)\}) = \min \{h(M, \{\mathbf{x}\}) : \mathbf{x} \in \mathbb{R}^2\} = r(M), \quad (3.13)$$

where  $h(A, B) \triangleq \max_{\mathbf{a} \in A} \min_{\mathbf{b} \in B} \|\mathbf{a} - \mathbf{b}\|$  is Hausdorff half deviation between compact sets  $A$  and  $B$ .

For any compact set  $M$ , there exists a unique Chebyshev center  $\mathbf{c}(M)$ , and it belongs to the convex hull  $\text{co} M$  of the set  $M$ . The value  $r(M)$  in (3.13) is called the Chebyshev radius of set  $M$ .

**Lemma 1.** *For any closed bounded set  $M \in \mathbb{R}^2$ , with  $r(M) > 0$  and any point  $\mathbf{x} \in \mathbb{R}^2$  the following estimate holds:*

$$r(M) \leq h(M, \{\mathbf{x}\}) - \frac{\|\mathbf{x} - \mathbf{c}(M)\|^2}{2h(M, \{\mathbf{x}\})}. \quad (3.14)$$

*Proof.* If point  $\mathbf{x}$  coincides with  $\mathbf{c}(M)$ , then inequality (3.14) becomes equality. Otherwise, we consider a nonzero vector  $\mathbf{z} = \mathbf{x} - \mathbf{c}(M)$  and construct a straight line  $l$ , which is perpendicular to  $\mathbf{z}$  and passes through the point  $\mathbf{c}(M)$ , as well as a semicircle  $\Lambda \subset \partial O(\mathbf{c}(M), r(M))$ , located on that half-plane relative to the line  $l$ , which does not contain  $\mathbf{x}$ . According to the properties of the Chebyshev center on any semicircle belonging to  $\partial O(\mathbf{c}(M), r(M))$ , there is always at least one point  $\mathbf{m} \in M$ . Indeed, if  $\Lambda \cap M = \emptyset$  holds for some semicircle  $\Lambda$ , then the similar statement  $\Lambda_\varepsilon \cap M = \emptyset$  holds for  $\varepsilon$ -neighborhood  $\Lambda_\varepsilon$ . This means that all points of the set  $M \cap \partial O(\mathbf{c}(M), r(M))$  belong to an arc of a circle with an angle  $\gamma < \pi$ . According to the properties of the Chebyshev center of a flat set, it always belongs to the convex hull  $M \cap \partial O(\mathbf{c}(M), r(M))$ . But if all the points of  $M \cap \partial O(\mathbf{c}(M), r(M))$  belong to an arc with an angle  $\gamma < \pi$ , then their convex hull does not contain the circle center. Thus, we have a contradiction.

Among all points  $\Lambda$ , the closest to  $\mathbf{x}$  are the intersections of  $\Lambda$  with  $l$  by construction. Therefore, an arbitrary point  $\mathbf{m} \in \Lambda \cap M$  obeys the estimation

$$\|\mathbf{m} - \mathbf{x}\|^2 \geq \|\mathbf{z}\|^2 + r^2(M). \quad (3.15)$$

We can easily transform estimation (3.15) to

$$\|\mathbf{m} - \mathbf{x}\| - r(M) \geq \frac{\|\mathbf{z}\|^2}{\|\mathbf{m} - \mathbf{x}\| + r(M)}.$$

It follows from (3.15) that  $\|\mathbf{m} - \mathbf{x}\| \geq r(M)$ , which means that the estimate can be increased

$$\|\mathbf{m} - \mathbf{x}\| - r(M) \geq \frac{\|\mathbf{z}\|^2}{2\|\mathbf{m} - \mathbf{x}\|}. \quad (3.16)$$

Since from the definition of the Hausdorff deviation  $h(M, \{\mathbf{x}\}) \geq \|\mathbf{m} - \mathbf{x}\|$  for any point  $\mathbf{m} \in M$ , then

$$h(M, \{\mathbf{x}\}) - r(M) \geq \frac{\|\mathbf{z}\|^2}{2h(M, \{\mathbf{x}\})}.$$

If we transfer  $h(M, \{\mathbf{x}\})$  to the right side of the inequality and make the reverse substitution of the vector  $\mathbf{z}$ , then we get (3.14). □

The basis for constructing a new array of coverage circle centers  $\widehat{S} = \{\widehat{\mathbf{s}}_i\}_{i=1}^n$  for the value  $S$  specified at the current step is the formula

$$\widehat{\mathbf{s}}_i = \begin{cases} k_c \mathbf{c}(D^{(i)}(M, S)) + (1 - k_c) \mathbf{s}_i, & D^{(i)}(M, S) \neq \emptyset, \\ \mathbf{s}_i, & D^{(i)}(M, S) = \emptyset. \end{cases}, \quad i = \overline{1, n}, \quad (3.17)$$

where  $k_c \in (0, 1]$  is a custom parameter. The meaning of the coefficient  $k_c$  is how quickly the coordinates of the covering circles change at each step. Increasing  $k_c$  makes it possible to increase the speed of the algorithm, but reduces its stability.

The proposed *Algorithm* consists of the following steps. The first step is to construct the initial position  $S^{(0)} \subset M$  of circles centers by stochastic methods. Then according to formula (3.17), iterative changes of the coordinates of the points are carried out to minimize value (2.1) for the current array  $S$ . The generalized Dirichlet zones, in accordance with theorem 1, are constructed by formula (3.12) as the intersection of  $M$  with half-planes, circles, and complements of disks. The stopping criterion is the fulfillment of the condition of sufficient proximity  $h(\widehat{S}, S) \leq h_0$  in the Hausdorff metric for the newly constructed  $\widehat{S}$  and the old  $S$  networks. The parameters  $h_0$  and  $k_c$  are set by the user.

The algorithm is improving, but it does not guarantee a global solution.

**Theorem 2** (The properties of the iterative algorithm). *For any compact set  $M$ , set of positive numbers  $\{\alpha_i\}_{i=1}^n$ ,  $k_c \in (0, 1]$  and set of  $n$  points  $S$  the following estimation holds*

$$R_M(\widehat{S}) \leq R_M(S), \quad (3.18)$$

where  $\widehat{S}$  is determined by formula (3.17).

*Proof.* To prove the theorem, we should show that for an arbitrary number  $i$ ,  $1 \leq i \leq n$ , for which  $D^{(i)}(M, S) \neq \emptyset$ , the following estimate holds

$$\max \left\{ \widehat{\varphi}^{(i)}(\mathbf{x}) : \mathbf{x} \in D^{(i)}(S, M) \right\} \leq \max \left\{ \varphi^{(i)}(\mathbf{x}) : \mathbf{x} \in D^{(i)}(S, M) \right\}, \quad (3.19)$$

where  $\widehat{\varphi}^{(i)}(\mathbf{x}) \triangleq \alpha_i^{-1} \|\mathbf{x} - \widehat{\mathbf{s}}_i\|$ .

Let  $F(\mathbf{s}) \triangleq h(D^{(i)}(S, M), \{\mathbf{s}\})$  be the function equal to the Hausdorff half-deviation of the compact set  $D^{(i)}(M, S)$  from a one-point set, containing one element  $\mathbf{s}$ . Definition 5 yields the estimate

$$F(\mathbf{s}_i) \geq r \left( D^{(i)}(S, M) \right) = F \left( \mathbf{c}(D^{(i)}(M, S)) \right). \quad (3.20)$$

The function  $F(\mathbf{s})$  can be represented as

$$F(\mathbf{s}) = \max_{\mathbf{g}} \{ \|\mathbf{s} - \mathbf{g}\| : \mathbf{g} \in D^{(i)}(S, M) \}.$$

It is easy to see that the function  $F(\mathbf{s})$  is convex. It follows from formula (3.17) that the point  $\widehat{\mathbf{s}}_i$  is a convex combination of points  $\mathbf{s}_i$  and  $\mathbf{c}(D^{(i)}(M, S))$ , which means that  $F(\cdot)$  obeys the estimate

$$F(\widehat{\mathbf{s}}_i) \leq k_c F \left( \mathbf{c}(D^{(i)}(M, S)) \right) + (1 - k_c) F(\mathbf{s}_i). \quad (3.21)$$

The inequality  $F(\widehat{\mathbf{s}}_i) \leq F(\mathbf{s}_i)$  follows from (3.20) and (3.21). Multiplying it by  $\alpha_i^{-1}$  we get estimate (3.19).

Definition 4 and formulas (2.1), (2.2) imply the equality

$$R_M(S) = \max_{i=\overline{1, n}} \max \left\{ \varphi^{(i)}(\mathbf{x}) : \mathbf{x} \in D^{(i)}(S, M) \right\} \quad (3.22)$$

and the estimation

$$R_M(\widehat{S}) \leq \max_{i=\overline{1, n}} \max \left\{ \widehat{\varphi}^{(i)}(\mathbf{x}) : \mathbf{x} \in D^{(i)}(S, M) \right\}. \quad (3.23)$$

Formulas (3.22) and (3.23) may contain empty generalized Dirichlet zones  $D^{(i)}(S, M)$ . However, since maximization with respect to  $i$  is performed in (3.22) and (3.23), the estimates are determined only by nonempty sets  $D^{(i)}(S, M)$ ,  $i = \overline{1, n}$ .

If we substitute estimates (3.19) into inequality (3.23) for all  $i = \overline{1, n}$  for which  $D^{(i)}(M, S) \neq \emptyset$ , then we obtain

$$R_M(\widehat{S}) \leq \max_{i=\overline{1, n}} \max \left\{ \varphi^{(i)}(\mathbf{x}) : \mathbf{x} \in D^{(i)}(S, M) \right\}.$$

This inequality and (3.22) imply the estimate (3.18).  $\square$



We approximate the generalized Dirichlet zones  $D^{(i)}(M, S)$ ,  $i = \overline{1, n}$ , by sets of points  $P^{(i)}$ . In the case when  $M$  is a convex polygon, the following characteristic points are included in the set  $P^{(i)}$ .

- 1) The vertices of the polygon  $M$ , which belong to  $D^{(i)}(M, S)$ .
- 2) The intersection points of the boundary  $\partial D^{(i)}(M, S)$  of the generalized Dirichlet zone and the boundary  $\partial M$  of the set  $M$ .
- 3) The intersection points of the boundary  $\partial D^{(i)}(M, S)$  of the generalized Dirichlet zone and the boundaries  $\partial D^{(j)}(M, S)$  and  $\partial D^{(k)}(M, S)$  of two mismatched Dirichlet zones  $i \neq j, i \neq k, j \neq k$ .
- 4) The intersection points of the boundary  $\partial D^{(i,j)}(M, S)$  of the domain of the dominance for  $i \neq j$  and the straight line  $\lambda$ , which contains the segment  $[s_i, s_j]$ , if these points belong to  $\partial D^{(i)}(M, S)$ .

As an approximation of the Chebyshev center of the set  $D^{(i)}(M, S)$  in formula (3.17) we take  $\mathbf{c}(P^{(i)})$ . Then we check the condition  $D^{(i)}(M, S) \subseteq O(\mathbf{c}(P^{(i)}), r(P^{(i)}))$ .

Note that to find sets of characteristic points, you need to check about  $n^3$  elements (if the number of covering circles is significantly greater than the number of the polygon vertices). This is due to the fact that three arbitrary generalized Dirichlet zones can have either one or two common points; each one must be considered. Their coordinates are found as intersections of the boundaries of the domain of dominance for points from  $S$ .

Now we give an estimate of the quality of the algorithm, based on the formula (3.17) at each step. For short we will omit arguments in  $D^{(i)}(S, M)$ .

**Theorem 3.** *Let we are given a compact set  $M \in \text{comp}(\mathbb{R}^2)$  and  $n$ -network  $S_n^{(k)}$ , which is a result of  $k$  iteration of the algorithm. Then for the network  $S_n^{(k+1)}$  obtained by formula (3.17) with  $k_c = 1$ , the estimation holds*

$$R_M(S_n^{(k+1)}) \leq R_M(S_n^{(k)}) - \frac{\left( \min \left\{ \alpha_i^{(-1)} \| \mathbf{s}_{k+1}^{(i)} - \mathbf{s}_k^{(i)} \| : i = \overline{1, n} \right\} \right)^2}{2R_M(S_n^{(k)})}. \quad (3.24)$$

*Proof.* Consider a certain Dirichlet zone  $D^{(i)}(\cdot) = \emptyset$ ,  $i \in \overline{1, n}$ . Let us show that the following estimate holds

$$h(D^{(i)}(\cdot), \{\mathbf{s}_{k+1}\}) \leq h(D^{(i)}(\cdot), \{\mathbf{s}_k\}) - \frac{\| \mathbf{s}_{k+1} - \mathbf{s}_k \|^2}{2h(D^{(i)}(\cdot), \{\mathbf{s}_k\})}. \quad (3.25)$$

If the points  $\mathbf{s}_k$  and  $\mathbf{s}_{k+1}$  coincide, then the inequality (3.25) takes the form of equality, and so it holds. Otherwise, by construction, the point  $\mathbf{s}_{k+1}$

coincides with the Chebyshev center  $\mathbf{c}(D^{(i)}(\cdot))$  of the zone  $D^{(i)}(\cdot)$ , and  $h(D^{(i)}(\cdot), \{\mathbf{s}_{k+1}\})$  equals to its Chebyshev radius. Therefore, (3.25) follows from the estimate (3.14) in lemma 1.

By construction, the  $R_M(S_n^{(k+1)})$  satisfies the estimate

$$R_M(S_n^{(k+1)}) \leq \max_{i=1,n} \alpha_i^{-1} h(D^{(i)}(\cdot), \{\mathbf{s}_{k+1}\}).$$

Taking into account inequalities (3.25), it can be reduced to

$$\begin{aligned} R_M(S_n^{(k+1)}) &\leq \max_{i=1,n} \left( \alpha_i^{-1} h(D^{(i)}(\cdot), \{\mathbf{s}_k\}) - \frac{\alpha_i^{-1} \|\mathbf{s}_{k+1} - \mathbf{s}_k\|^2}{2h(D^{(i)}(\cdot), \{\mathbf{s}_k\})} \right) \leq \\ &\leq \max_{i=1,n} \alpha_i^{-1} h(D^{(i)}(\cdot), \{\mathbf{s}_k\}) - \min_{i=1,n} \frac{\alpha_i^{-1} \|\mathbf{s}_{k+1} - \mathbf{s}_k\|^2}{2h(D^{(i)}(\cdot), \{\mathbf{s}_k\})} \leq \\ &\leq \max_{i=1,n} \alpha_i^{-1} h(D^{(i)}(\cdot), \{\mathbf{s}_k\}) - \frac{\min_{i=1,n} \alpha_i^{-1} \|\mathbf{s}_{k+1} - \mathbf{s}_k\|^2}{2 \max_{i=1,n} h(D^{(i)}(\cdot), \{\mathbf{s}_k\})} \leq \\ &\leq R_M(S_n^{(k)}) - \frac{\min_{i=1,n} \alpha_i^{-1} \|\mathbf{s}_{k+1} - \mathbf{s}_k\|^2}{2R_M(S_n^{(k)})}, \end{aligned}$$

that is equivalent to (3.24).  $\square$

#### 4. Computational experiment

The authors develop software for constructing coverings of a bounded set by circles of various radii. It is based on methods of computational geometry: finding the intersection and union of polygons and determining the Chebyshev center of the polygon. Theorem 2 guarantees that applying the algorithm does not deteriorate the properties of coverings. Theorem 3 gives an estimate of the algorithm speed.

Testing of the algorithm proposed in the previous section was carried out using the PC of the following configuration: Intel (R) Core i5-3570K (3.4 GHz, 8 GB RAM) and Windows 10 operating system. Each experiment was carried out with 5 ÷ 10 runs of the software, in each of which 100 ÷ 200 iterations were performed to change the coordinates of the centers of the covering elements. The executed time is about 15 minutes. As covered sets, we deal with polygons, including non-convex ones.

An indirect indicator  $\sigma(\Xi_n)$  of the covering quality is the ratio

$$\sigma(\Xi_n) = \frac{\sum_{i=1}^n \mu(M)}{\mu(O(\mathbf{s}_i, \alpha_i r))} \quad (4.1)$$

of the sum of the areas of the circles included in the coverage  $\Xi_n$  to the area  $\mu(M)$  of the figure  $M$ . The parameter  $\sigma(\Xi_n)$  is called the covering density. Note that it differs from the classical definition of density, where one take into account only the area of the part of circles that intersect  $M$ .

The quality index (4.1) can be easily calculated for figures of various geometries. Moreover, it is invariant with respect to the compression/extension and plane motion transformations. It can be expressed in terms of the parameter  $r$  as

$$\sigma(\Xi_n) = \frac{\mu(M)}{\pi r^2 \sum_{i=1}^n \alpha_i^2}.$$

In all the examples presented below, the solution is found by repeatedly launching the developed software. The coordinates of the centers of the circles corresponding to the minimum parameter  $r$  are used to restart the computational scheme with the introduction of random perturbations.

**Example 1.** Let the set  $M = \{(x, y) \in \mathbb{R}^2 : y \geq 0, x + y \leq 1, -x + y \leq 1\}$  be the right triangle with vertices  $(-1, 0), (0, 1), (1, 0)$ . It is required to find the optimal covering  $\Xi_{11}$  of the triangle  $M$  by combining 11 circles whose radii are proportional to the numbers  $\alpha_i = 1.5$  for  $1 \leq i \leq 3$  and  $\alpha_i = 1$  for  $4 \leq i \leq 11$ ; and  $\Xi_{12}$  with radii that are proportional to the numbers  $\alpha_i = 1.4$  for  $1 \leq i \leq 2$  and  $\alpha_i = 1$  for  $3 \leq i \leq 12$ .

The resulting set of covering circle centers of  $\Xi_{11}$ :

$$S_{11} = \{(0.4919, 0.2504), (-0.3319, 0.4741), (-0.7607, 0.1551), \\ (-0.3621, 0.1018), (0.2961, 0.6176), (0.8138, 0.0383), (0.0328, 0.7231), \\ (0.0807, 0.4287), (-0.0773, 0.8251), (0.2015, 0.1053), (-0.0791, 0.1441)\}.$$

Here  $r \approx 0.1912$ , the density of covering  $\sigma(\Xi_{11}) \approx 1.6935$ .

The resulting set of covering circle centers of  $\Xi_{12}$ :

$$S_{12} = \{(-0.1651, 0.4476), (-0.6898, 0.1718), (0.1239, 0.0572), \\ (0.2318, 0.6898), (0.6370, 0.1947), (-0.2118, 0.0572), (0.3422, 0.2042), \\ (0.4699, 0.4517), (0.2150, 0.3830), (-0.0238, 0.8243), \\ (0.4688, 0.0088), (0.8229, 0.0088)\}.$$

Here  $r \approx 0.1773$ , the density of covering  $\sigma(\Xi_{12}) \approx 1.7776$ .

**Example 2.** Let the set

$$M = \{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} \leq 2, \min\{|x|, |y|\} \leq 1\}$$

be the non-convex dodecagon. It is required to find the optimal covering  $\Xi_7$  of the dodecagon  $M$  by combining 7 circles whose radii are proportional

to the numbers  $\alpha_i = 1.25$  for  $1 \leq i \leq 2$  and  $\alpha_i = 1$  for  $3 \leq i \leq 7$ ; and  $\Xi_8$  with radii that are proportional to the numbers  $\alpha_i = 1.4$  for  $1 \leq i \leq 3$  and  $\alpha_i = 1$  for  $4 \leq i \leq 8$ .

The resulting set of covering circle centers of  $\Xi_7$ :

$$S_7 = \{(-1.4876, -0.0204), (-0.0156, -1.4970), (0.0099, 0.2295), \\ (-1.1758, 1.4887), (0.2729, 1.4970), (1.3757, 0.3739), (1.1176, -1.0571)\}.$$

Here  $r \approx 0.8844$ , the density of covering  $\sigma(\Xi_7) \approx 1.6637$ .

The resulting set of covering circle centers of  $\Xi_8$ :

$$S_8 = \{(0.3227, -1.0558), (-1.0726, -1.5), (-1.1286, -0.0484), \\ (-0.6667, 1.4695), (1.4354, -0.5), (0.4349, 1.5), (0.2164, 0.3753), \\ (1.4354, 0.5)\}.$$

Here  $r \approx 0.7545$ , the density of covering  $\sigma(\Xi_8) \approx 1.6216$ .

**Example 3.** Let the set  $M = \{(x, y) \in \mathbb{R}^2: |x| \leq 1, |y| \leq 1\}$  be the square with sides parallel to the coordinate axes and equal to 2. It is required to find the optimal covering  $\Xi_9$  of the square  $M$  by combining 9 circles whose radii are proportional to the numbers  $\alpha_i = 1.4$  for  $1 \leq i \leq 2$ ,  $\alpha_i = 1.2$  for  $3 \leq i \leq 4$ , and  $\alpha_i = 1$  for  $5 \leq i \leq 9$ .

The resulting set of covering circle centers of  $\Xi_9$ :

$$S_9 = \{(0.4555, 0.2012), (-0.7076, 0.4418), (0.7212, 0.6702), \\ (0.0134, 0.9486), (-0.9141, -0.5582), (-0.3822, -0.9497), \\ (-0.2336, -0.33), (0.9159, -0.5579), (0.4483, -0.7644)\}.$$

Here  $r \approx 0.4501$ , the density of covering  $\sigma(\Xi_9) \approx 1.8775$ . Figure 1 shows the covering  $\Xi_9$ .

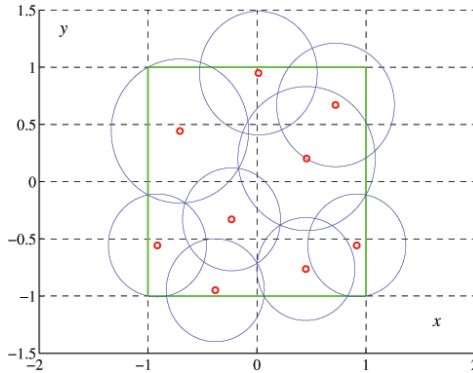


Figure 1. Covering of the square by 9 circles.

In order to verify the algorithms, a series of experiments was carried out for the total number of circles  $n = 8$ . Radii can be equal to two values  $R$  and  $r$ , while the ratio is  $R/r = 1.5$ . Cases from 0/8 to 7/1 are considered (the first numeral shows the number of small circles, the second – large ones). Table 1 presents the results of the calculations.

Table 1

Covering of the square by 8 circles

No	Number of small circles	Number of large circles	Radius $r$	Density $\sigma$
1	8 (0)	0 (8)	0.5212	1.7068
2	7	1	0.4677	1.5892
3	6	2	0.4386	1.5864
4	5	3	0.4164	1.6001
5	4	4	0.4092	1.7096
6	3	5	0.3851	1.6598
7	2	6	0.3717	1.6819
8	1	7	0.3701	1.8020

Note that the radii of the circles decrease monotonously with an increase in the number of large circles. And when we switch from 2/6 case to 1/7 one, the difference is observed only in the third digit. One can also see that the density of the covering behaves non-monotonously. There are two local maximums 6/2 and 3/5 and two local maximums 4/4 and 1/7. One of the maximums appears if we supplement table 1 with the case 0/8, which coincides with 8/0. Thus, the hypothesis that in the best coverage, the ratio of the number of small and large circles should be inversely proportional to their radii was not confirmed.

## 5. Conclusions

We considered the problems of covering a bounded set on a plane by a given number of circles whose radii, generally speaking, are different and proportional with fixed coefficients to a parameter  $r$ . It is the objective function to be minimized. We proved a theorem on the structure of the influence zone of a point (generalized Dirichlet zone), which is the center of the covering circle. An iterative algorithm for solving the considered problem was proposed, the relaxation property was proved, and a speed estimate was obtained.

Further research is aimed both at increasing the dimension of the problem, and at increasing the number of types of circles.

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## О покрытии ограниченных множеств наборами кругов различных радиусов

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**Аннотация.** Рассмотрена задача о построении оптимального покрытия плоской фигуры объединением кругов. Радиусы кругов, вообще говоря, различны. Каждый из них равен произведению некоторого положительного коэффициента на общий для всех параметр  $r$ , который и является целевой функцией, подлежащей минимизации. Проведено аналитическое исследование задачи. Получены выражения, позволяющие описать обобщенные зоны Дирихле для рассмотренного случая. Показано, что они существенно отличаются от классических зон Дирихле. Предложена итерационная процедура коррекции координат центров кругов, образующих покрытие, которая основана на отыскании чебышевских центров областей влияния точек. Показано, что она не ухудшает свойства покрытия. Предложен вычислительный алгоритм, использующий метод мултистарта для генерации начальных положений точек и итерационную процедуру. Выполнена его реализация в виде компьютерной программы. Проведены численные эксперименты по построению оптимальных покрытий наборами кругов при различных коэффициентах, определяющих радиус каждого из них. Рассмотрены случаи двух и трех различных типов кругов. В качестве покрываемых множеств взяты многоугольники: как выпуклые, так и невыпуклые, выполнена визуализация вычислений. Проведен анализ результатов расчетов, который позволил сделать содержательные выводы о свойствах построенных покрытий.

**Ключевые слова:** оптимизация, покрытие кругами, обобщенная зона Дирихле, чебышевский центр, итерационный алгоритм, вычислительный эксперимент.

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