Серия «Математика»
2019. Т. 30. С. 31—44

Онлайн-доступ к журналу:
http://mathizv.isu.ru

И З В Е С Т И Я
Иркутского государственного университета

УДК 517.977
MSC 93D15, 93C05, 49J15
DOI https://doi.org/10.26516/1997-7670.2019.30.31

Stabilization of Coupled Linear Systems Via Bounded Distributed Feedbacks

N. M. Dmitruk
Belarusian State University, Minsk, Republic of Belarus

Abstract. This article deals with a stabilization problem for a team of linear interconnected systems via bounded feedbacks. Effective approaches to stabilization of constrained systems from model predictive control theory are developed for the decentralized case when each system of the group is controlled by its local controller. We propose formulations of local optimal control problems and an algorithm based on them that constructs a distributed feedback guaranteeing asymptotic stability of the group.

Keywords: stabilization, feedback, distributed control.

1. Introduction

In recent years, control problems for teams of interacting dynamical systems has received a significant attention from the research community [6;7], which is motivated by a large amount of practical applications — these are control problems for teams of mobile robots, unmanned aerial vehicles, energy systems, transport systems, etc. In such applications classical control theory methods may not be applicable, since they assume centralized control of the whole team, often representing a large-scale system. Besides, they do not account for networked or communication restrictions (e.g., delays in the communication between systems) within the team. In these cases distributed control techniques are needed.

One approach to tackle stabilization problems, popular in theoretical research and in practice, is Model Predictive Control [15] (MPC), and Distributed Model Predictive Control (DMPC) [10; 11] for interconnected systems. Within the DMPC framework many approaches have been proposed for systems with coupled dynamics [9; 16] and multi-agent systems

[13; 14]. The underlying idea is to break a large-scale control problem into sub-problems (local problems) where only inputs of the local system are optimized. For multi-agent systems a stabilization problem is most studied [11; 13], however, other control objectives, such as consensus and synchronization, are also of great practical interest (see e.g. [14]).

In this paper we consider a stabilization problem for a team of linear time-invariant systems with coupled dynamics subject to delays in communication between the systems. The goal is to achieve asymptotic stability [2] of the team via distributed feedback control. The proposed algorithms develop ideas of centralized model predictive control methods based on linear programming [8; 15], related stabilization methods based on optimal damping problems [1; 5] and distributed feedback control schemes for optimal control problems developed in [3; 4; 12]. The focus is on constructing local optimization problems and analyzing information to be communicated between the systems in order to establish a rather small amount of data that is sufficient for the algorithm implementation.

2. Problem formulation

We consider a team of q linear time-invariant control systems with coupled dynamics of the form

$$\dot{x}_i = A_i x_i + \sum\nolimits_{j\in I_i} A_{ij}x_j + B_i u_i,\ x_i(0) = x_{i0}, \tag{2.1}$$

where $x_i = x_i(t) \in \mathbb{R}^{n_i}$ denotes the state, $u_i = u_i(t) \in \mathbb{R}^{r_i}$ denotes the control of the i-th system at time t, $i \in I = \{1, 2, \dots, q\}$, $A_i = A_{ii}$, B_i, A_{ij}, $j \in I_i = I \setminus \{i\}$, $i \in I$, are given matrices of respective dimensions. The matrix A_i characterizes system's self-dynamics, B_i is the input of the i-th system, the matrices A_{ij} characterize dynamical coupling between the systems in the team.

As feasible inputs we use sampled-data functions $u_i(t)$, $t \geq 0$, with the sampling time $h > 0$: $u_i(t) \equiv u_i(s)$, $t \in [s, s + h[$, $s \in T_h = \{0, h, \dots\}$.

Along with the team of systems (2.1) we consider its representation

$$\dot{x} = Ax + Bu,\ x(0) = x_0, \tag{2.2}$$

where $x = (x_1^T, \dots, x_q^T)^T \in \mathbb{R}^n$, $u = (u_1^T, \dots, u_q^T)^T \in \mathbb{R}^r$, $A = (A_{ij}, i, j \in I)$, $B = \mathrm{diag}(B_i, i \in I)$ is block diagonal, $n = \sum_{i\in I} n_i$, $r = \sum_{i\in I} r_i$.

Let $B_{(i)} = (0, \dots, B_i^T, \dots 0)^T \in \mathbb{R}^{n\times r_i}$. In the following we assume that for each $i \in I$ the pair $(A, B_{(i)})$ is controllable in the class of sampled-data inputs.

A function $u(x)$, $x \in \mathbb{R}^n$, is called a discrete feedback if for each x_0 the trajectory $x(t)$, $t \geq 0$, of the closed-loop system

$$\dot{x} = Ax + Bu(x),\ x(0) = x_0, \tag{2.3}$$

is a recursive solution of the linear equation $\dot{x} = Ax + Bu(x(s))$, $x(s) = x(s-0)$ $(x(-0) = x_0)$, on the intervals $t \in [s, s+h[$, $s \in T_h$. Obviously, the closed-loop system (2.3) has a unique solution.

Let D denote a region around the origin $x = 0$, and let $L > 0$ define a feasible input set in the form $U = \{u \in \mathbb{R}^r : \ ||u||_\infty \leq L\}$.

Definition 1. [1] *A discrete feedback $u(x)$, $x \in D$, is called a bounded stabilizing discrete feedback for (2.2) if: 1) $u(x) \in U$, $x \in D$; $u(0) = 0$; 2) for every $x_0 \in D$ the states $x(s)$, $s \in T_h$, of system (2.3) stay in D; 3) the trivial solution $x(t) \equiv 0$, $t \geq 0$, is asymptotically stable in D.*

Obviously, the feedback $u(x)$, $x \in D$, with properties 1)–3) is not uniquely defined. In this paper we consider two approaches. In Section 3 we review *centralized stabilization* based on MPC methods [8; 15] and related works [1; 5]. In Section 4 we propose a new approach to *distributed stabilization* which combines the methods from [1;5] and ideas from [3;4;12].

Throughout this paper the following notations are used: $x(t_1|t_0, x_0, u(\cdot))$ denotes the state at time instant t_1 of system (2.2) with the initial condition $x(t_0) = x_0$ and input $u(\cdot) = (u(t), t \in [t_0, t_1])$; $F(t) = e^{At}$, $t \geq 0$; $F_i(t) \in \mathbb{R}^{n \times n_i}$ denotes the corresponding block of the matrix $F(t)$, i.e. $F(t) = (F_1(t), \ldots, F_q(t))$; $D(s) = F(t_f - s - h) \int_0^h F(t)Bdt$, $D_i(s) = F(t_f - s - h) \int_0^h F_i(t)B_i dt$; $\mathbb{1}_p = (1, \ldots, 1) \in \mathbb{R}^p$. The following norms of the vector $y \in \mathbb{R}^p$ are used: $||y||_\infty = \max\{|y_1|, \ldots, |y_p|\}$, $||y||_Q^2 = y^T Q y$, $Q > 0$.

3. Centralized stabilization

In the centralized stabilization case the team has a single central controller, which, based on the team's current state, chooses a control input for all systems in the team. In what follows the team's current state is denoted by $x^*(\tau)$ and is assumed to be available for complete and accurate measurements. We stress that it may differ from the states of the mathematical model (2.2) due to inaccuracies of mathematical modeling, presence of disturbances, and other uncertainties.

As discussed in the introduction, a popular approach to solving stabilization problems is MPC. The overall idea of all MPC methods is based on repetitive solution at each current discrete time instant $\tau \in T_h$ the so-called predictive optimal control problem subject to a finite time interval $[0, t_f]$ $(t_f = Nh, N \in \mathbb{N})$, and the initial condition for the predictive model (2.2) coinciding with the current state $x^*(\tau)$. The general formulation of the predictive problem (in the centralized case) has the form

$$\begin{aligned} &J^0(x^*(\tau)) = \min J(u), \\ &\dot{x} = Ax + Bu, \ x(0) = x^*(\tau), \\ &x(t_f) \in X_f, \ u(t) \in U, \ t \in [0, t_f], \end{aligned} \tag{3.1}$$

where $J(u)$ is some cost; $X_f \subset D$ is the terminal set, $\{0\} \in X_f$. Problem (3.1) may also include path constraints if those are imposed on the transient trajectories.

Let $u^0(t|x^*(\tau))$, $t \in [0, t_f]$, denote the optimal open-loop input of problem (3.1).

The MPC algorithm can be described as follows: at each time instant $\tau \in T_h$, the controller solves the optimal control problem (3.1) and feeds the first value $u^0(0|x^*(\tau))$ of its optimal open-loop input to the team on the interval $[\tau, \tau + h[$. As a result, we obtain the so-called discrete feedback realization

$$u^*(t) \equiv u^0(x^*(\tau)) := u^0(0|x^*(\tau)), \quad t \in [\tau, \tau + h[, \quad \tau \in T_h.$$

Asymptotic stability of the closed-loop system is achieved by a proper choice of the cost $J(u)$ of the predictive problem (3.1) and a suitable terminal condition at time instant t_f. Significant attention in the literature is given to MPC schemes with quadratic costs $J(u)$ of the form

$$J(u) = \int_0^{t_f} ||x(t)||_Q^2 + ||u(t)||_R^2 \, dt + ||x(t_f)||_P^2, \ Q, R, P > 0,$$

and terminal sets X_f being ellipsoids for wich there exists a local linear feedback $u_{loc}(x) = Kx \in U$, $x \in X_f$, such that $A + BK$ is a Hurwitz matrix. This approach allows us to reduce problem (3.1) to a quadratically constrained quadratic program and to solve it efficiently, e.g. using the interior-point methods. Simplest approaches (see, e.g. [15]) use the terminal constraint $x(t_f) = 0$, i.e. $X_f = \{0\}$.

For linear systems a linear cost of the form

$$J(u) = \int_0^{t_f} ||Qx(t)||_\infty + ||Ru(t)||_\infty \, dt + ||Px(t_f)||_\infty$$

is popular, and allows to reduce the predictive problem (3.1) to a multiparametric linear program [8].

In this paper bounded stabilizing feedbacks are constructed according to the approach in [1], where we use

$$J(u) = \max_{t \in [0, t_f]} ||u(t)||_\infty, \quad X_f = \{0\}.$$

Hence the predictive problem is the optimal damping problem. For an arbitrary initial state $z \in \mathbb{R}^n$ it has the form

$$\mathcal{P}(z): \qquad \rho(z) = \min_u \max_{t \in [0, t_f]} ||u(t)||_\infty, \tag{3.2}$$
$$\dot{x} = Ax + Bu, \ x(0) = z, \quad x(t_f) = 0.$$

Under the controllability assumption of Section 2 and for a control horizon $t_f \geq nh/r$ every problem $\mathcal{P}(z)$ of family (3.2) has a solution that is denoted by $u^0(t|z)$, $t \in [0, t_f]$.

Let $D = \{z \in \mathbb{R}^n : \ \rho(z) \leq L\}$. In the following we assume that $x_0 \in D$.

In [1; 5] the following result is proved

Proposition 1. *A function $u^0(x) = u^0(0|x)$, $x \in D$, is a bounded stabilizing feedback for system (2.2).*

Algorithm 1 (centralized stabilizing control construction):
1) Set $\tau = 0$, $x^*(\tau) = x_0$.
2) Find an optimal open-loop input $u^0(t|x^*(\tau))$, $t \in [0, t_f]$, to the problem $\mathcal{P}(x^*(\tau))$.
3) Apply input $u^*(t) \equiv u^0(x^*(\tau)) := u^0(0|x^*(\tau))$, $t \in [\tau, \tau + h[$ to system (2.2).
4) Set $\tau := \tau + h$, return to Step 2).

The algorithm for centralized stabilization is specified as Algorithm 1. Now we briefly discuss how the problem $\mathcal{P}(x^*(\tau))$ is solved during Step 2) of Algorithm 1. In the class of sampled-data inputs it is equivalent to a linear program

$$\rho^*(\tau) = \min_{\rho, \mathbf{u}} \rho, \quad D\mathbf{u} = -g(x^*(\tau)), \quad \mathbf{u} - \rho \mathbb{1}_{rN} \leq 0, \ -\mathbf{u} - \rho \mathbb{1}_{rN} \leq 0, \quad (3.3)$$

where $\rho \in \mathbb{R}$, $\mathbf{u} = (u^T(0), u^T(h), \ldots, u^T(t_f - h))^T \in \mathbb{R}^{rN}$, $g(z) = F(t_f)z$; $D = (D(0), D(h), \ldots, D(t_f - h)) \in \mathbb{R}^{n \times rN}$.

The linear program (3.3) has $rN+1$ variables, n equality constraints and $2rN$ inequality constraints. The paper [1] proposes to reduce the problem dimension using the following change of variables: $\xi = 1/\rho$, $\mathbf{v} = \mathbf{u}/\rho$ ($\rho > 0$ for $x^*(\tau) \neq 0$). This results in a linear program

$$\xi^*(\tau) = \max_{\xi, \mathbf{v}} \xi, \quad g(x^*(\tau))\xi + D\mathbf{v} = 0, \quad ||\mathbf{v}||_\infty \leq 1, \quad (3.4)$$

with $rN + 1$ variables, n equality constraints and geometric constraints for the variables $\mathbf{v}$. Now we have $\rho^*(\tau) = 1/\xi^*(\tau)$. Since problem (3.2) has to be solved at each $\tau \in T_h$ for the current state $x^*(\tau)$ in time less than h, solving problem (3.4) is preferable to solving problem (3.3).

Using the optimal damping problem (3.2) for predictions as compared to the classical MPC approaches allows us to propose a rather simple approach to distributed feedback control, which is described in the next section.

4. Distributed stabilization

In the distributed stabilization case it is assumed that each system (2.1) has its own (local) controller that generates a (distributed) bounded stabilizing feedback only for the associated system. The local controller constructs the inputs as in centralized case, i.e. on the base of solution of the (local) predictive optimal control problem. This problem is denoted by $\mathcal{P}_i(x_i, \mathbf{z})$, where x_i is the state of the i-th system, $\mathbf{z}$ is information on other systems' behavior. In the particular control process it is assumed that by the current time instant $\tau \in T_h \setminus \{0\}$ information arriving from systems $k \in I_i$ consists of 1) the state $x_k^*(\tau - h)$, and 2) the input $u_k^*(\tau - h)$ applied to system k at the previous time $\tau - h$.

The above assumption means that communication is delayed by one sampling period h, and the current position of the control process, as it is available to the i-th controller at time instant τ, is $(x_i^*(\tau), \mathbf{z}^*(\tau))$, where $\mathbf{z}^*(\tau) = \{x_k^*(\tau - h), u_k^*(\tau - h), k \in I\}$.

The solution of the problem $\mathcal{P}_i(\tau) := \mathcal{P}_i(x_i^*(\tau), \mathbf{z}^*(\tau))$ is denoted by $u_i^d(t|\tau)$, $t \in [0, t_f]$, and is referred to as the local optimal open-loop input of system i predicted at time τ. For every τ we define an overall open-loop input $u^d(t|\tau) = (u_k^d(t|\tau),\ k \in I)$, $t \in [0, t_f]$, as an input composed of all local optimal open-loop inputs.

Following [3] and taking into account the centralized problem $\mathcal{P}(x^*(\tau))$ formulation, the local predictive problem $\mathcal{P}_i(\tau)$ of the i-th controller at time $\tau \in T_h \setminus \{0\}$ is formulated in the form

$$
\begin{aligned}
\mathcal{P}_i(\tau): \qquad & \rho_i(\tau) = \min_{u_i} \max_{t\in[0,t_f]} ||u_i(t)||_\infty, \qquad (4.1)\\
& \dot{x}_i = A_i x_i + \sum\nolimits_{j\in I_i} A_{ij} x_j + B_i u_i,\\
& \dot{x}_k = A_k x_k + \sum\nolimits_{j\in I_k} A_{kj} x_j + B_k u_k^d(t+h|\tau-h),\ k \in I_i,\\
& x_i(0) = x_i^*(\tau),\ x_k(0) = x_k^d(\tau|\tau-h),\ k \in I_i,\ x(t_f) = 0.
\end{aligned}
$$

In problem (4.1) the optimization variable is the input u_i, and the inputs u_k of all other systems $k \in I_i$ are held as fixed parameters equal to their local optimal open-loop inputs $u_k^d(t|\tau-h)$, $t \in [h, t_f]$, of problems $\mathcal{P}_k(\tau-h)$, predicted at the previous time $\tau - h$ and assumed trivial on the intervals $[t_f, t_f + h]$: $u_k^d(t|\tau - h) := 0$, $t \in [t_f, t_f + h]$, $k \in I_i$.

The initial state of the i-th system in problem (4.1) is its current state $x_i^*(\tau)$. For the initial states of all other systems $k \in I_i$ we use the components $x_k^d(\tau|\tau - h)$ of the state $x^d(\tau|\tau - h) = x(t|\tau - h, x^*(\tau - h), u^*(\tau - h))$ of system (2.2) with the initial condition $x(\tau-h) = x^*(\tau-h)$ and the input $u(t) \equiv u^*(\tau - h)$, $t \in [\tau - h, \tau]$.

The proposed formulation (4.1) has a drawback: to form the constraints the controller of the i-th system needs to know the local optimal open-loop inputs $u_k^d(t|\tau-h)$ on the whole interval $[0,t_f]$. In the following we show that information communicated during the control process can be reduced to the data $\mathbf{z}^*(\tau)$ chosen above.

We note that the terminal state $x(t_f)$ of system (4.1) can be represented as a sum

$$x(t_f)=\bar{x}(t_f)+\sum\nolimits_{k\in I_i} y_k^d(\tau), \tag{4.2}$$

where

$$y_k^d(\tau)=F_k(t_f)x_k^d(\tau|\tau-h)+\int_0^{t_f-h}F_k(t_f-t)B_k u_k^d(t+h|\tau-h)dt,\ k\in I,$$

and $\bar{x}(t_f)$ is the terminal state of the following system

$$\begin{aligned}\dot{\bar{x}}_i&=A_i\bar{x}_i+\sum\nolimits_{j\in I_i}A_{ij}\bar{x}_j+B_iu_i,\quad \bar{x}_i(0)=x_i^*(\tau),\\ \dot{\bar{x}}_k&=A_k\bar{x}_k+\sum\nolimits_{j\in I_k}A_{kj}\bar{x}_j,\quad \bar{x}_k(0)=0,\ k\in I_i.\end{aligned}$$

It is easy to establish that for $u^*(\tau-h)=u^d(0|\tau-h)$ the following equality holds

$$\sum\nolimits_{k\in I} y_k^d(\tau)=F(h)x(t_f|0,x^*(\tau-h),u^d(\cdot|\tau-h)). \tag{4.3}$$

Let $\tau=h$, and $u^d(t|0)=u^0(t|x_0)$, $t\in[0,t_f]$. Obviously,

$$\sum_{k\in I} y_k^d(h)=F(h)x(t_f|0,x^*(0),u^d(\cdot|0))=0,$$

since $u^d(\cdot|0)$ is the solution of the centralized problem $\mathcal{P}(x_0)$.

Assume that for some $\tau\in T_h\setminus\{0,h\}$ we have $\sum_{k\in I}y_k^d(\tau)=0$. Then (4.2) takes the form $x(t_f)=\bar{x}(t_f)-y_i^d(\tau)$ that allows us to reformulate the problem $\mathcal{P}_i(\tau)$ in the equivalent form

$$\begin{aligned}\bar{\mathcal{P}}_i(\tau):\qquad &\rho_i(\tau)=\min\max_{t\in[0,t_f]}||u_i(t)||_\infty, \qquad (4.4)\\ &\dot{\bar{x}}_i=A_i\bar{x}_i+\sum\nolimits_{j\in I_i}A_{ij}\bar{x}_j+B_iu_i,\\ &\dot{\bar{x}}_k=A_k\bar{x}_k+\sum\nolimits_{j\in I_k}A_{kj}\bar{x}_j,\\ &\bar{x}_i(0)=x_i^*(\tau),\ \bar{x}_k(0)=0,\ \bar{x}(t_f)=y_i^d(\tau).\end{aligned}$$

The following equalities hold

$$x(t_f|0,x^*(\tau),u^d(\cdot|\tau))=\sum\nolimits_{k\in I}\bar{x}(t_f|0,x_k^*(\tau),u_k^d(\cdot|\tau))=\sum\nolimits_{k\in I}y_k^d(\tau)=0.$$

Hence, from (4.3) we have $\sum_{k\in I}y_k^d(\tau+h)=0$.

Summarizing, we have established that 1) the overall open-loop input $u^d(t|\tau)$, $t \in [0, t_f]$ steers system (2.2) from $x(0) = x^*(\tau)$ to the origin in time t_f, i.e. it is feasible in the centralized problem $\mathcal{P}(x^*(\tau))$.
2) For all $\tau \in T_h \setminus \{0\}$ the equality $\sum_{k\in I} y_k^d(\tau) = 0$ holds.
3) The local problem may be formulated as (4.4).

Obviously, to form problem (4.4) the i-th controller needs the states $x_k^*(\tau - h)$ and applied inputs $u_k^*(\tau - h)$ from other systems $k \in I_i$, which in turn composes information $\mathbf{z}^*(\tau)$.

When solving problem (4.4) numerically, one has to reduce it to a linear program

$$\xi_i(\tau) = \max_{\xi_i, \mathbf{v}_i} \xi_i, \quad g_i(\tau)\xi_i + D_i\mathbf{v}_i = 0, \quad ||\mathbf{v}_i||_\infty \le 1, \tag{4.5}$$

where $\xi_i \in \mathbb{R}, \mathbf{v}_i = (v_i^T(0), v_i^T(h), \ldots, v_i^T(t_f - h))^T \in \mathbb{R}^{r_i N}$,

$$D_i = (D_i(0), D_i(h), \ldots, D_i(t_f - h)) \in \mathbb{R}^{n\times r_i N},$$

$$g_i(\tau) = F_i(t_f)x_i^*(\tau) - y_i^d(\tau).$$

Note that (4.5) has $r_i N + 1$ variables and n equality constraints, and its dimension does not depend on the number of systems in the team.

The algorithm for distributed stabilization is specified as Algorithm 2. The algorithm constructs the distributed feedback $u^d(x, \mathbf{z}) = (u_i^d(x_i, \mathbf{z}),$ $i \in I)$ as a function of a position $(x, \mathbf{z})$, and its realization in a particular control process

$$u_i^*(t) \equiv u_i^d(x_i^*(\tau), \mathbf{z}^*(\tau)) := u_i^d(0|\tau),\ t \in [\tau, \tau + h[,\ \tau \in T_h,\ i \in I. \tag{4.6}$$

Algorithm 2 (distributed stabilizing control construction):
1) Set $\tau = 0$, $x^*(\tau) = x_0$.
2) Find a solution $u^0(t|x_0)$, $t \in [0, t_f]$, to the centralized problem $P(x_0)$. For each $k \in I$ set $u_k^d(t|0) = u_k^0(t|x_0)$, $t \in [0, t_f]$.
For each system $i \in I$ (in parallel):
3) Apply input $u_i^*(t) \equiv u_i^d(x_i^*(\tau), \mathbf{z}^*(\tau)) := u_i^d(0|\tau),\ t \in [\tau, \tau + h[$.
4) Communicate $x_i^*(\tau)$, $u_i^*(\tau)$ to all systems $k \in I_i$.
5) Set $\tau := \tau + h$, and obtain current state measurement $x_i^*(\tau)$.
6) Solve problem (4.4) and find $u_i^d(t|\tau)$, $t \in [0, t_f]$. Return to Step 3).

Proposition 2. *A distributed feedback $u^d(x, \mathbf{z})$ is a bounded stabilizing feedback for the overall system (2.2) with the region of attraction D.*

Proof. 1. We need to establish that an input

$$u_i(t) = \begin{cases} u_i^d(t + h|\tau - h), & t \in [0, t_f - h], \\ 0, & t \in [t_f - h, t_f[, \end{cases} \tag{4.7}$$

is a feasible open-loop input of the problem $\bar{\mathcal{P}}_i(\tau)$. Indeed, since system (2.2) is not affected by disturbances, we have $x^d(\tau|\tau-h) = x^*(\tau)$. By construction of $y_i^d(\tau)$ in problem (4.4), the input $u_i(t)$, $t \in [0, t_f[$, satisfies the terminal constraint, which means that $\bar{\mathcal{P}}_i(\tau)$ is feasible.

2. Function (4.7) in the problem $\bar{\mathcal{P}}_i(\tau)$ has the cost equal to $\rho_i(\tau-h)$ that is not less than the optimal value $\rho_i(\tau)$ of this problem. Therefore, for all $i \in I$ we have $\rho_i(\tau) \le \rho_i(\tau-h)$.

3. The cost of the cenralized problem $\mathcal{P}(x^*(\tau))$ at the input $u^d(\cdot|\tau) = (u_k^d(\cdot|\tau), k \in I)$ equals to $\rho^d(\tau) = \max_{k\in I} \rho_k(\tau)$, that yields $\rho^d(\tau) \le \rho^d(\tau - h)$, $\tau = 2h, 3h, \ldots$; $\rho^d(h) \le \rho(x_0)$.

This implies that 1) any trajectory of system (2.2) with $x_0 \in D$ and input (4.6), $i \in I$, stays in D, and 2) $u^*(\tau) \in U$.

4. Following [1], we show that if $x^*(\tau) \ne 0$ then $\rho^d(\tau + t_f) < \rho^d(\tau)$, i.e. at least after N steps of Algorithm 2 we obtain a strict decrease of the cost of the problem $\mathcal{P}(x^*(\tau))$. Assume the opposite: for any $l = \overline{1, N}$ the equality $\rho^d(\tau) = \rho^d(\tau + lh)$ holds. Then there exists $i_0 \in I$ such that $\rho^d(\tau) = \rho_{i_0}^d(\tau) = \rho_{i_0}^d(\tau + lh)$, $l = \overline{1, N}$, and the optimal open-loop input in the problem $\bar{\mathcal{P}}_{i_0}(\tau + lh)$ is a function $u_{i_0}(t) = u_{i_0}^d(t + lh|\tau)$, $t \in [0, t_f - lh[$, $u_{i_0}(t) = 0$, $t \in [t_f - lh, t_f[$. For the time instant $\tau + t_f$ we obtain the optimality of the trivial input, which implies that $\rho^d(\tau) = \rho^d(\tau + t_f) = \rho_{i_0}^d(\tau + t_f) = 0$, and therefore, $x^*(\tau) = x^*(\tau + t_f) = 0$. Further proof repeats the arguments given in [1]. $\square$

5. Examples

To illustrate and compare the stabilization algorithms based on centralized and distributed feedbacks we consider two examples.

Example 1. Consider the team of two systems:

$$\dot{x}_1 = \begin{pmatrix} 0 & 1 \\ -11 & 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u_1, \tag{5.1}$$

$$\dot{x}_2 = \begin{pmatrix} 0 & 1 \\ -10.25 & 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 & 0 \\ -0.25 & 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} u_2.$$

Choose the following parameters: $L = 2$, $t_f = 3$, $h = 0.1$. As initial states at time instant $\tau = 0$ we choose $x_1(0) = x_{10} = (-1, 0)^T$, $x_2(0) = x_{20} = (0.5, 0)^T$. We verify that the initial state $x_0 = (x_{10}^T, x_{20}^T)^T$ of system (5.1) is in the domain D by solving the problem $\mathcal{P}(x_0)$, and obtaining $\rho(x_0) = 1.52197255 < L$.

On the base of the predictive problem $P(x^*(\tau))$, $\tau \in T_h$, Algorithm 1 constructed a realization of the centralized stabilizing feedback. On the

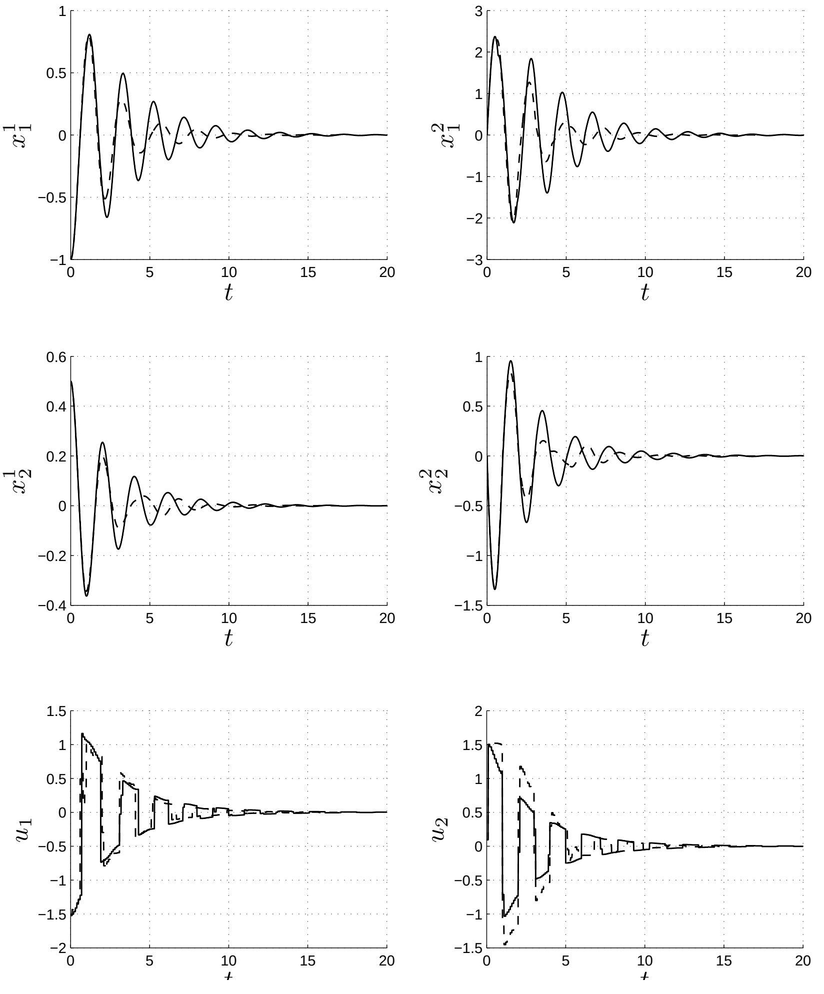

Figure 1. Trajectories and realizations of stabilized and distributed feedbacks in example 1

base of two predictive problems $\mathcal{P}_1(\tau)$, $\mathcal{P}_2(\tau)$, $\tau \in T_h$, Algorithm 2 constructed a realization of the distributed stabilizing feedback for the example under consideration.

Figure 1 presents fragments (for $0 \leq t \leq 20$) of the trajectories and feedback realizations. Solid lines correspond to centralized solution and dash lines correspond to distributed solution.

For the centralized stabilization the neighborhood $||x^*(\tau)|| < 10^{-5}$ was reached by the time instant $\tau = 37.5$, by that time the total control impulse was 8.3651. For the distributed stabilization the same neighborhood was

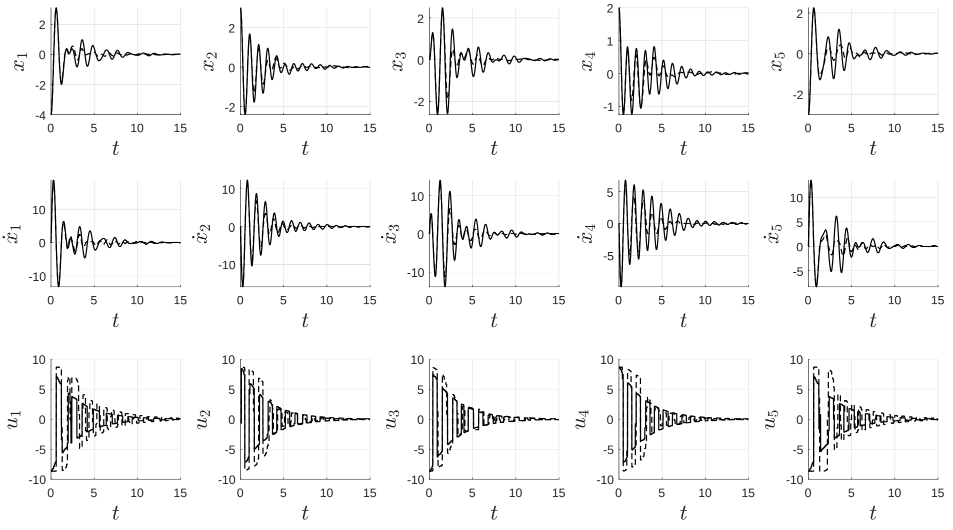

Figure 2. Trajectories and realizations of stabilized and distributed feedbacks in example 2

reached faster, by the time instant $\tau = 28.5$. This is due to the control inputs of greater amplitude. By the time instant $\tau = 28.5$ the total control impulse was already 8.9213.

Example 2. Consider a team consisting of five coupled oscillating systems:

$$\begin{aligned} \ddot{x}_1 &= -2kx_1 + kx_2 + u_1, \\ \ddot{x}_i &= -2kx_i + kx_{i-1} + kx_{i+1} + u_i, \ i = 2, 3, 4, \\ \ddot{x}_5 &= -2kx_5 + kx_4 + u_5. \end{aligned} \tag{5.2}$$

We choose the following parameters' values: $k = 10$, $L = 10$, $t_f = 3$, $h = 0.1$. We assume that at time $\tau = 0$ all systems are stationary in different states: $x_1(0) = -4$, $x_2(0) = 3$, $x_3(0) = 0$, $x_4(0) = 2$, $x_5(0) = -3$. The initial state belongs to the region D since $\rho(x_0) = 8.6621 < L$.

Figure 2 presents fragments (for $0 \leq t \leq 15$) of the trajectories and feedback realizations. In the centralized control process team (5.2) reached the neighborhood $||x^*(\tau)|| < 10^{-5}$ by the time instant $\tau = 42.2$, the total input impulse was 125.7846. In the distributed control process same neighborhood was reached in comparable time, by the time instant $\tau = 42.3$, however, the total impulse of the distributed inputs was 162.9104.

6. Conclusion

This paper proposes an algorithm for constructing distributed bounded stabilizing feedbacks for a team of linear time-invariant coupled systems. The algorithm is based on a parallel solution at each time instant of local predictive optimal control problems associated with each system in the team and having lower dimensions compared to the centralized predictive problem. Local problem's solution yields the feedback realization only for the respective system. The proposed algorithm guarantees feasibility of all local problems during the control process, communication of a small amount of data, and asymptotic stability of the overall team.

References

1. Balashevich N.V. Stabilization of Linear Systems by Bounded Controlling Actions. *Journal of Automation and Information Sciences*, 2009, vol. 41, no. 5, pp. 16–28.
2. Barbashin E.A. *Introduction to the Theory of Stability.* Wolters-Noordhoff, 1970.
3. Gabasov R., Dmitruk N.M., Kirillova F.M. Decentralized optimal control of a group of dynamical objects. *Comput. Math. and Math. Phys.*, 2008, vol. 48, pp. 561-576.
4. Gabasov R., Dmitruk N.M., Kirillova F.M. Decentralized optimal control of dynamical systems under uncertainty. *Comput. Math. and Math. Phys.*, 2011, vol. 51, pp. 1128-1145.
5. Gabasov R., Kirillova F.M., Kostyukova O.I. Methods of stabilizing dynamic systems. *Izv. Ross. Akad. Nauk, Tekh. Kibern.*, 1994, no. 3.
6. Kaljaev I.A., Gajduk A.R., Kapustjan S.G. *Modeli i algoritmy kollektivnogo upravlenija v gruppah robotov* [Models and algorithms of collective control in groups of robots] Moscow, Fizmatlit Publ., 2009. (In Russian)
7. Kurzhanskii A. B. On a team control problem under obstacles. *Proc. of the Steklov Institute of Mathematics*, 2015, vol. 291, no. 1, pp. 128-142.
8. Bemporad A., Borelli F., Morari M. Model predictive control based on linear programming – the explicit solution. *IEEE Transactions on Automatic Control*, 2002, vol. 47, no. 12, pp. 1974-1985.
9. Camponogara E. et. al. Distributed model predictive control. *IEEE Control Systems Magazine*, 2002, vol. 22, no. 1, pp. 44-52.
10. Christofides P. D. et al. Distributed model predictive control: A tutorial review and future research directions. *Computers & Chemical Eng.*, 2013, vol. 51, pp. 21-41.
11. *Distributed Model Predictive Control Made Easy.* Eds. J. M. Maestre, R. R. Negenborn. Springer, 2014.
12. Dmitruk N. Robust Optimal Control of Dynamically Decoupled Systems via Distributed Feedbacks. *Optimization in the Natural Sciences. Communications in Computer and Information Science*, 2015, vol. 499, pp. 95-106.
13. Keviczky T., Borrelli F., Balas G.J. Decentralized receding horizon control for large scale dynamically decoupled systems. *Automatica*, 2006, vol. 42, pp. 2105-2115.
14. Müller M. A., Reble M., Allgöwer F. Cooperative control of dynamically decoupled systems via distributed model predictive control. *Internat. Journal of Robust and Nonlinear Control*, 2012, vol. 22, no. 12, pp. 1376-1397.

15. Rawlings J.B., Mayne D.Q. Model Predictive Control: Theory and Design. Madison, Nob Hill Publishing, 2009.
16. Stewart B. T. et. al. Cooperative distributed model predictive control. *Systems & Control Letters*, 2010, vol. 59, pp. 460-469.

Natalia Dmitruk, Candidate of Sciences (Physics and Mathematics), Associate Professor, Belarusian State University, 4, Nezavisimosti av., Minsk, 220030, Republic of Belarus, tel.: +375 (17) 2095074, e-mail: dmitrukn@bsu.by, ORCID iD https://orcid.org/0000-0003-1845-4927

Received 29.10.19

Стабилизация линейных взаимосвязанных систем ограниченными децентрализованными обратными связями

Н. М. Дмитрук

Белорусский государственный университет, Минск, Республика Беларусь

Аннотация. Рассматривается задача стабилизации группы линейных взаимосвязанных систем ограниченными обратными связями. Эффективные подходы к стабилизации при наличии ограничений из теории управления по прогнозирующей модели развиваются на децентрализованный случай, когда каждая система группы управляется своим локальным регулятором. Предлагаются формулировки локальных задач оптимального управления и основанный на них алгоритм, который строит децентрализованную обратную связь, обеспечивающую асимптотическую устойчивость группы.

Ключевые слова: стабилизация, обратная связь, децентрализованное управление.

Список литературы

1. Балашевич Н. В. Стабилизация линейных систем ограниченными управляющими воздействиями // Проблемы управления и информатики. 2009. № 3. С. 20–31.
2. Барбашин Е. А. Введение в теорию устойчивости. М. : Наука, 1967.
3. Габасов Р., Дмитрук Н. М., Кириллова Ф. М. Оптимальное децентрализованное управление группой динамических объектов // Журнал вычислительной математики и математической физики. 2008. Т. 48, № 4. С. 593–609.
4. Габасов Р., Дмитрук Н. М., Кириллова Ф. М. Оптимальное децентрализованное управление динамическими системами в условиях неопределенности // Журнал вычислительной математики и математической физики. 2011. Т. 51, № 7. С. 1209–1227.
5. Габасов Р., Кириллова Ф. М., Костюкова О. И. К методам стабилизации динамических систем // Известия РАН. Сер. Техническая кибернетика. 1994. № 3. С. 67–77.
6. Каляев И. А., Гайдук А. Р., Капустян С. Г. Модели и алгоритмы коллективного управления в группах роботов. М. : Физматлит, 2009.

7. Куржанский А. Б. О задаче группового управления в условиях препятствий // Труды ИММ УрО РАН. 2014. Т. 20, № 3. С. 166—179.
8. Bemporad A., Borelli F., Morari M. Model predictive control based on linear programming – the explicit solution // IEEE Transactions on Automatic Control. 2002. Vol. 47, N 12. P. 1974–1985.
9. Distributed model predictive control / E. Camponogara [et al.] // IEEE Control Systems Magazine. 2002. Vol. 22, N 1. P. 44–52.
10. Distributed model predictive control: A tutorial review and future research directions / P. D. Christofides [et al.] // Computers & Chemical Eng. 2013. Vol. 51. P. 21–41.
11. Distributed Model Predictive Control Made Easy / eds. J. M. Maestre, R. R. Negenborn. Springer, 2014.
12. Dmitruk N. Robust Optimal Control of Dynamically Decoupled Systems via Distributed Feedbacks // Optimization in the Natural Sciences. Communications in Computer and Information Science. Vol. 499. Springer, 2015. P. 95–106.
13. Keviczky T., Borrelli F., Balas G. J. Decentralized receding horizon control for large scale dynamically decoupled systems // Automatica. 2006. Vol. 42. P. 2105–2115.
14. Müller M. A., Reble M., Allgöwer F. Cooperative control of dynamically decoupled systems via distributed model predictive control // Internat. Journal of Robust and Nonlinear Control. 2012. Vol. 22. N 12. P. 1376–1397.
15. Rawlings J. B., Mayne D. Q. Model Predictive Control: Theory and Design. Madison : Nob Hill Publishing, 2009.
16. Cooperative distributed model predictive control / B. T. Stewart [et al.] // Systems & Control Letters. 2010. Vol. 59. P. 460–469.

Наталия Михайловна Дмитрук, кандидат физико-математических наук, доцент, факультет прикладной математики и информатики, Белорусский государственный университет, Республика Беларусь, 220030, г. Минск, просп. Независимости, 4; тел.: +375 (17) 2095074, e-mail: dmitrukn@bsu.by, ORCID iD https://orcid.org/0000-0003-1845-4927

Поступила в редакцию 29.10.19