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Global Existence of a Solution for a Multiscale Model Describing Moisture Transport in Concrete Materials *

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Abstract. In the previous study [5] we proved the existence of a solution locally in time for a two-scale problem which is given as a mathematical model for moisture transport arising in a concrete carbonation process. The two-scale model consists of a diffusion equation of the relative humidity in a macro domain and the free boundary problems describing a wetting and drying process in infinite micro domains. In this paper, by improving the diffusion equation of the relative humidity based on the experimental result [3; 10], we construct a globally-in-time solution of the two scale model. For the global existence, we obtain uniform estimates and uniform boundedness of the solution with respect to time and use the method of extending local solutions.

Keywords: two-scale model, free boundary problem, quasilinear parabolic equation, moisture transport.

1. Introduction

In our recent work [5], we proposed a two-scale model describing moisture transport phenomena arising in a concrete carbonation process, and showed that our two-scale problem is solvable locally in time. In this paper, we improve an equation consisted of the two-scale model and prove the existence of our concept of solution globally in time.

Based on the setting of [5], let us describe our model. Our model consists of a macro domain Ω, where Ω is a bounded domain in $\mathbb{R}^3$ which is occupied by concrete, and a micro domain for each $x \in \Omega$. In the macro domain Ω

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we consider the relative humidity $h = h(t, x)$, where t is a time variable. On the other hand, we assume that the micro domain is the hole for each $x \in \Omega$, and consider the hole as an interval $(0, L)$, where L is the depth of the hole. This interval $(0, L)$ indicates the water drop region $(0, s(t, x))$ and the air region $(s(t, x), L)$, and the boundary L denotes the edge of the hole in contact with h. Here, we regard the degree of saturation $s = s(t, x)$ and the size of the water drop region, and assume that the relative humidity u in one hole acts in the following non-cylindrical domain $Q_s(T)$ defined by

$$Q_s(T) := \{(t, x, z)|(t, x) \in (0, T) \times \Omega, s(t, x) < z < L\}.$$

Our two scale model, which we denote by (P), is as follows: Find a triplet $(h, s, u) = (h(t, x), s(t, x), u(t, x, z))$ satisfying the following equations, boundary and initial conditions,

$$\rho_v h_t - \mathrm{div}(g(h)\nabla h) = s(1 - f(h))v \text{ in } Q(T) := (0, T) \times \Omega, \tag{1.1}$$
$$h = h_b \text{ on } S(T) := (0, T) \times \partial\Omega, \tag{1.2}$$
$$\rho_v u_t - k u_{zz} = 0 \text{ on } Q_s(T), \tag{1.3}$$
$$u(t, x, L) = h(t, x) \text{ for } (t, x) \in Q(T), \tag{1.4}$$
$$k u_z(t, x, s(t, x)) = (\rho_w - \rho_v u(t, x, s(t, x)))s_t(t, x) \text{ for } (t, x) \in Q(T), \tag{1.5}$$
$$s_t(t, x) = a(u(t, x, s(t, x)) - \varphi(s(t, x))) \text{ for } (t, x) \in Q(T), \tag{1.6}$$
$$h(0, x) = h_0(x) \text{ for } x \in \Omega, \tag{1.7}$$
$$s(0, x) = s_0(x) \text{ for } x \in \Omega, \tag{1.8}$$
$$u(0, x, z) = u_0(x, z) \text{ for } (x, z) \in \Omega_{s_0}, \tag{1.9}$$

where $\Omega_{s_0} = \{(x, z) \in \Omega \times \mathbb{R} | s_0(x) < z < L\}$, ρ_w and ρ_v are the densities of H_2O in water regions and in air regions, respectively, g is a continuous function on $(0, \infty)$, f and v are given functions on $\mathbb{R}$ and on $Q(T)$, respectively, k and a are positive constants, φ is a continuous function on $\mathbb{R}$, h_0 is an initial condition on Ω, s_0 is an initial position of the free boundary s and u_0 is an initial condition on (s_0, L).

From the physical point of view, (1.1) is the diffusion equation of h which is originally proposed by [7; 8], g is a diffusion coefficient. Also, v represents the concentration of CO_2 and $s(1 - f(h))v$ represents the quantity of water generated by chemical reaction according to the level of the humidity, where f is a monotone function increasing on $[0, 1]$ such that $f(0) > 0$ and $f(1)$ is almost 1. The forcing term is proposed by [3;10] based on the experimental result that CO_2 cannot be dissolved completely in low humidity, while the water in the pores hinders the diffusion of CO_2 in high humidity, and $1 - f(h)$ is the reduction factor depending on h.

The system (1.3)-(1.6) with (1.8) and (1.9) is a free boundary problem describing a wetting and drying process in the hole for each $x \in \Omega$ and is originally proposed by [9]. The equations (1.3) and (1.5) are derived from the mass conservation for H_2O in the air region and near the free boundary, respectively. Also, (1.4) means that each hole is exposed to air at the end of the hole, and (1.6) is the growth rate of the water drop region and φ represents the rate of change from water to air in the hole.

The problem (P) is quite close to the two-scale model in [5]. Indeed, the two-scale setting and the system {(1.3)-(1.6), (1.8), (1.9)} is exactly same. The difference between (P) and the model in [5] is the diffusion equation of the relative humidity h: we consider (1.1) in (P), while we studied in [5] the following diffusion equation of the relative humidity:

$$\rho_v h_t - \mathrm{div}(g(h)\nabla h) = sv \text{ in } Q(T), \tag{1.10}$$

where ρ_v, g, and v are the same constant and functions as in (P). As pointed out in [5], for the global existence it needs to satisfy that $0 \le h \le h_*$ on $[0,T]$, where h_* is a positive constant with $h_* < 1$. However, it is difficult for the solution h satisfying (1.10) to guarantee such boundedness on $[0,T]$ for $T > 0$. To overcome this difficulty we improve (1.10) based on the experimental result [3; 10], and arrive to a more realistic equation (1.1). Under this improvement, we show the smallness of h satisfying (P) on $[0,T]$ for $T > 0$ and establish the existence of a globally-in-time solution of (P).

2. Notations and Assumptions

In this paper we use the following notations. In general, for a Banach space X we denote by $|\cdot|_X$ its norm. Particularly, we denote by $H = L^2(\Omega)$, and the norm and the inner product of H are simply denoted by $|\cdot|_H$ and $(\cdot,\cdot)_H$, respectively. Also, for $\Omega \subset \mathbb{R}^n$ for $n = 1$ or $n = 3$, $H^1(\Omega)$, $H^1_0(\Omega)$ and $H^2(\Omega)$ are the usual Sobolev spaces.

Throughout this paper, we assume the following conditions:

(A1) Ω is a bounded domain of $\mathbb{R}^3$ which has the boundary $\partial\Omega$ in the class of C^2.

(A2) k, a, κ_0 are positive constants satisfying $\kappa_0 < 1$.

(A3) $G \in C^3((0,+\infty))$ is such that $g(r) := G'(r) \ge g_0$ for $r > 0$, where g_0 is a positive constant and put

$$C_g = \sup_{r\in[\kappa_0,+\infty)} |g'(r)| + \sup_{r\in[\kappa_0,+\infty)} |g''(r)|.$$

(A4) $f \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, $0 \le f \le 1$ on $\mathbb{R}$ and $f(r) \equiv 1$ for $r \ge h^*$, where h^* is a positive constant satisfying $h^* < 1$, and put $C_f = |f'|_{L^\infty(\mathbb{R})}$.

(A5) $v \in L^\infty(Q(T))$ and $v_t \in L^2(0,T;H^1(\Omega))$ with $v \ge 0$ a.e. on $Q(T)$.

(A6) $h_b \in C^2(\overline{Q(T)})$ and $h_{bt} \in L^2(0,T;H^2(\Omega))$ with

$$\kappa_0 \le h_b \le h^* \text{ on } Q(T), \tag{2.1}$$

where h^* is the same constant as in (A4).

(A7) $h_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ with

$$\kappa_0 \le h_0 \le h^* \text{ on } \Omega, \tag{2.2}$$

where h^* is the same constant as in (A4), and Δh_0 is bounded a.e. on Ω. Also, h_0 satisfies that $h_0 = h_b(0)$ a.e. on $\partial\Omega$.

(A8) $\varphi \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, $\varphi = 0$ on $(-\infty, 0]$, $\varphi \le 1$ on $\mathbb{R}$, $\varphi' > 0$ on $(0, L]$ and for $b_0 > 0$ $\varphi(L) - b_0 > 0$. Also, we denote by $\hat{\varphi}$ the primitive function of φ with $\hat{\varphi}(0) = 0$ and put $C_\varphi = |\varphi'|_{L^\infty(\mathbb{R})}$.

(A9) Two positive constants ρ_w and ρ_v satisfy

$$\rho_w \ge \rho_v(C_\varphi + 2), \quad 9aL\rho_v^2 \le k\rho_w.$$

(A10) $s_0 \in H$ such that $0 \le s_0 \le L - \delta_0$ for $\delta_0 > 0$ a.e. on Ω, and $u_0 \in L^\infty(\Omega_{s_0})$ and the function $x \to |u_0(x)|_{H^1(s_0,L)}$ is bounded a.e. on Ω and $u_0(x, L) = h(x, 0)$ for $x \in \Omega$ and $0 \le u_0 \le 1$ a.e. on Ω_{s_0}.

Definition 1. *Let h and s be functions on $Q(T)$ and u be a function on $Q_s(T)$, respectively, for $T > 0$. We say that a triplet (h, s, u) is a solution of (P) on $[0, T]$ if the conditions (S1)-(S9) hold:*

(S1) $h \in W^{1,2}(0, T; H) \cap L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ *with* $h \ge 0$ *a.e. on* $Q(T)$.

(S2) s, $s_t \in L^\infty(Q(T))$, $0 \le s < L$ *on* $Q(T)$, $u \in L^\infty(Q_s(T))$, u_t, $u_{zz} \in L^2(Q_s(T))$ *and* $(t, x) \in Q(T) \to |u_z(t, x, \cdot)|_{L^2(s(t,x),L)}$ *is bounded.*

(S3) $\rho_v h_t - \Delta G(h) = s(1 - f(h))v$ *a.e. in* $Q(T)$.

(S4) $h = h_b$ *a.e. on* $S(T)$.

(S5) $\rho_v u_t - k u_{zz} = 0$ *on* $Q_s(T)$.

(S6) $u(t, x, L) = h(t, x)$ *for a.e.* $(t, x) \in Q(T)$.

(S7) $k u_z(t, x, s(t, x)) = (\rho_w - \rho_v u(t, x, s(t, x)))s_t(t, x)$ *for a.e.* $(t, x) \in Q(T)$.

(S8) $s_t(t, x) = a(u(t, x, s(t, x)) - \varphi(s(t, x)))$ *for a.e.* $(t, x) \in Q(T)$.

(S9) $h(0, x) = h_0(x)$, $s(0, x) = s_0(x)$ *for* $x \in \Omega$,
$u(0, x, z) = u_0(x, z)$ *for* $(x, z) \in \Omega_{s_0}$.

Our main result of this paper is concerned with the existence of a globally-in-time solution for the problem (P).

Theorem 1. *Let $T > 0$. If (A1)-(A10) hold, then (P) has a unique solution (h, s, u) on $[0, T]$ with $\kappa_0 \le h \le h^*$ a.e. on $Q(T)$ and $0 \le u \le 1$ a.e. on $Q_s(T)$.*

3. Mathematical model for moisture transport

For $T > 0$ and $\delta \in (0, L)$ we set

$$X(T,\delta) := \{s \in W^{1,2}(0,T;H) |\ 0 \le s \le L - \delta, \\ |s_t| \le 2a \text{ a.e. on } Q(T), s(0) = s_0 \text{ in } \Omega\}.$$

First, for given $\tilde{s} \in X(T,\delta)$, we prove the existence of a solution of

$$(\mathrm{AP})(\tilde{s}) := (\mathrm{AP})(\tilde{s}, h_b, h_0)$$

on $[0,T]$. To do so, we put

$$D_K(T) := \{z \in C(0,T;H) |\ |z_t|_{L^2(0,T;H)} + |z|_{L^\infty(0,T;H^1(\Omega))} \le K\},$$

where K is a positive constant determined later. For given $\tilde{h} \in D_K(T)$, we consider $(\mathrm{AP})(\tilde{s}, \tilde{h}) := (\mathrm{AP})(\tilde{s}, \tilde{h}, h_b, h_0)$:

$$\begin{cases} \rho_v h_t - \mathrm{div}(g(h)\nabla h) = \tilde{s}(1 - f(\tilde{h}))v \text{ in } Q(T), \\ h = h_b \text{ on } S(T), \\ h(0,x) = h_0(x) \text{ for } x \in \Omega. \end{cases}$$

By (A4) and (A5) we easily see that $\tilde{s}(1-f(\tilde{h}))v \in L^2(0,T;H) \cap L^\infty(Q(T))$. Hence, by using the result of [1; 2] we have a solution h of $(\mathrm{AP})(\tilde{s}, \tilde{h})$ such that $h \in W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, $h(t) - h_b(t) \in H^1_0(\Omega)$ for a.e. $t \in [0.T]$ and

$$k_0 \le h \le p(t) \text{ a.e. on } \Omega \text{ for } t \in [0,T], \tag{3.1}$$

where $P(t) = L|v|_{L^\infty(Q(T))}t + h^*$ for $t \in [0,T]$. Next, we note the useful property of a solution of $(\mathrm{AP})(\tilde{s}, \tilde{h})$.

Lemma 1. *Let $T > 0$ and h be solutions of $(AP)(\tilde{s}, \tilde{h})$ on $[0,T]$ for given $\tilde{s} \in X(T,\delta)$ and $\tilde{h} \in D_K(T)$. Then, there exists an increasing function $M_i(T) > 0$ $(1 \le i \le 2)$ with respect to T which is independent of $\tilde{s}$ and δ such that*

$$(i)\quad |h|_{W^{1,2}(0,T;H)} + |h|_{L^\infty(0,T;H^1(\Omega))} \le M_1(T),$$
$$(ii)\quad |\nabla h|_{L^\infty(Q(T))} \le M_2(T).$$

The estimate (i) is obtained by the standard calculation because $|\tilde{s}(1 - f(\tilde{h}))v| \le L|v|_{L^\infty(Q(T))}$ by $\tilde{s} \in X(T,\delta)$, (A4) and (A5). Also, from $\tilde{s}(1 - f(\tilde{h})) \in L^\infty(Q(T))$ we can apply the same argument of Lemmas 3.1-3.4 in [1] and get $M_2(T)$ satisfying (ii). In particular, we note that these constants are independent of the choice of $\tilde{h}$.

Next, for fixed $\tilde{s} \in X(T,\delta)$ we prove the continuous dependence of a solution h of $(\mathrm{AP})(\tilde{s}, \tilde{h})$ for given $\tilde{h} \in D_K(T)$.

Lemma 2. *Let h_1 and h_2 be solutions of $(AP)(\tilde{s}, \tilde{h}_1)$ and $(AP)(\tilde{s}, \tilde{h}_2)$ for given $\tilde{h}_1$, $\tilde{h}_2 \in D_K(T)$, respectively. Then, there exists $C(T)$ depending on T such that*

$$\begin{aligned}&|h_1(t) - h_2(t)|_H^2 + \int_0^t |\nabla(h_1(\tau) - h_2(\tau))|_H^2 d\tau \\ \leq& C(T) \int_0^t |\tilde{h}_1(\tau) - \tilde{h}_2(\tau)|_H^2 d\tau \text{ for } t \in [0, T].\end{aligned}$$

Proof. By the subtraction of the equations for h_1 and h_2, it holds that

$$\begin{aligned}&\frac{\rho_v}{2}\frac{d}{dt}|h_1(t) - h_2(t)|_H^2 + \int_\Omega \nabla(G(h_1(t)) - G(h_2(t)))\nabla(h_1(t) - h_2(t))dx \\ =& \int_\Omega \left[\tilde{s}(t)\Big(1 - f(\tilde{h}_1(t)) - (1 - f(\tilde{h}_2(t)))\Big)v(t)\right](h_1(t) - h_2(t))dx. \quad (3.2)\end{aligned}$$

Here, by using (ii) of Lemma 1, we can estimate the second term of the left hand side of (3.2) as follows.

$$\begin{aligned}&\int_\Omega \nabla(G(h_1) - G(h_2))\nabla(h_1 - h_2)dx \\ =& \int_\Omega (g(h_1)\nabla(h_1 - h_2) + (g(h_1) - g(h_2))\nabla h_2)\nabla(h_1 - h_2)dx \\ \geq& g_0|\nabla(h_1 - h_2)|_H^2 - M_2(T)\int_\Omega |g(h_1) - g(h_2)||\nabla(h_1 - h_2)|dx \\ \geq& \frac{g_0}{2}|\nabla(h_1 - h_2)|_H^2 - \frac{M_2^2(T)C_g^2}{2g_0}|h_1 - h_2|_H^2, \quad (3.3)\end{aligned}$$

where g_0 and C_g are the same positive constants as in (A3), and $M_2(T)$ is the same constant as in Lemma 1. Also, by (A4) and Hölder inequality, we observe that

$$\begin{aligned}&\int_\Omega \left[\tilde{s}(t)\Big(1 - f(\tilde{h}_1(t)) - (1 - f(\tilde{h}_2(t)))\Big)v(t)\right](h_1(t) - h_2(t))dx \\ \leq& LC_f|v|_{L^\infty(Q(T))}|h_1(t) - h_2(t)|_H|\tilde{h}_1(t) - \tilde{h}_2(t)|_H. \quad (3.4)\end{aligned}$$

Hence, by substituting (3.3) and (3.4) into (3.2) we obtain

$$\begin{aligned}&\frac{\rho_v}{2}\frac{d}{dt}|h_1(t) - h_2(t)|_H^2 + \frac{g_0}{2}|\nabla(h_1(t) - h_2(t))|_H^2 \\ \leq& \frac{L^2C_f^2|v|_{L^\infty(Q(T))}^2}{2}|\tilde{h}_1(t) - \tilde{h}_2(t)|_H^2 \\ &+ \left(\frac{L^2C_f^2|v|_{L^\infty(Q(T))}^2}{2} + \frac{M_2^2(T)C_g^2}{2g_0}\right)|h_1(t) - h_2(t)|_H^2 \text{ for a.e. } t \in [0, T].\end{aligned}$$

Therefore, by Gronwall's inequality, we see that there exists a positive constant $C(T)$ depending on T such that Lemma 2 holds. □

From Lemma 2, for fixed $\tilde{s} \in X(T,\delta)$, we also see that the solution h of (AP)$(\tilde{s},\tilde{h})$ for given $\tilde{h} \in D_K(T)$ is unique. Now, we prove the existence of a solution of (AP)$(\tilde{s})$ for given $\tilde{s} \in X(T,\delta)$. By taking $K = M_1(T)$, we define the mapping $\Gamma : D_K(T) \to D_K(T)$ by $\Gamma(\tilde{h}) = h$, where h is a unique solution of (AP)$(\tilde{s},\tilde{h})$ for given $\tilde{h}$. Then, by Lemma 2 we see that Γ is continuous with respect to $C(0,T;H)$. Therefore, since $D_K(T)$ is compact in $C(0,T;H)$, by Schauder's fixed point theorem, we can find $h \in D_K(T)$ such that $\Gamma(h) = h$. This means that (AP)$(\tilde{s})$ has a unique solution on $[0,T]$ for fixed $\tilde{s} \in X(T,\delta)$.

Next, we prove the following boundedness of a solution h of (AP)$(\tilde{s})$ for given $\tilde{s} \in X(T,\delta)$.

Lemma 3. *Let $T > 0$, $\delta \in (0,L)$ and h be a solution of (AP)$(\tilde{s})$ for given $\tilde{s} \in X(T,\delta)$. Then, $\kappa_0 \le h(t) \le h^*$ for $t \in [0,T]$ a.e. on Ω, where h^* is the same constant as in (A4).*

Proof. First, we prove that $h \ge \kappa_0$ for $t \in [0,T]$ a.e. on Ω. By (2.1) it holds that $[-h_b + \kappa_0]^+ = 0$ a.e. on $Q(T)$. Then, we have that

$$\frac{\rho_v}{2}\frac{d}{dt}|[-h(t)+\kappa_0]^+|_H^2 + g_0\int_\Omega |\nabla[-h(t)+\kappa_0]^+|^2 dx$$
$$\le \int_\Omega -\tilde{s}(t)(1-f(h(t)))v(t)[-h(t)+\kappa_0]^+ dx \text{ for a.e. } t \in [0,T].$$

Here, $\tilde{s}(1-f(h))v \ge 0$ a.e. on $Q(T)$ so that it follows that

$$\frac{\rho_v}{2}\frac{d}{dt}|[-h(t)+\kappa_0]^+|_H^2 + g_0\int_\Omega |\nabla[-h(t)+\kappa_0]^+|^2 dx \le 0 \text{ for a.e. } t \in [0,T].$$

Clearly, the second term is positive. Hence, we obtain

$$\frac{d}{dt}|[-h(t)+\kappa_0]^+|_H^2 \le 0 \text{ for a.e. } t \in [0,T].$$

This result and (2.2) implies that $h(t) \ge \kappa_0$ for $t \in [0,T]$ a.e. on Ω. Next, we show that $h \le h^*$ for $t \in [0,T]$ a.e. on Ω. Since $[h_b - h^*]^+ = 0$ a.e. on $Q(T)$ by (2.1), it holds that

$$\frac{\rho_v}{2}\frac{d}{dt}|[h(t)-h^*]^+|_H^2 + g_0\int_\Omega |\nabla[h(t)-h^*]^+|^2 dx$$
$$\le \int_\Omega \tilde{s}(t)(1-f(h(t)))v(t)[h(t)-h^*]^+ dx \text{ for a.e. } t \in [0,T].$$

We note that the second term is positive and the right hand side is equal to 0 because $f(h) = 1$ by (A4) if $h > h^*$. Hence, we obtain that

$$\frac{d}{dt}|[h(t)-h^*]^+|_H^2 \le 0 \text{ for a.e. } t \in [0,T].$$

Finally, by integrating over $[0,t]$ for $t \in [0,T]$ and (2.2) we see that $h \le h^*$ for $t \in [0,T]$ a.e. on Ω. Thus, Lemma 3 is proved. □

At the end of this section, we give some properties of a solution h of (AP)($\tilde{s}$) for given $\tilde{s} \in X(T, \delta)$.

Lemma 4. *(i) Let $T > 0$, $\delta \in (0, L)$ and h be a solution of (AP)($\tilde{s}$) on $[0, T]$ for given $\tilde{s} \in X(T, \delta)$. Then, there exists an increasing function $M_i(T) > 0$ $(1 \leq i \leq 4)$ with respect to T which is independent of $\tilde{s}$ such that*

$$|h|_{W^{1,2}(0,T;H)} + |h|_{L^\infty(0,T;H^1(\Omega))} \leq M_1(T), \tag{3.5}$$

$$|\nabla h|_{L^\infty(Q(T))} \leq M_2(T), \tag{3.6}$$

$$|h_t|_{L^\infty(0,T;H)} + |\nabla h_t|_{L^2(0,T;H)} \leq M_3(T), \tag{3.7}$$

$$|h_t|_{L^\infty(Q(T))} \leq M_4(T), \tag{3.8}$$

where $M_1(T)$ and $M_2(T)$ are the same as in Lemma 1.

(ii) Let h_1 and h_2 be solutions of (AP)($\tilde{s}_1$) and (AP)($\tilde{s}_2$) for $\tilde{s}_1$, $\tilde{s}_2 \in X(T, \delta)$, respectively. Then, there exists $M_5(T) > 0$, $M_6(T) > 0$ such that

$$\begin{aligned} &|\nabla(h_1(t) - h_2(t))|_H^2 + \int_0^t |\Delta(h_1(\tau) - h_2(\tau))|_H^2 d\tau \\ &\leq M_5(T)|\tilde{s}_1 - \tilde{s}_2|^2_{L^2(0,T;H)} \text{ for } t \in [0, T], \end{aligned} \tag{3.9}$$

and

$$|h_{1t} - h_{2t}|_{L^2(0,T;H)} \leq M_6(T)|\tilde{s}_1 - \tilde{s}_2|_{L^2(0,T;H)} \tag{3.10}$$

As mentioned in Lemma 1, since $|\tilde{s}(1 - f(h))v| \leq L|v|_{L^\infty(Q(T))}$ (3.5) and (3.6) are obtained, and these estimates are independent of the choice of $\tilde{s}$. Also, by (A4), (A5) and (3.5), we see that $(\tilde{s}(1 - f(h))v)_t \in L^2(0, T; H)$. On account of this, (3.7) can be obtained by using the same argument of [1]. Moreover, with the help of (A4), (A5), (A7) and (3.7) it holds that $(\tilde{s}(1 - f(h))v)_t \in L^6(Q(T))$ and $\tilde{s}_0(1 - f(h_0))v(0) \in L^\infty(\Omega)$. From this result, we can have (3.8) by the technique of [6] or the same way of the proof of Lemma 3.2 in [5]. The continuous dependence estimate (3.9) and (3.10) are derived from the subtraction of the equations h_1 and h_2 by using (3.6) and the same proof of Lemma 3.3 of [5]. In this paper, we omit the precise proofs.

4. Free boundary problem

In this section, we note the obtained result for the free boundary problem with a given $\bar{h}$ on $Q(T)$ for $T > 0$. First, we give a definition of a solution to (FBP)($\bar{h}, s_0, u_0$) satisfying (1.3)-(1.6), (1.8), and (1.9) for each $x \in \Omega$. Throughout this section, we use the notation $Q_{s(x)}(T)$ for the following domain: For each $x \in \Omega$,

$$Q_{s(x)}(T) := \{(t, z) | t \in (0, T), s(t, x) < z < L\}.$$

Definition 2. *Let $T > 0$ and $x \in \Omega$, $s = s(x) = s(\cdot, x)$ be a function on $[0, T]$ and $u = u(x) = u(\cdot, x, \cdot)$ be a function on $Q_{s(x)}(T)$. We say that a pair (s, u) is a solution of (FBP)$(\bar{h}, s_0, u_0) :=$ (FBP)$(\bar{h}(\cdot, x), s_0(x), u_0(x, \cdot))$ on $[0, T]$ if the following conditions (D1)-(D6) hold:*

(D1) $s(x) \in W^{1,\infty}(0, T), 0 \le s(x) < L$ on $[0, T]$, $u(x) \in L^\infty(Q_{s(x)}(T))$, $u_t(x), u_{zz}(x) \in L^2(Q_{s(x)}(T))$, $t \in [0, T] \mapsto |u_z(t, x, \cdot)|_{L^2(s(t,x),L)}$ is bounded.

(D2) $\rho_v u_t(x) - k u_{zz}(x) = 0$ on $Q_{s(x)}(T)$.

(D3) $u(t, x, L) = \bar{h}(t)$ for a.e. $t \in [0, T]$.

(D4) $k u_z(t, x, s(t)) = (\rho_w - \rho_v u(t, x, s(t, x))) s_t(t, x)$ for a.e. $t \in [0, T]$.

(D5) $s_t(t, x) = a(u(t, x, s(t, x)) - \varphi(s(t, x)))$ for a.e. $t \in [0, T]$.

(D6) $s(0, x) = s_0(x)$, $u(0, x, z) = u_0(x, z)$ for $z \in [s_0(x), L]$.

To handle (FBP)$(\bar{h}, s_0, u_0)$, by introducing the following function $\tilde{u}$

$$\tilde{u}(t, x, y) = u(t, x, (1 - y)s(t, x) + yL) \text{ for } (t, y) \in (0, T) \times [0, 1], \tag{4.1}$$

we transform (FBP)$(\bar{h}, s_0, u_0)$, initially posed in a non-cylindrical domain, to the following problem $\widetilde{\text{(FBP)}}(\bar{h}, s_0, \tilde{u}_0)$ in a cylindrical domain:

$$\rho_v \tilde{u}_t(x) - \frac{k}{(L - s(t, x))^2} \tilde{u}_{yy}(x) = \frac{\rho_v (1 - y) s_t(x)}{L - s(t, x)} \tilde{u}_y(x) \text{ in } (0, T) \times (0, 1),$$
$$\tilde{u}(t, x, 1) = \bar{h}(t, x) \text{ for } t \in [0, T],$$
$$\frac{k}{L - s(t, x)} \tilde{u}_y(t, x, 0) = (\rho_w - \rho_v \tilde{u}(t, x, 0)) s_t(t, x) \text{ for } t \in [0, T],$$
$$s_t(t, x) = a(\tilde{u}(t, x, 0) - \varphi(s(t, x))) \text{ for } t \in [0, T],$$
$$s(0, x) = s_0(x), \ \tilde{u}(0, x, y) = u_0(x, (1 - y)s_0(x) + yL) =: \tilde{u}_0(x, y) \text{ for } y \in [0, 1].$$

Here, the condition (D1) is equivalent to the following (S):

$$\text{(S)} \begin{cases} s(x) \in W^{1,\infty}(0, T), \ 0 \le s(x) < L \text{ on } [0, T], \\ \tilde{u}(x) \in W^{1,2}(0, T; L^2(0, 1)) \cap L^\infty(0, T; H^1(0, 1)) \cap L^\infty((0, T) \times (0, 1)) \\ \cap L^2(0, T; H^2(0, 1)). \end{cases}$$

The following result for $\widetilde{\text{(FBP)}}(\bar{h}, s_0, u_0)$ is already obtained in [4].

Theorem 2. *Let $\bar{h} \in W^{1,2}(0, T; H) \cap L^2(0, T; H^2(\Omega))$, $\bar{h}_t \in L^\infty(Q(T))$ with $0 \le \bar{h} \le \bar{h}_*$ a.e. on $Q(T)$, where $\bar{h}_*$ is a positive constant with $\bar{h}_* < 1$. Also, assume that $s_0 \in H$ is such that $0 \le s_0 \le L - \delta$ for $\delta > 0$ a.e. on Ω, and $\tilde{u}_0 \in L^\infty(\Omega \times (0, 1))$ and the function $x \to |\tilde{u}_0(x)|_{H^1(0,1)}$ is bounded a.e. on Ω and $\tilde{u}_0(x, 1) = \bar{h}(x, 0)$ for a.e. $x \in \Omega$ and $0 \le \tilde{u}_0 \le 1$ a.e. on $\Omega \times (0, 1)$. Then, (i) and (ii) hold:*

(i) For any $T > 0$, $\widetilde{(FBP)}(\bar{h}(\cdot, x), s_0(x), \tilde{u}_0(x, \cdot))$ has a unique solution $(s(\cdot, x), \tilde{u}(\cdot, x, \cdot))$ on $[0, T]$ such that $\tilde{u} \in L^\infty(\Omega; W^{1.2}(0, T; L^2(0, 1))) \cap$

$L^\infty(\Omega;L^\infty(0,T;H^1(0,1)))\cap L^\infty(\Omega;L^2(0,T;H^2(0,1)))\cap L^\infty(\Omega;L^\infty((0,1)))$, $s\in L^\infty(\Omega;W^{1,\infty}(0,T))$, $0\le\tilde u\le 1$ a.e. on $\Omega\times(0,1)$ for $t\in[0,T]$ and $|s_t|\le 2a$ a.e. on $(0,T)\times\Omega$. Also, for $(s(\cdot,x),u(\cdot,x,\cdot))$ with $u(\cdot,x,\cdot)=\tilde u\left(\cdot,x,\frac{\cdot-s(\cdot,x)}{L-s(\cdot,x)}\right)$ on $Q_{s(x)}(T)$, there exists a positive constant C_1 such that

$$\begin{aligned}&\frac{\rho_v}{2}\int_0^t\int_{s(\tau)}^L|u_\tau(\tau)|^2dzd\tau+\frac{k}{2}\int_{s(t)}^L|u_z(t)|^2dz\\ \le&\frac{k}{2}\int_{s_0}^L|u_{0z}|^2dz+\frac{k}{2}\int_0^t|s_\tau(\tau)||u_z(\tau,s(\tau))|^2d\tau\\ &+C_1\int_0^t(|s_\tau(\tau)|^2+|\bar h_\tau(\tau,x)|^2)d\tau+C_1 \text{ for } t\in[0,T] \text{ a.e. on } \Omega. \quad (4.2)\end{aligned}$$

Moreover, let C_2 be a positive constant obtained by (4.2) which satisfies $\int_{s(t)}^L|u_z(t)|^2dz\le C_2$ for $t\in[0,T]$, then there exists a positive constant δ_* depending on k, a, $\bar h_*$, C_φ and L and C_2 such that

$$0\le s\le L-\delta_* \text{ for } t\in[0,T] \text{ a.e. on } \Omega. \quad (4.3)$$

(ii) For a.e. $x\in\Omega$, let $(s_1(\cdot,x),\tilde u_1(\cdot,x,\cdot))$ and $(s_2(\cdot,x),\tilde u_2(\cdot,x,\cdot))$ be solutions of $\widetilde{(FBP)}(\bar h_1(\cdot,x),s_0(x),\tilde u_0(x,\cdot))$ and $\widetilde{(FBP)}(\bar h_2(\cdot,x),s_0(x),\tilde u_0(x,\cdot))$ on $[0,T]$, respectively, then it holds that

$$\begin{aligned}&|\tilde u_1(t)-\tilde u_2(t)|^2_{L^2(\Omega\times(0,1))}+\int_0^t|\tilde u_{1y}(\tau)-\tilde u_{2y}(\tau)|^2_{L^2(\Omega\times(0,1))}d\tau\\ &+|s_1-s_2|^2_{L^\infty(0,t;H)}\le C_3|\bar h_1-\bar h_2|^2_{W^{1,2}(0,t;H)} \text{ for } t\in[0,T], \quad (4.4)\end{aligned}$$

where C_3 is a positive constant depending on k, a, $\bar h_*$, C_φ, ρ_w, ρ_v, L and δ_*.

5. Local and Global existence

5.1. Local existence

In this section, we prove the local existence of a solution of (P). First, for fixed $T>0$ and $\delta'\in(0,L)$ we define the mapping $\Lambda: X(T,\delta')\to X(T,\delta')$ for $\delta'\in(0,L)$ as follows: for each $\tilde s\in X(T,\delta')$ we denote by h the unique solution of (AP)($\tilde s$) on $[0,T]$. Then, by Lemmas 3 and 4, h satisfies the assumption of Theorem 2 with $\bar h_*=h^*$. Next, we denote by s the unique solution $(\tilde u,s)$ of $\widetilde{(\mathrm{FBP})}(h,s_0,\tilde u_0)$ on $[0,T]$. Here, by (i) of Lemma 4 and Theorem 2 it holds that $0\le s(t)\le L-\delta_*$ for $t\in[0,T]$ a.e. on Ω and $|s_t|\le 2a$ a.e. on $Q(T)$, where δ_* is a positive constant obtained by Theorem

2. By Lemma 4 the estimate of $|h|_{W^{1,2}(0,T;L^\infty(\Omega))}$ is independent of δ' so that we can choose $\delta_* \in (0,L)$ independent of δ' in (i) of Theorem 2. Thus, we take $\delta' = \delta_*$ and infer that $\Lambda : X(T',\delta_*) \to X(T',\delta_*)$ for any $0 < T' \le T$.

Now, we prove that there exists $T' < T$ such that Λ is a contraction mapping on $W^{1,2}(0,T';H)$. Let h_i be a solution of (AP)$(\tilde{s}_i)$ on $[0,T']$ and $(\tilde{u}_i, s_i) = (\tilde{u}_i, \Lambda(\tilde{s}_i))$ be a solution of $\widetilde{(\mathrm{FBP})}(h_i, s_0, \tilde{u}_0)$ on $[0,T']$ for $i = 1,2$. We note the result by Sobolev's embedding theorem in one dimension:

$$\begin{aligned}&|\tilde{u}_1(t,x,0) - \tilde{u}_2(t,x,0)|^2\\ \le &C_e|\tilde{u}_1(t,x) - \tilde{u}_2(t,x)|_{L^2(0,1)}|\tilde{u}_1(t,x) - \tilde{u}_2(t,x)|_{H^1(0,1)},\end{aligned} \tag{5.1}$$

where C_e is a positive constant by Sobolev's embedding theorem. Then, by (4.4) and (5.1) it holds that

$$\begin{aligned}&\int_0^{T'} |\Lambda(\tilde{s}_1)_t - \Lambda(\tilde{s}_2)_t|_H^2 dt\\
=&\int_0^{T'} |a(\tilde{u}_1(t,\cdot,0) - \tilde{u}_2(t,\cdot,0) - (\varphi(\Lambda(\tilde{s}_1)) - \varphi(\Lambda(\tilde{s}_2))|_H^2 dt\\
\le&2a^2\left(\int_0^{T'} |\tilde{u}_1(t,\cdot,0) - \tilde{u}_2(t,\cdot,0)|_H^2 dt + C_\varphi^2 \int_0^{T_1} |\Lambda(\tilde{s}_1) - \Lambda(\tilde{s}_2)|_H^2 dt\right)\\
\le&2C_e a^2\left(\int_0^{T'} |\tilde{u}_1 - \tilde{u}_2|_{L^2(\Omega\times(0,1))}|\tilde{u}_{1y} - \tilde{u}_{2y}|_{L^2(\Omega\times(0,1))} dt\right.\\
&\left.+\int_0^{T'} |\tilde{u}_1 - \tilde{u}_2|^2_{L^2(\Omega\times(0,1))} dt\right) + 2a^2C_\varphi^2 \int_0^{T'} |\Lambda(\tilde{s}_1) - \Lambda(\tilde{s}_2)|_H^2 dt\\
\le&2C_e a^2\sqrt{T'}C_3|h_1 - h_2|^2_{W^{1,2}(0,T';H)} + 2C_e a^2 T' C_3 |h_1 - h_2|^2_{W^{1,2}(0,T';H)}\\
&+ 2a^2C_\varphi^2 \int_0^{T'} |\Lambda(\tilde{s}_1) - \Lambda(\tilde{s}_2)|_H^2 dt.\end{aligned} \tag{5.2}$$

By using (3.10) and

$$|\Lambda(\tilde{s}_1(t)) - \Lambda(\tilde{s}_2(t))|_H^2 \le T' \int_0^{T'} |\Lambda(\tilde{s}_1(\tau))_\tau - \Lambda(\tilde{s}_2(\tau))_\tau|_H^2 d\tau$$

we can obtain from (5.2) that

$$\begin{aligned}&|\Lambda(\tilde{s}_1) - \Lambda(\tilde{s}_2)|^2_{W^{1,2}(0,T';H)}\\
\le&(1+T'^2)\sqrt{T'}\left[\tilde{C}_1(T')|\Lambda(\tilde{s}_1)-\Lambda(\tilde{s_2})|^2_{W^{1,2}(0,T';H)}+\tilde{C}_2(T')|\tilde{s}_1-\tilde{s}_2|^2_{L^2(0,T';H)}\right],\end{aligned}$$

where $\tilde{C}_1(T')$ and $\tilde{C}_2(T')$ are positive constants depending on a, C_e, C_3 as in (4.4), $M_6(T')$ as in (3.10) and T'. Hence, for a small $0 \le T' < T$ such that $\sqrt{T'}(1 + T'^2)(\tilde{C}_1(T') + \tilde{C}_2(T')) < \frac{1}{2}$, we see that Λ is a contraction mapping on $X(T', \delta_*)$. Therefore, by Banach's fixed point theorem there exists one and only one $s \in X(T', \delta_*)$ such that $\Lambda(s) = s$ in $X(T', \delta_*)$. This means that (h, u, s) with $u(t, x, z) = \tilde{u}\left(t, x, \frac{z-s(t,x)}{L-s(t,x)}\right)$ for $(t, x, z) \in Q_s(T')$ is a unique solution of (P) on $[0, T']$.

5.2. Global existence

In this section, we establish a globally-in-time solution of (P). Let define

$$T^* := \sup\{T' > 0 | \text{ (P) has a solution } (h, s, u) \text{ on } [0, T']\}.$$

By the local existence result, it is clear that $T^* > 0$. Now, we assume $T^* < T$. Then, by Lemma 3, (i) of Lemma 4 and Theorem 2 it holds that for any $t < T^*$,

$$\kappa_0 \le h(t) \le h^* \text{ a.e. on } \Omega, \; 0 \le u(t) \le 1 \text{ a.e. on } \Omega \times [s(t, \cdot), L], \tag{5.3}$$

$$|h|_{W^{1,2}(0,t;H)} + |h|_{L^\infty(0,t;H^1(\Omega))} \le M_1(T), \tag{5.4}$$

$$|s_t| \le 2a, \; |\nabla h| \le M_2(T), \; |h_t| \le M_4(T) \text{ a.e. on } Q(T^*), \tag{5.5}$$

$$\int_0^t \int_{s(\tau)}^L |u_\tau(\tau)|^2 dz d\tau + \int_{s(t)}^L |u_z(t)|^2 dz \le M_7(T) \text{ a.e. on } \Omega, \tag{5.6}$$

where $M_7(T)$ is a positive constant depending on u_{0z}, ρ_v, ρ_w, k, a, $M_4(T)$ and C_1. Then, by (i) of Theorem 2 and (5.6) we see that there exists a positive constant δ_* depending on $M_7(T)$ such that

$$0 \le s \le L - \delta_* \text{ for } t \in [0, T^*) \text{ a.e. on } \Omega. \tag{5.7}$$

Next, by $h_t - \Delta G(h) = s(1 - f(h))v$ a.e. on $Q(T^*)$, $|s(1 - f(h))v| \le L|v|_{L^\infty(Q(T))}$ and (5.5) we infer that $\Delta G(h(t))$ is bounded in $L^\infty(\Omega)$ for a.e. $t \in [0, T^*)$. Hence, from this result, (5.3), and $g_0 \le g$ we have that

$$|\Delta h(t)|_{L^\infty(\Omega)} \le C \text{ for a.e. } t \in [0, T^*), \tag{5.8}$$

where C is a positive constant. Also, by (A6) and (5.4), $h(t) - h_b(t)$ is bounded in $H^1_0(\Omega)$ for a.e. $t \in [0, T^*)$. Moreover, by (4.1), (5.6) and (5.7) it follows that for $t \in [0, T^*)$,

$$\begin{aligned}
&\int_0^t \int_0^1 |\tilde{u}_t(\tau)|^2 dy dt = \int_0^t \int_{s(\tau)}^L \left| u_t(\tau) + u_z(\tau) \frac{(L - z)s_t(\tau)}{L - s(\tau)} \right|^2 \frac{1}{L - s(\tau)} dz dt \\
&\le \frac{2}{\delta_*} \int_0^t \int_{s(\tau)}^L |u_t(\tau)|^2 dz dt + \frac{2(4La)^2}{\delta_*^3} \int_0^t \int_{s(\tau)}^L |u_z(\tau)|^2 dz dt \\
&\le \left(\frac{2}{\delta_*} + \frac{2(4La)^2 T}{\delta_*^3} \right) M_7(T) \text{ a.e. on } \Omega,
\end{aligned} \tag{5.9}$$

and

$$\int_0^1 |\tilde{u}_y(t)|^2 dy = \int_{s(t)}^L |u_z(t)(L-s(t))|^2 \frac{1}{L-s(t)} dz$$
$$\leq L \int_{s(t)}^L |u_z(t)|^2 dz \leq LM_7(T) \text{ a.e. on } \Omega. \tag{5.10}$$

Now, let N be a subset of $[0,T^*)$ with $|N|=0$ such that for $t \in [0,T^*) \setminus N$, it holds that $\nabla h(t)$ and $\Delta h(t)$ are bounded in $L^\infty(\Omega)$, $h(t)-h_b(t)$ is bounded in $H_0^1(\Omega)$, and $\tilde{u}(t,x,1) = h(t,x)$ for a.e. $x \in \Omega$. Since $h(t)$, $s(t)$ and $\tilde{u}(t)$ also satisfy (5.3), (5.4) and (5.7) for $t \in [0,T^*) \setminus N$, we can take a sequence $\{t_n\} \subset [0,T^*) \setminus N$ such that $t_n \to T^*$ as $n \to \infty$ and for some $h(T^*)$, $s(T^*) \in H$, $\eta_i \in L^\infty(\Omega)(1 \leq i \leq 3)$ and $\tilde{u}(T^*) \in L^2(\Omega \times (0,1))$ the following convergences hold:

$$h(t_n) \to h(T^*) \text{ in } H, \text{ weakly in } H^1(\Omega), \tag{5.11}$$
$$\partial_i h(t_n) \to \eta_i \text{ weakly - * in } L^\infty(\Omega), \tag{5.12}$$
$$h(t_n) - h_b(t_n) \to h(T^*) - h_b(T^*) \text{ weakly in } H_0^1(\Omega), \tag{5.13}$$
$$s(t_n) \to s(T^*) \text{ weakly in } H, \tag{5.14}$$
$$\tilde{u}(t_n) \to \tilde{u}(T^*) \text{ weakly in } L^2(\Omega \times (0,1)) \text{ as } n \to \infty, \tag{5.15}$$

where ∂_i is the weak derivative with respect to x_i.

Lemma 5. *Let $h(T^*)$, $s(T^*) \in H$ and $\tilde{u}(T^*) \in L^2(\Omega \times (0,1))$ be functions satisfying (5.11)-(5.15). Then, it holds that*

(i) $h(T^*) \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$, $\kappa_0 \leq h(T^*) \leq h^*$ a.e. on Ω, $\Delta h(T^*)$ is bounded a.e. on Ω and $h(T^*) = h_b(T^*)$ a.e. on $\partial\Omega$.

(ii) $0 \leq s(T^*) \leq L - \delta_*$ a.e. on Ω, where δ_* is the same as in (5.7).

(iii) $0 \leq u(T^*) \leq 1$ a.e. on $\Omega_{s(T^*)}$, the function $x \to |u(T^*,x)|_{H^1(s(T^*),L)}$ is bounded a.e. on Ω and $u(T^*,x,L) = h(T^*,x)$ for a.e. $x \in \Omega$.

Proof. (i) By (5.3) and (5.11) it is clear that $\kappa_0 \leq h(T^*) \leq h^*$ a.e. on Ω. Also, by (5.11) and (5.12), we have that $\eta_i = \partial_i h(T^*) \in L^\infty(\Omega)$. Next, we show that $h(T^*) \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ and $\Delta h(T^*)$ is bounded a.e. on Ω. Let define $-\Delta h(T^*)$ in the following distribution sense:

$$\langle -\Delta h(T^*), \varphi \rangle = \int_\Omega \nabla h(T^*) \nabla \varphi dx \text{ for } \varphi \in C_0^\infty(\Omega).$$

Then, for $\varphi \in C_0^\infty(\Omega)$ it holds that

$$\langle -\Delta h(T^*), \varphi \rangle = \int_\Omega -h(T^*) \Delta \varphi dx = \lim_{n \to \infty} \int_\Omega -h(t_n) \Delta \varphi dx$$
$$= \lim_{n \to \infty} \int_\Omega \nabla h(t_n) \nabla \varphi dx = \lim_{n \to \infty} \int_\Omega -\Delta h(t_n) \varphi dx.$$

Hence, by (5.8) it follows that $\langle -\Delta h(T^*), \varphi\rangle \le C|\varphi|_{L^1(\Omega)}$ for $\varphi \in C_0^\infty(\Omega)$. Accordingly, by Hahn-Banach's theorem, there exists $l \in L^\infty(\Omega)$ such that

$$\langle l, \varphi\rangle_{L^\infty(\Omega),L^1(\Omega)} = \langle -\Delta h(T^*), \varphi\rangle \text{ for } \varphi \in C_0^\infty(\Omega). \qquad (5.16)$$

Consequently, by (5.16) and the regularity result of elliptic problems, we see that $h(T^*) \in H^2(\Omega)$, and therefore, $l = -\Delta h(T^*) \in H$. Then, by $l \in L^\infty(\Omega)$ we also obtain that $\Delta h(T^*) \in L^\infty(\Omega)$. Thus, from these result (i) holds.

(ii) Since $0 \le s(t_n) \le L - \delta_*$ a.e. on Ω, $[s(t_n) - (L - \delta_*)]^+ = 0$ and $-[s(t_n) - (L - \delta_*)]^+ \le -(s(t_n) - (L - \delta_*))$, it follows that

$$\begin{aligned} &|[s(T^*) - (L - \delta_*)]^+|_H^2 \\ =&([s(T^*) - (L - \delta_*)]^+, s(T^*) - (L - \delta_*) - [s(t_n) - (L - \delta_*)]^+)_H \\ \le&([s(T^*) - (L - \delta_*)]^+, s(T^*) - (L - \delta_*) - (s(t_n) - (L - \delta_*))_H. \end{aligned}$$

Hence, by (5.14), we derive that $|[s(T^*) - (L - \delta_*)]^+|_H^2 = 0$ which implies that $s(T^*) \le L - \delta_*$ a.e. on Ω. Similarly, we have that $0 \le s(T^*)$ a.e. on Ω.

(iii) By using (4.1) and (5.9), it holds that for the function $\tilde{u}$,

$$|\tilde{u}(t_n) - \tilde{u}(t_m)|_{L^2(0,1)}^2 \le |t_n - t_m| \left(\frac{2}{\delta_*} + \frac{2(4La)^2T}{\delta_*^3}\right) M_7(T) \text{ a.e. on } \Omega.$$

This implies that $\tilde{u}(t_n)$ is a Cauchy sequence in $L^2(\Omega \times (0,1))$, namely, by (5.15) we see that

$$\tilde{u}(t_n) \to \tilde{u}(T^*) \text{ in } L^2(\Omega \times (0,1)) \text{ as } n \to \infty. \qquad (5.17)$$

By (5.3) and (5.17) it is clear that $0 \le \tilde{u}(T^*) \le 1$ a.e. on $\Omega \times (0,1)$. Also, from (5.11) and (5.17) there exists a subsequence $\{n_k\} \subset \{n\}$ and $M_1 \subset \Omega$ with $|M_1| = 0$ such that

$$h(t_{n_k}, x) \to h(T^*, x) \text{ and } \tilde{u}(t_{n_k}, x) \to \tilde{u}(T^*, x) \text{ in } L^2(0,1) \qquad (5.18)$$

as $k \to \infty$ for $x \in \Omega \setminus M_1$. Moreover, by (5.10), there exists $M_2 \subset \Omega$ with $|M_2| = 0$ such that $\tilde{u}_y(t_{n_k})$ is bounded in $L^2(0,1)$ on $\Omega \setminus M_2$. Then, for $x \in \Omega \setminus (M_1 \cup M_2)$, we can take a subsequence $\{t_{n_k}(x)\} \subset \{t_{n_k}\}$ such that for some $\xi(x) \in L^2(0,1)$,

$$\tilde{u}_y(t_{n_k}(x), x) \to \xi(x) \text{ weakly in } L^2(0,1). \qquad (5.19)$$

Therefore, by (5.18) and (5.19), we see that $\xi(x) = \tilde{u}_y(T^*, x)$ in $L^2(0,1)$ on $\Omega \setminus (M_1 \cup M_2)$, namely, $\tilde{u}(T^*) \in H^1(0,1)$ a.e. on Ω.

Finally, we prove that $\tilde{u}(T^*, x, 1) = h(T^*, x)$ a.e. on Ω. From (5.18) we see that $\tilde{u}(t_{n_k}(x), x) \to \tilde{u}(T^*, x)$ in $L^2(0,1)$ and $h(t_{n_k}(x), x) \to h(T^*, x)$ as

$k \to \infty$ and $\tilde{u}(T^*, x) \in H^1(0,1)$ for $x \in \Omega \setminus (M_1 \cup M_2)$. Here, by (5.1) it holds that

$$\begin{aligned} &|\tilde{u}(t_{n_k}(x), x, 1) - \tilde{u}(T^*, x, 1)|^2 \\ \le & C_e |\tilde{u}(t_{n_k}(x), x) - \tilde{u}(T^*, x)|_{H^1(0,1)} |\tilde{u}(t_{n_k}(x), x) - \tilde{u}(T^*, x)|_{L^2(0,1)}. \end{aligned}$$

Hence, by the convergence of $\tilde{u}(t_{n_k}(x))$ in $L^2(0,1)$ and the fact that $\tilde{u}(t_{n_k}(x))$ is bounded in $H^1(0,1)$ for $x \in \Omega \setminus (M_1 \cup M_2)$ we have that $\tilde{u}(t_{n_k}(x), x, 1) \to \tilde{u}(T^*, x, 1)$ as $k \to \infty$, and therefore $\tilde{u}(T^*, x, 1) = h(T^*, x)$ a.e. on Ω. Thus, we see that $u(T^*, x, z) = \tilde{u}\left(T^*, x, \frac{z - s(T^*, x)}{L - s(T^*, x)}\right)$ satisfies (iii). □

By Lemma 5 we can consider $h(T^*)$, $s(T^*)$ and $u(T^*)$ as an initial data. Finally, by repeating the argument of the local existence, we see that the solution can be extended beyond T^*. This is a contradiction of the definition of T^*, and therefore, T^* must coincide with T. Thus, we can show that (P) has a solution globally in time, and Theorem 1 is proved.

6. Conclusion

We studied a two-scale problem as a mathematical model describing moisture transport phenomena arising in concrete carbonation process. Under suitable assumptions, we established a globally-in-time solution of our two-scale model by the method of extending local solutions.

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Существование глобального решения для многомерной модели влагопереноса в бетонных материалах

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Аннотация. В предыдущем исследовании [5] мы доказали существование локального по времени решения для двумерной задачи, которая дается в качестве математической модели влагопереноса, возникающего в процессе карбонизации бетона. Двумерная модель состоит из уравнения диффузии относительной влажности в макро-области и задач со свободной границей, описывающих процесс смачивания и сушки в бесконечных микро-областях. В этой статье, улучшая уравнение диффузии относительной влажности на основе экспериментального результата [3; 10], мы строим глобальное решение двумерной модели. Для доказательства существования глобального решения мы получили равномерные оценки и равномерную ограниченность решения по времени и использовали метод расширения локальных решений.

Ключевые слова: двумерная модель, задача со свободной границей, квазилинейное параболическое уравнение, уравнение влагопереноса.

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