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On the Behaviour at Infinity of Solutions to Nonlocal Parabolic Type Problems

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Abstract. The paper deals with possible behaviour at infinity of solutions to the Cauchy problem for a parabolic type equation whose elliptic part is the generator of a Markov jump process , i.e. a nonlocal diffusion operator. The analysis of the behaviour of the solutions at infinity is based on the results on the asymptotics of the fundamental solutions of nonlocal parabolic problems. It is shown that such fundamental solutions might have different asymptotics and decay rates in the regions of moderate, large and super-large deviations. The asymptotic formulae for the said fundamental solutions are then used for describing classes of unbounded functions in which the studied Cauchy problem is well-posed. We also consider the question of uniqueness of a solution in these functional classes.

Keywords: nonlocal operators, parabolic equations, fundamental solution, Markov jump process with independent increments.

1. Introduction and Statement of the Problem

1.1. Introduction. The Contact Model

Parabolic type equations with a nonlocal elliptic operator on the right-hand side play an important role in the analysis of a population evolution in models of mathematical biology and population dynamics. The presence of a nonlocal operator on the right-hand side of the equation reflects the fact that the interaction in these models has a nonlocal character. Let us describe one of these models, the so-called continuum contact model in $\mathbb{R}^d$, see e.g. [7–9]. It is a continuous time birth and death Markov process in continuum defined on the space of infinite (but locally finite) configurations $\gamma \in \Gamma$ lying in the space $\mathbb{R}^d$: $\gamma \subset \mathbb{R}^d$. The process is characterized by the birth and death rates. Each point $x \in \gamma$ of a configuration γ might create an offspring y independently on other points of the configuration. The offspring location is distributed in the space with the density $a(x-y)$ (so-called dispersal kernel), and we assume $\int_{\mathbb{R}^d} a(z)dz = 1$. In addition any point of the configuration has an independent exponentially distributed random life time determined by the mortality rate $m(x) > 0$, and in the general case the mortality rate is a spatially inhomogeneous function $m(x) \geq 0$. The generator of the dynamics of this process takes the form

$$LF(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} a(x-y) \left(F(\gamma \cup y) - F(\gamma)\right) dy + \sum_{x\in\gamma} m(x) \left(F(\gamma \backslash x) - F(\gamma)\right).$$

The case of homogeneous mortality $m(x) \equiv \kappa$ has been studied in details in the paper [7]. The most interesting case is $\kappa = 1$ - the critical regime, when a family of stationary distributions exist.

One of the remarkable property of the contact model is the fact that the evolution equation on the first correlation function (so-called density of configurations) is decoupled and can be considered separately. That is the case only for the first correlation function, evolutions of the higher order correlation functions have more complicated hierarchical structure involving lower order correlation functions. The evolution problem has the form

$$\frac{\partial u}{\partial t} = Au, \quad u = u(t,x), \quad x \in \mathbb{R}^d, \ t \geq 0, \quad u(0,x) = u_0(x) \geq 0, \quad (1.1)$$

where

$$Au(x) = -m(x)u(x) + \int_{\mathbb{R}^d} a(x-y)u(y)dy. \quad (1.2)$$

If $m(x) \equiv 1$, then the operator A takes the form

$$Au(x) = -u(x) + \int_{\mathbb{R}^d} a(x-y)u(y)dy = \int_{\mathbb{R}^d} a(x-y)(u(y)-u(x))dy. \quad (1.3)$$

We notice that correlation functions in the contact model, as well as in other models of the population dynamics, need not vanish at infinity, and in some models they can even grow. Thus to study the behaviour of correlation functions we have to consider the evolution equations (1.1)-(1.2) in suitable classes of bounded or increasing functions.

1.2. Estimates of fundamental solutions to some parabolic type problems

In this section we consider some important classes of parabolic type equations and give a short review of known results on the asymptotic behaviour of the corresponding fundamental solutions.

The fundamental solution of the classical heat equation

$$\partial_t u - \Delta u = 0,$$

where Δ is the Laplace operator in $\mathbb{R}^d$, is given by the Gauss-Weierstrass function

$$p_t\left(x,y\right) = \frac{1}{\left(4\pi t\right)^{d/2}} \exp\Big(- \frac{\left|x-y\right|^2}{4t}\Big). \tag{1.4}$$

For a general parabolic equation

$$\partial_t u - Lu = 0,$$

where L is a uniformly elliptic second-order operator in divergence form, the Aronson estimates, see [1], for the fundamental solution are well-known:

$$p_t\left(x,y\right) \asymp \frac{C}{t^{d/2}} \exp\Big(- \frac{\left|x-y\right|^2}{ct}\Big),$$

where the sign $\asymp$ means that both $\leq$ and $\geq$ inequalities hold with probably different constants $c > 0$ and $C > 0$.

The fundamental solution of parabolic type equation

$$\partial_t u + \left(-\Delta\right)^{\alpha/2} u = 0, \quad 0 < \alpha < 2,$$

where $\left(-\Delta\right)^{\alpha/2}$ is an integro-differential operator of the form

$$\left(-\Delta\right)^{\alpha/2} f\left(x\right) = c_{d,\alpha}\, \text{p.v.} \int_{\mathbb{R}^d} \frac{f\left(x\right) - f\left(y\right)}{\left|x-y\right|^{d+\alpha}} dy, \tag{1.5}$$

has been studied using the subordination techniques, see [2; 5]. The following asymptotic relation holds:

$$p_t\left(x,y\right) \asymp \frac{C}{t^{d/\alpha}} \left(1 + \frac{\left|x-y\right|}{t^{1/\alpha}}\right)^{-(d+\alpha)}$$

1.3. The statement of the problem

In this paper we are concerned with parabolic type equations, where instead of the elliptic differential operator L we consider its nonlocal analog, namely, the convolution type operator A given by

$$Af(x) = \int_{\mathbb{R}^d} (f(y) - f(x))\, a(x-y)\, dy, \tag{1.6}$$

where the convolution kernel $a(x)$ is a nonnegative, even, bounded, integrable function with bounded second moments:

$$a(x) \geq 0; \quad a(x) = a(-x); \qquad a(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d), \tag{1.7}$$

$$\int_{\mathbb{R}^d} a(x)dx = 1, \quad \int_{\mathbb{R}^d} |x|^2 a(x)dx < \infty. \tag{1.8}$$

In particular, condition (1.8) implies that the matrix $\sigma = \{\sigma_{ij}\}$ with

$$\sigma_{ij} = \int_{\mathbb{R}^d} x_i x_j a(x) dx$$

is well defined and positive definite. It follows from (1.7) that $a(x) \in L^2(\mathbb{R}^d)$, and for its Fourier transform $\hat{a}(p)$ we have:

$$\hat{a}(p) \in C_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \quad \max_{\mathbb{R}^d} \hat{a}(p) = \hat{a}(0) = 1, \quad \hat{a}(p) \to 0 \text{ as } |p| \to \infty.$$

Moreover, we assume that the convolution kernel $a(x)$ has a light tail at infinity:

$$a(x) \leq c e^{-b|x|^p} \qquad \text{with some } b > 0 \quad \text{and} \quad p \geq 1. \tag{1.9}$$

Since A is a bounded operator in $L^2(\mathbb{R}^d)$, its heat semigroup e^{tA} admits the following representation:

$$e^{tA} = e^{-t} e^{ta*} = e^{-t} \sum_{k=0}^{\infty} t^k \frac{a^{*k}}{k!} = e^{-t}\mathrm{Id} + e^{-t} \sum_{k=1}^{\infty} t^k \frac{a^{*k}}{k!}.$$

This sum contains the singular part $e^{-t}\mathrm{Id}$ and the regular part

$$v(x,t) = e^{-t} \sum_{k=1}^{\infty} t^k \frac{a^{*k}(x)}{k!}. \tag{1.10}$$

Therefore, the fundamental solution of (1.6) has the form

$$u(x,t) = e^{-t}\delta(x) + v(x,t), \tag{1.11}$$

and for any $f \in L^2(\mathbb{R}^d)$ the solution to the nonlocal Cauchy problem

$$\partial_t u - Au = 0, \qquad u\big|_{t=0} = f \tag{1.12}$$

has the form

$$u(x,t) = e^{-t} f(x) + (v * f)(x,t), \tag{1.13}$$

where v is defined by (1.10). Notice the similarity of the representation (1.13) for the solution of nonlocal problem (1.12) and the Poisson integral for the classical Cauchy problem. In the present work we study unbounded at infinity solutions of problem (1.12) using formula (1.13) and the asymptotical estimates of the function $v(x,t)$. Some particular cases of convolution kernels have been considered earlier in [4]. Our approach applies to the generic convolution kernels that satisfy the above conditions.

It should also be noticed that there is a crucial difference between the nonlocal operators defined in (1.5) and in (1.6). Namely, in contrast with the nonlocal operator in (1.5) the operator A defined in (1.6) has an integrable kernel $a(x-y)$. It is also useful to note the probabilistic interpretation of the function $v(x,t)$. Under conditions (1.7), (1.8) the operator A given by (1.6) is a generator of a continuous time Markov jump process. If this process starts at zero, then formula (1.11) determines transition probabilities of the process at time t, and $v(x,t)$ is the density of the process under the condition that at least one jump has been made.

2. Asymptotic estimates of $v(x,t)$ as $t \to \infty$

The asymptotic behaviour of the function $v(x,t)$ depends crucially on the relation between $|x|$ and t. We consider four regions in space-time (x,t):
1) $|x| \le rt^{1/2}(1+o(1))$ (standard deviations region)
2) $|x| = r\, t^{\frac{1+\delta}{2}}(1+o(1)),\ 0<\delta<1$ (moderate deviations region)
3) $|x| = rt(1+o(1))$ $(\delta = 1)$ (large deviations region)
4) $|x| = rt^{\frac{1+\delta}{2}}(1+o(1)),\ \delta>1$ ("extra-large" deviations region)

Theorem 1 (see [6]). *Assume that $a(x)$ satisfies conditions (1.7) - (1.9). Then for the function $v(x,t)$ the following asymptotic relations hold as $t \to \infty$ in regions of standard and moderate deviations:*
1) if $|x| \le rt^{\frac{1}{2}}$ for some $r>0$, then

$$v(x,t) = \frac{c(\sigma)}{t^{\frac{d}{2}}}\ e^{-\frac{(\sigma^{-1}x,x)}{2t}}\,(1+o(1))\,, \tag{2.1}$$

where $c(\sigma)>0$ is a constant depending on the covariance matrix σ;
2) if $x = rt^{\frac{1+\delta}{2}}(1+o(1))$ with $0<\delta<1$ and $r \in R^d\backslash\{0\}$, then

$$v(x,t) = e^{-\frac{(\sigma^{-1}x,x)}{2t}(1+o(1))} = e^{-\frac{1}{2}(\sigma^{-1}r,r)\,t^{\delta}(1+o(1))}. \tag{2.2}$$

3) If $x = rt(1+o(1))$ (the region of large deviations), then we get

$$v(x,t) \le e^{-\Phi(r)t(1+o(1))}, \qquad t \to \infty. \tag{2.3}$$

The rate function $\Phi(r)$ possesses the following important properties: $\Phi(0)=0$, $\Phi(r)>0$ for $r \neq 0$, Φ is a convex function,

$$\Phi(r) = \frac{1}{2}(\sigma^{-1}r, r)(1+o(1)), \qquad as \;\; |r| \to 0. \tag{2.4}$$

In addition, if $p=1$, then

$$\Phi(r) = b|r|(1+o(1)), \quad |r| \to \infty, \tag{2.5}$$

and if $a(x)$ has a compact support, then

$$\Phi(r) \ge \frac{1}{\mu}|r|\ln|r| \qquad |r| \to \infty, \tag{2.6}$$

where μ depends on the support of $a(x)$.

If the function $a(x)$ satisfies the following two-sided estimate

$$C_2 e^{-b|x|^p} \le a(x) \le C_1 e^{-b|x|^p}, \quad p \ge 1,$$

then the following asymptotic formula holds

$$v(x,t) = e^{-\Phi(r)t(1+o(1))}, \qquad t \to \infty. \tag{2.7}$$

Here the function $\Phi(r)$ for $p=1$ is defined by (2.5), and if $p>1$, then

$$\Phi(r) = \frac{p}{p-1}\big(b(p-1)\big)^{1/p}|r|(\ln|r|)^{\frac{p-1}{p}}(1+o(1)), \qquad |r| \to \infty. \tag{2.8}$$

Theorem 2. *In the region of "extra-large" deviations, when $|x| \gg t$, the following estimate holds for all sufficiently large t:*

$$v(x,t) \le \exp\Big\{ -c|x|\big(\ln|\frac{x}{t}|\big)^{\frac{p-1}{p}}\Big\}. \tag{2.9}$$

If $a(x)$ has a compact support, then in the region of "extra-large" deviations and large t:

$$v(x,t) \le \exp\Big\{ -c|x|\ln|\frac{x}{t}|\Big\}. \tag{2.10}$$

It should be noted that the Gaussian form of the asymptotics (2.1) in the region of standard deviations is the immediate consequence of the local limit theorem for processes with independent increments. Formula (2.1) can also be derived from the asymptotic representation of the corresponding Fourier transform, see e.g. [3]. In the moderate deviations region the asymptotics of the fundamental solution still coincide with that in the standard deviations region, but only in the logarithmic order. For the pre-exponential factor we can only state the sub-exponential rate of decay. Crucial modifications of the Gaussian form of the asymptotics occurs in the region of large deviations, when $x = rt$, see formulae (2.3), (2.7). It is there, at the distances of order t, that the nonlocal character of the operator A starts to play an important role. As seen from (2.4), the fundamental solution is still close to the Gaussian function for small r, but it differs essentially from the corresponding Gaussian function for sufficiently large r, see (2.5), (2.8), (2.6). In the "extra-large" deviations region this difference is further enhanced. As follows from estimates (2.9), (2.10) the nonlocal fundamental solution $v(x,t)$ has more heavy tail at infinity than the classical heat kernel (1.4).

3. Classes of unbounded solutions

Let us observe that the formula (1.13) makes sense for a wider class of initial functions $f(x)$ than the class $L^2\left(\mathbb{R}^d\right)$. For the classical heat equation one can take as the initial data a function $f(x)$ growing at infinity. Then using the representation for the solution through the Poisson integral one can conclude, see e.g. [10], that if $f(x)$ is a continuous function satisfying estimate

$$|f(x)| \le Ce^{bx^2}, \quad b > 0, \tag{3.1}$$

then the solution exists as $0 < t < 1/4b$. Moreover, the solution also satisfies an estimate of type (3.1), and it is unique in this class.

A similar statement holds for the nonlocal parabolic type problems considered in this paper. It is clear that the admissible growth of the initial condition will be determined by the behavior of the fundamental solution at infinity, i.e. in the region of "extra-large" deviations.

We need the following lemma.

Lemma 1. *There exists a constant $\hat{c} > 0$ such that in the region $\{(x,t) : t > 0,\ \frac{|x|}{t} \gg 1\}$ the following estimate holds*

$$v(x,t) \le \exp\Big\{ - \hat{c}|x| \Big(\ln \Big|\frac{x}{t}\Big|\Big)^{\frac{p-1}{p}} \Big\} \tag{3.2}$$

for $p \ge 1$, and

$$v(x,t) \le \exp\Big\{ - \hat{c}|x| \Big(\ln \Big|\frac{x}{t}\Big|\Big)\Big\} \tag{3.3}$$

if $a(x)$ has a compact support.

Proof. For the proof of (3.3) we use representation (1.10). According to estimates (3.60)-(3.61) from [6], there exist constants $\alpha_p > 0$ and $\varkappa > 0$ such that for all sufficiently large x and all k with $1 \le k \le \alpha_p\,|x|$ we have:

$$a^{*k}(x) \le \exp\Big\{ - \varkappa \frac{|x|^p}{k^{p-1}}\Big\}.$$

If k satisfies $1 \le k \le |x|\big(\log\big(\frac{|x|}{t}\big)\big)^{-\frac{1}{p}}$, then

$$a^{*k}(x) \le \exp\Big\{-\varkappa|x|\Big[\Big(\log\Big(\frac{|x|}{t}\Big)\Big)^{\frac{1}{p}}\Big]^{p-1}\Big\} = \exp\Big\{-\varkappa|x|\Big(\log\Big(\frac{|x|}{t}\Big)\Big)^{\frac{p-1}{p}}\Big\}.$$

We also have

$$\sum_{k\le|x|\left(\log\left(\frac{|x|}{t}\right)\right)^{-\frac{1}{p}}} \frac{t^k\, e^{-t}}{k!} \le 1.$$

Notice that the relation $|x| \gg t$ implies $|x|\big(\log\big(\frac{|x|}{t}\big)\big)^{-\frac{1}{p}} \gg t$. If $k \ge |x|\big(\log\big(\frac{|x|}{t}\big)\big)^{-\frac{1}{p}}$, then, by the Stirling formula,

$$\frac{t^k}{k!} \le \exp\Big\{-k\log\Big(\frac{k}{t}\Big)+k\Big\} \le \exp\Big\{-\frac{1}{2}|x|\Big(\log\Big(\frac{|x|}{t}\Big)\Big)^{-\frac{1}{p}}\log\Big(\frac{|x|}{t}\Big)\Big\}$$
$$\le \exp\Big\{-\frac{1}{2}|x|\Big(\log\Big(\frac{|x|}{t}\Big)\Big)^{\frac{p-1}{p}}\Big\}.$$

Combining the last three estimates yields the desired inequality (3.2).

The proof of (3.3) relies on similar arguments. Since $a(\cdot)$ has a finite support, then

$$v(x,t) = e^{-t}\sum_{k>\mu|x|}\frac{t^k a^{*k}(x)}{k!} \le C_1 e^{-t}\exp\Big\{\max_{k>\mu|x|t} S_0(k,t)\Big\},$$

where μ is a constant that depends on the size of the support of $a(\cdot)$, and $S_0(k,t) = k\ln\frac{t}{k} + k$. One can easily check that $S_0(k,t)$ is a decreasing function of k, if $k > t$. Since $|x| \gg t$, we have

$$\max_{k>\mu|x|} S_0(k,t) = S_0(\mu|x|,t) = -\mu|x|\ln\frac{|x|}{t}(1+o(1)).$$

This yields estimate (3.3). □

Remark 1. It should be noted that in the formulation of Lemma the value of t might be finite and arbitrary small. In addition, for all sufficiently large x the constant $\hat{c}$ in estimate (3.3) is greater than the corresponding constant c in bounds (2.9) - (2.10).

We now turn to the main result of the paper.

Theorem 3. *Let conditions* (1.7) - (1.9) *on the function* $a(x)$ *be satisfied, and assume that the initial condition* $f(x)$ *is a continuous function such that*

$$|f(x)| \le K e^{b|x|(\ln|x|)^{\frac{p-1}{p}}}, \qquad \textit{with } 0 < 2b < c \tag{3.4}$$

if $p \ge 1$, *or*

$$|f(x)| \le K e^{b|x|\ln|x|}, \qquad \textit{with } \ 0 < 2b < c, \tag{3.5}$$

in the case, when $a(x)$ *has a compact support. Here* c *is the same constant as in* (2.9) *or* (2.10), *respectively.*

Then for any $t > 0$ *there exists the solution of Cauchy problem* (1.12) *defined by formula* (1.13). *This solution satisfies the upper bound*

$$|u(x,t)| \le K_p(t) e^{\tilde{c}|x|(\ln|x|)^{\frac{p-1}{p}}}, \tag{3.6}$$

as $p \ge 1$, *or*

$$|u(x,t)| \le K_\infty(t) e^{\tilde{c}|x|\ln|x|} \tag{3.7}$$

in the case, when $a(x)$ *has a compact support, where* $\tilde{c}$ *is a constant such that* $2b < \tilde{c} < c$.

Moreover, the solution of the Cauchy problem is unique in the class of functions that satisfy growth condition (3.6) *or* (3.7), *respectively.*

Proof. First we prove the existence of solution of the Cauchy problem in the class of functions that satisfy estimate (3.6) (or (3.7)). As follows from representation (1.13) it is sufficient to estimate $(v*f)(x,t)$. In what follows we consider the case of $a(x)$ with a compact support. If the function $a(x)$ meets the general bound (1.9) with some $p \ge 1$, the reasoning will be similar. We first estimate $(v*f)(x,t)$ for $|x| > \varkappa t^{\frac{c}{c-2b}}$ with $\varkappa = \varkappa(c,b) = 2^{\frac{c+2b}{c-2b}}$:

$$\begin{aligned}(v*f)(x,t) &= \int_{\mathbb{R}^d} v(z,t) f(x-z)dz \\ &= \int\limits_{|z|\le|x|} v(z,t)f(x-z)dz + \int\limits_{|z|>|x|} v(z,t)f(x-z)dz.\end{aligned} \tag{3.8}$$

For the first integral in (3.8) we have

$$\int\limits_{|z|\le|x|} v(z,t)f(x-z)dz \le \int\limits_{|z|\le|x|} v(z,t)Ke^{b|x-z|\ln|x-z|}dz$$

$$\le K \int\limits_{|z|\le|x|} v(z,t)e^{2b|x|\ln 2|x|}dz \le Ke^{\tilde{c}|x|\ln|x|} \int\limits_{|z|\le|x|} v(z,t)dz \le Ke^{\tilde{c}|x|\ln|x|},$$

where $2b < \tilde{c} < c$. We have used here the inequality $|x-z| \le 2|x|$ that is valid for $|z| \le |x|$, and the estimate $|x|\ln 2|x| < (1+\varepsilon)|x|\ln|x|$, that holds for sufficiently large $|x|$ and small $\varepsilon > 0$.

For estimating the second integral on the right-hand side in (3.8) we use the inequality $|x-z| \le 2|z|$ and the asymptotic formula (2.10) for $v(z,t)$

in the region of "extra-large" deviations $|z| \ge |x| > \varkappa t^{\frac{c}{c-2b}}$:

$$\int\limits_{|z|>|x|} v(z,t)f(x-z)dz \le \int\limits_{|z|>|x|>\varkappa t^{\frac{c}{c-2b}}} e^{-c|z|\ln|\frac{z}{t}|} K e^{2b|z|\ln 2|z|} dz.$$

If $|z| > 2^{\frac{c+2b}{c-2b}}\, t^{\frac{c}{c-2b}}$, then

$$-c|z|\ln 2|z| + 2b|z|\ln 2|z| + c|z|\ln 2t = -|z|\big((c-2b)\ln|2z| - c\ln 2t\big) \le -\alpha|z| \tag{3.9}$$

with some $\alpha > 0$. Consequently,

$$\int\limits_{|z|>\varkappa t^{\frac{c}{c-2b}}} v(z,t)f(x-z)dz < K \int\limits_{|z|>\varkappa t^{\frac{c}{c-2b}}} e^{-\alpha|z|}dz = K_1(t) < \infty. \tag{3.10}$$

Thus, for $|x| > \varkappa t^{\frac{c}{c-2b}}$ the following estimate holds:

$$|u(x,t)| \le K_1(t) e^{\tilde{c}|x|\ln|x|} \tag{3.11}$$

with $2b < \tilde{c} < c$.

In the case, when $|x| < \varkappa t^{\frac{c}{c-2b}}$, we estimate each of integral on the right-hand side of (3.8) separately. For the first integral we get the same estimate as above:

$$\int\limits_{|z|\le|x|} v(z,t)f(x-z)dz \le \int\limits_{|z|\le|x|} v(z,t)K e^{b|x-z|\ln|x-z|}dz \le K e^{\tilde{c}|x|\ln|x|}.$$

We divide the second integral into two integrals:

$$\int\limits_{|z|>|x|} v(z,t)f(x-z)dz \tag{3.12}$$
$$= \int\limits_{|x|<|z|<\varkappa t^{\frac{c}{c-2b}}} v(z,t)f(x-z)dz + \int\limits_{|z|>\varkappa t^{\frac{c}{c-2b}}} v(z,t)f(x-z)dz.$$

Considering the inequality $|x-z| \le 2|z|$, for the first integral on the right-hand side we obtain

$$\int\limits_{|x|<|z|<\varkappa t^{\frac{c}{c-2b}}} v(z,t)e^{2b|z|\ln 2|z|}dz \le e^{2b\varkappa t^{\frac{c}{c-2b}}\ln 2\varkappa t^{\frac{c}{c-2b}}|} \int\limits_{\mathbb{R}^d} v(z,t)dz = K_2(t).$$

The second integral admits the same bound as above:

$$\int\limits_{|z|>\varkappa t^{\frac{c}{c-2b}}} v(z,t)f(x-z)dz < K_1(t).$$

Thus, for $|x| < \varkappa t^{\frac{c}{c-2b}}$ we also obtain the desired estimate (3.11).

In the case of kernels $a(\cdot)$ satisfying condition (1.9) the existence of a solution in the class of functions for which estimate (3.6) holds can be proved in a similar way. There is only one difference. Namely, in this case we should show that for all sufficiently large $|z|$ the following inequality holds:

$$-c|z|\big(\ln\frac{|z|}{t}\big)^q + 2b|z|\big(\ln 2|z|\big)^q \le -\alpha|z|, \quad q = \frac{p-1}{p} \in [0,1). \tag{3.13}$$

This inequality ensures that the integral in (3.10) is finite. To justify (3.13) it suffices to show that for$|z| > \varkappa t^\gamma$ with some $\varkappa > 0$ and $\gamma > 0$ we have

$$\Big(\ln\frac{|2z|}{2t}\Big)^q > \frac{2b}{c}\big(\ln 2|z|\big)^q. \tag{3.14}$$

The validity of the latter inequality can be easily checked if we let $\gamma = \big(1-(\frac{2b}{c})^{1/q}\big)^{-1}$ and $\varkappa = 2^\gamma$.

The next step is to prove the uniqueness of solution of the Cauchy problem for nonlocal parabolic equation. The proof is based on Holmgren's principle and follows the line of Section 7.7 in [10]. Let $t_0 > 0$, and assume that $u(x,0) = 0,\ x \in \mathbb{R}^d$. We want to prove that this initial condition implies that $u(x,t) \equiv 0$ in the whole strip $[0,t_0) \times \mathbb{R}^d$.

Consider the adjoint Cauchy problem

$$\frac{\partial w}{\partial t} = -\int_{\mathbb{R}^d} a(x-y)(w(y)-w(x))dy, \qquad w\big|_{t=t_0} = \psi(x),$$

with the terminal condition $\psi(x)$, where $\psi \in \mathcal{D}(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d)$ belongs to the space of smooth functions with compact support. The solution of this problem admits the representation similar to that in (1.13). It reads

$$w(x,t) = e^{-(t_0-t)}\psi(x) + (v * \psi)(x, t_0 - t), \quad t < t_0, \tag{3.15}$$

where the function $v(x,t)$ was defined by (1.10). Since ψ has a compact support, the solution (3.15) satisfies

$$|w(x,t)| \le C_\psi \max_{y\in \operatorname{supp}\psi} v(x-y, t_0-t) \tag{3.16}$$

for $t < t_0$ and $|x| \ge R > 0$, where R is sufficiently large.

From (3.3), (3.16) and Remark 1 it follows that

$$|w(x,t)| \le C_\psi \exp\Big\{-c|x|\Big(\ln\frac{|x|}{\max\{1, t_0-t\}}\Big)\Big\}, \quad 0 < t < t_0. \tag{3.17}$$

The corresponding estimate also holds in the case of $a(x)$ with a light tail at infinity with $p \ge 1$.

Thus, the estimate (3.17) implies that the integral

$$\langle u(x,t),\, w(x,t)\rangle = \int_{\mathbb{R}^d} u(x,t)w(x,t)dx$$

exists and converges uniformly for $0 < t < t_0$. Moreover, the integrals obtained by replacing u and w with their derivatives with respect to t also converge uniformly for $0 < t < t_0$. Therefore, the function

$$\chi(t) = \langle u(x,t),\, w(x,t)\rangle$$

is continuous at $t \in [0, t_0)$. Our assumption $u(x,0) = 0,\ x \in \mathbb{R}^d$, implies that $\chi(0) = 0$. The function $\chi(t)$ is differentiable on the intervbal $t \in (0, t_0)$, and by the symmetry condition on $a(x-y)$ we have

$$\begin{aligned}\frac{d\chi(t)}{dt} &= \int_{\mathbb{R}^d} \Big(\frac{\partial u(x,t)}{\partial t} \cdot w(x,t) + \frac{\partial w(x,t)}{\partial t} \cdot u(x,t)\Big) dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y)(u(y,t) - u(x,t))w(x,t)dydx \\ - &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y)(w(y,t) - w(x,t))u(x,t)dydx = 0.\end{aligned} \tag{3.18}$$

Thus, $\chi(t) \equiv const = 0$.

The formula (3.15) for $w(x,t)$ yields

$$\int_{\mathbb{R}^d} u(x,t_0)\psi(x)dx = \lim_{t\to t_0-0} \int_{\mathbb{R}^d} u(x,t)w(x,t)dx.$$

Since

$$\lim_{t\to t_0-0} \int_{\mathbb{R}^d} u(x,t)w(x,t)dx = \lim_{t\to t_0-0} \chi(t) = 0,$$

then $\int_{\mathbb{R}^d} u(x,t_0)\psi(x)dx = 0$ for every $\psi \in \mathcal{D}(\mathbb{R}^d)$. Consequently, $u(x,t) \equiv 0$ for $0 \le t \le t_0$. □

4. Conclusions

In this work we described some classes of initial conditions for which the nonlocal parabolic problems studied here are well-posed. It was shown that the initial conditions of exponential and even slightly stronger growth at infinity are admissible. Moreover, the critical growth condition is characterized by the behaviour at infinity of the convolution kernel of the corresponding nonlocal operator. In particular, it follows from our estimates that the class of admissible initial conditions for the studied here nonlocal Cauchy problem is more narrow than that for the classical heat equation. Such a difference between the structures of the classes of admissible initial conditions is caused by the fact that the fundamental solution of the nonlocal problem decays at infinity slower than the usual heat kernel, the difference in the behaviour becoming apparent at the distance of order t.

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О поведении на бесконечности решений нелокальных задач параболического типа

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Аннотация. Изучается возможное поведение на бесконечности решений задачи Коши для уравнений параболического типа, в которых в качестве эллиптического оператора берётся генератор марковского скачкообразного процесса, т. е. оператор нелокальной диффузии. Исследование поведения решений на бесконечности базируется на асимптотике фундаментального решения нелокальных параболических задач. Показано, что такое фундаментальное решение имеет разную асимптотику и скорость убывания в областях умеренных, больших и супер-больших уклонений. На основании этих асимптотических формул описаны классы неограниченных функций, в которых корректны рассматриваемые задачи Коши. Обсуждается также единственность решения в этих классах функций.

Ключевые слова: нелокальные операторы, параболические уравнения, фундаментальное решение, марковский скачкообразный процесс с независимыми приращениями.

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