Серия «Математика»
2019. Т. 28. С. 21—35

Онлайн-доступ к журналу:
http://mathizv.isu.ru

И З В Е С Т И Я
Иркутского
государственного
университета

УДК 517.926
MSC 34A09
DOI https://doi.org/10.26516/1997-7670.2019.28.21

The Index and Split Forms of Linear Differential-Algebraic Equations *

M. V. Bulatov
Matrosov Institute for System Dynamics and Control Theory of SB RAS, Irkutsk, Russian Federation

V. F. Chistyakov
Matrosov Institute for System Dynamics and Control Theory of SB RAS, Irkutsk, Russian Federation

Abstract. We consider linear systems of ordinary differential equations (ODE) with rectangular matrices of coefficients, including the case when the matrix before the derivative of the desired vector function is not full rank for all argument values from the domain. Systems of this type are usually called differential-algebraic equations (DAEs). We obtained criteria for the existence of nonsingular transformations splitting the system into subsystems, whose solution can be written down analytically using generalized inverse matrices. The resulting solution formula is called a generalized split form of a DAE and can be viewed as a certain analogue of the Weierstrass-Kronecker canonical form. In particular, it is shown that arbitrary DAEs with rectangular coefficient matrices are locally reducible to a generalized split form. The structure of these forms (if it is defined on the integration segment) completely determines the structure of general solutions to the systems. DAEs are commonly characterizes by an integer number called index, as well as by the solution space dimension. The dimension of the solution space determines arbitrariness of the the general solution manifold. The index determines how many times we should differentiate the entries on which the solution to the problem depends. We show the ways of calculating these main characteristics.

Keywords: differential-algebraic equations, canonical form, split form, solution space, index, singular points.

* This research was supported by the program of fundamental scientific researches of the SB RAS No.18-01-00643 A, 18-51-54001 Vietnam.

1. Problem statement

Since the beginning of 1970s, analysis of complex electrical and hydraulic networks, as well as electronic circuits, often employed systems that include interconnected ordinary differential equations (ODEs) of various orders and algebraic equations. Such systems can be written as vector equations with a singular matrix multiplying the leading term (cf. [2]- [15]). Consider a first order system

$$\Lambda_1 x := A(t)\dot{x} + B(t)x = f,\ t \in T = [\alpha, \beta], \tag{1.1}$$

where $A(t),\ B(t)$ are $(m \times n)$-matrices, $x \equiv x(t),\ f \equiv f(t)$ is the desired and the given vector functions, correspondingly, $\dot{x} := dx(t)/dt$. It is assumed that all entries are smooth enough, and

$$\operatorname{rank} A(t) < \min\{m, n\}\ \forall t \in T. \tag{1.2}$$

System (1.1) is said to be closed, if the number of equations is equal to the number of components in the desired vector function ($m = n$); if $m > n$, the system is called overedetermined; otherwise, if $m < n$, we say that the system is underdetermined. For a closed system, condition (1.2) is similar to the equality $\det A(t) \equiv 0,\ t \in T$. In what follows, systems of type (1.1) with constant coefficients are called stationary systems (SS).

By the solution to (1.1) on T we understand a vector function $x(t) \in \mathbf{C}^1(T)$ that turns (1.1) into identity on T.

If (1.1) satisfies (1.2), then (1.1) is commonly referred to as a differential algebraic equation (DAE) [4] or an algebraic differential system [3]. Sometimes such systems are called singular systems [2], or descriptor systems [1], [10]. The research in this area has been carried out for 40 years (see [11] and the references listed therein). However, some issues for non-closed DAEs remain relevant and are addressed below.

Remark 1. For simplicity, the dependence on t sometimes can be omitted, if this does not cause confusion. Inclusions $V(t) \in \mathbf{C}^i(T),\ i > 1$, $V(t) \in \mathbf{C}(T)$, where $V(t)$ is a matrix or a vector function, mean that all derivatives of all elements of $V(t)$ are continuous up to order i, or simply continuous.

Definition 1. *The solution space of the homogeneous system (1.1) is finite-dimensional on T, if there exists an $(n \times d)$-matrix $X_d(t) \in \mathbf{C}^1(T)$ with the smallest possible d, such that any linear combination $x(t, c) = X_d(t)c$, where vector c takes all values from $\mathbf{R}^d$, satisfies $\Lambda_1 x(t, c) \equiv 0$, and there are no other solutions to $\Lambda_1 x = 0$ on T different to $x(t, c)$. The kernel of the operator Λ_1 is finite-dimensional (*$\dim \ker \Lambda_1 < \infty$*), if the solution space of (1.1) is finite-dimensional: $d < \infty$. The number d will be called the solution space, or the kernel dimension.*

The linear spaces $\ker \Lambda_1 = \{u \in \mathbf{C}^1(T) : \Lambda_1 u = 0\}$, $\mathrm{im}\, \Lambda_1 u = \{\phi \in \mathbf{C}(T) : \Lambda_1 u = \phi$, for which $u \in \mathbf{C}^1(T)\}$, are called the kernel and the image of the operator Λ_1, correspondingly.

Definition 2. *The closed system (1.1) has a Cauchy type solution on T, if it is solvable for any vector function $f(t) \in \mathbf{C}^{kn}(T)$ and its solutions can be written down as a linear combination*

$$x(t, c) = X_d(t)c + \psi(t), \tag{1.3}$$

where $X_d(t)$ is an $(n\times d)$-matrix from $\mathbf{C}^k(T)$ with the property $\mathrm{rank}\, X_d(t) = d$ $\forall t \in T$, *c is a vector of arbitrary constants, $\psi(t)$ is a vector function with the property $\Lambda_1\psi(t) = f(t)$, $t \in T$. Additionally, any subsegment $[\alpha_0, \beta_0] \subseteq T$ does not contain solutions different to $x(t, c)$.*

Remark 2. If (1.1) is closed, $A(t)$, $B(t)$, $f \in \mathbf{C}(T)$, $\det A(t) \neq 0$ $\forall t \in T$, then its general solution is expressed by the Cauchy formula $x(t, c) = X_n(t)c + \psi(t)$, $\psi(t) = \int_\alpha^t K(t, s)f(s)ds$, $t \in T$, [7] where $X_n(t)$ is the Cauchy matrix of the system

$$\dot{x} = -A^{-1}(t)B(t)x, K(t, s) = X_n(t)X_n^{-1}(s)A^{-1}(s).$$

DEAs with the Cauchy type solutions inherit very important properties of linear systems in the Cauchy form: 1) the solution sets on T and on $T_0 \subset T$ coincide (there is no "memory"); 2) if a solution to a DAE passes through the point ($b \in \mathbf{R}^n$, $\zeta \in T$), then this solution is unique on T. For an equation with aftereffect, for example for the Fredholm equation $x(t) = \int_\alpha^\beta x(s)ds + 1$, $t \in T$, the solution is $x(t) = 1/(1 - \tau)$, $\tau = \beta - \alpha$ and it changes (or may not exist) on $T_0 \subset T$.

Definition 3. *If there exist operators*

$$\Omega_l = \sum_{j=0}^{l} L_j(t)(d/d)^j,\ \Omega_r = \sum_{j=0}^{r} R_j(t)(d/d)^j,$$

where $L_j(t)$ are $(m\times m)$-matrices and $R_j(t)$ are $(n\times n)$-matrices from $\mathbf{C}(T)$ with the properties

$$\Omega_l \circ \Lambda_k y = \tilde{A}(t)\dot{y} + \tilde{B}(t)y,\ \Lambda_k \circ \Omega_r y = \bar{A}(t)\dot{y} + \bar{B}(t)y,$$

where $y \equiv y(t)$ is an arbitrary vector function from $\mathbf{C}^{\nu+1}(T)$, $\nu = \{l \text{ or } r\}$, $\tilde{A}(t)$, $\tilde{B}(t), \bar{A}(t), \bar{B}(t)$ are some $(m \times n)$-matrices from $\mathbf{C}(T)$, $\mathrm{rank}\, \tilde{A}(t) = \mathrm{rank}\, \bar{A}(t) = \min\{m, n\}$ $\forall t \in T$, *then such operators are called the left and the right regularizing operators (LRO and RRO) for the DAE (1.1), and the smallest possible l and r are said to be the right index and the left index, correspondingly.*

Below, we will need the formula that follows from the Leibniz rule for product differentiation

$$\mathrm{d}_i[MF] = \mathcal{M}_i[M]\mathrm{d}_i[F], \tag{1.4}$$

where $M \equiv M(t),\ F \equiv F(t)$ are some matrices of suitable dimension from $\mathbf{C}^i(T)$, $\mathrm{d}_i[M] = \left(M^\top\ \ (d/dt)M^\top\ \cdots\ (d/dt)^i M^\top\right)^\top$, $\top$ denotes transposition, $\mathcal{M}_i[M] = \|\tilde{M}_{pj}\|_{p=\overline{0,i},j=\overline{0,i}}$, $\tilde{M} \equiv 0$, if $j > p$, and $\tilde{M}_{pj} = C_j^p M^{(p)}$, if $j \le p$, $C_p^j = p!/j!(p-j)!$ are binomial coefficients.

Definition 4. *System (1.1) together with its derivatives of order up to* i $\mathrm{d}_i[\Lambda_1 x - f] = 0,\ t \in T$, *where* $\mathrm{d}_i[.]$ *is defined by (1.4), is said to be the* i*-extended system (1.1).*

Using (1.4), write down the i-extended system as

$$D_i[A(t),B(t)]\mathrm{d}_{i+1}[x] = \left(\mathcal{M}_i[B(t)]\ \ O\right) + \left(O\ \ \mathcal{M}_i[A(t)]\right)\mathrm{d}_{i+1}[x] = \mathrm{d}_i[f(t)], \tag{1.5}$$

where the matrix $D_i[A(t),B(t)]$ is of dimension $[(i+1)m \times (i+2)n]$, the zero matrices O are of dimension $[(i+1)m \times n]$. In what follows, we will use the splitting

$$D_i[A(t),B(t)] = \left(\tilde{B}_i(t)\ \ \Gamma_i[A(t),B(t)]\right), \tag{1.6}$$

where $\Gamma_i[A(t),B(t)]$ is a block triangular matrix with the blocks $A(t)$ on the diagonal, $\tilde{B}_i(t) = \mathrm{d}_i[B(t)]$.

Definition 5. *If there exists the operator* $\tilde{\Omega}_l$*, for which the LRO is defined and* $\tilde{\Omega}_l \circ \Lambda_k y = \tilde{A}(t)\dot{y} + \tilde{B}(t)y\ \forall y \equiv y(t) \in \mathbf{C}^{l+1}(T)$*, where* $\tilde{A}(t) \not\equiv 0,\ t \in T$*, then the isolated points* $t_\nu \in T$: $\operatorname{rank} \tilde{A}(t_\nu) < \min\{m,n\}$ *are called the singular points of (1.1).*

Example 1. Consider the following three systems

$$1)\ \begin{pmatrix} -\mathrm{s}(t) & -\mathrm{c}(t) \\ \mathrm{s}(t) & \mathrm{c}(t) \end{pmatrix}\dot{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = f;\ 2)\ \begin{pmatrix} -\mathrm{s}(t) & \mathrm{s}(t) \\ -\mathrm{s}(t) & \mathrm{s}(t) \end{pmatrix}\dot{y} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y = f,$$

$$3)\ \begin{pmatrix} -\mathrm{s}(t) & \mathrm{s}(t) \\ -\mathrm{s}(t) & \mathrm{s}(t) \end{pmatrix}\dot{z} + \begin{pmatrix} 1 & 0 \\ 1 & \mathrm{c}(t) \end{pmatrix} z = f,\ t \in T,$$

where $x = (x_1\ x_2)^\top,\ y = (y_1\ y_2)^\top,\ z = (z_1\ z_2)^\top, f = (f_1\ f_2)^\top$, $\mathrm{s}(t) = \sin^2(\omega(t)),\ \mathrm{c}(t) = \cos^2(\omega(t))$, $\omega(t)$ is a given smooth function on T. Here, in Examples 1) and 3), roots of the functions $\cos(2\omega(t))$, $\mathrm{c}(t)$, $\mathrm{s}(t)$ are singular points in terms of Definition 5:

$$\tilde{\Omega}_1 = \operatorname{diag}\{1, (d/dt)\}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix};\ \det\ \tilde{A}(t) = \begin{pmatrix} -\mathrm{s}(t) & -\mathrm{c}(t) \\ 1 & 1 \end{pmatrix} = \cos(2\omega(t)),$$

$$\tilde{\Omega}_1 = \text{diag}\{1, (d/dt)\} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; \ \det\ \tilde{A}(t) = \begin{pmatrix} -\mathrm{s}(t) & -\mathrm{s}(t) \\ 0 & \mathrm{c}(t) \end{pmatrix} = -\mathrm{s}(t)\mathrm{c}(t).$$

Even though the points where the rank of $A(t)$ changes may be located anywhere on T, there exists $y = f - A(t)\dot{f}\ \forall f \in \mathbf{C}^2(T)$. We can assume $\Omega_2 = (d/dt)[E_2 - A(t)(d/dt)]$ as the LRO.

Now perform a more detailed analysis by summing the first and the second DAE. We obtain

$$\cos(2\omega(t))\dot{x}_1 + x_1 = \tilde{f}_1,\ x_2 + x_1 = \tilde{f}_2,\ 0 \cdot \dot{y}_1 + y_1 = \bar{f}_1,\ y_2 - y_1 = \bar{f}_2,$$

$$-\mathrm{s}(t)\dot{z}_1 + \mathrm{s}(t)\dot{z}_2 + z_1 = f_1,\ \mathrm{c}(t)z_2 = \bar{f}_2,$$

where $\tilde{f}_1 = f_1 + (\mathrm{c}(t)\ \mathrm{c}(t))\dot{f},\ \tilde{f}_2 = f_2 + f_1,\ \bar{f}_1 = f_1 - (-\mathrm{s}(t)\ \mathrm{s}(t))\dot{f},\ \bar{f}_2 = f_2 - f_1$. Components of solutions can behave differently for different $\omega(t)$. In particular roots of the equation $\mathrm{c}(t) = 0$ may exist on T, whereas roots of $\mathrm{s}(t) = 0$ and $cos(2\omega(t)) = 0$ may not.

Remark 3. For systems (1.1) with the properties

$$\det A(t) \neq 0\ \forall t \neq \zeta,\ t, \zeta \in T, \det A(\zeta) = 0,$$

the point ζ is always singular, the classic existence and uniqueness theorem is violated there. Earlier, more powerful methods for studying properties of solutions in the neighborhood of ζ were developed (see [7], [12]- [14] and the references listed therein). Generally, we cannot predict if a DAE has singular points by evaluating the behavior of the rank of matrices $A(t), (A(t)|B(t))$. Before applying any of the methods from the papers metioned above, the following two problems should be solved : 1) find singular points; 2) categorize them as it was shown in Example 1, where the DAE under study was split into subsystems (algebraic and ordinary differential). Such splitting cannot always be done, even in theory (below we give a example).

Below, we will need information on properties of variable matrices and algebraic equations that correspond to them.

Definition 6. *(see, for example, [2]). The $(n \times m)$-matrix $M^-(t)$ is said to be a semi-inverse of the $(m \times n)$-matrix $M(t)$, $t \in T$, if for any $t \in T$ $M(t)M^-(t)M(t) = M(t)$.*

A semi-inverse is defined for any $t \in T$ and for any $(m \times n)$-matrix $M(t)$.The theory of generalized inverse matrices can be found in a number of monographs (see, for example, [2]). If $M(t)$ is square and non-singular, then $M^{-1}(t) = M^-(t)$. According to [15], there exists the matrix $M^-(t) \in \mathbf{C}^q(T)$, if $M \in \mathbf{C}^q(T)$ and rank $M(t) = r = const \forall t \in T$. If rank $M(t) \neq const, t \in T$, then at least one element of the matrix $M^-(t)$ has a second type discontinuity in the point $t \in T$ where the rank changes.

The system of algebraic equations $M(t)x = \phi(t),\ t \in T$, is solvable if and only if

$$\Pi(t)\phi(t) = 0,\ \Pi(t) = \Big[E_m - M(t)M^-(t)\Big] t \in T, \tag{1.7}$$

and its solution can be written as

$$x = M^-(t)\phi(t) + \tilde{\Pi}(t)u(t),\ \tilde{\Pi}(t) = \Big[E_n - M^-(t)M(t)\Big], \tag{1.8}$$

where the expressions $\Pi(t),\ \tilde{\Pi}(t)$ are projectors, $u(t)$ is an arbitrary vector function. The system has (cf. [2, c.34]) constant solutions

$$x = \mathcal{G}^-\theta + [E_n - \mathcal{G}^-\mathcal{G}]\mathrm{c}, \tag{1.9}$$

where $\mathcal{G} = \int\limits_\alpha^\beta M^\top(s)M(s)ds,\ \theta = \int\limits_\alpha^\beta M^\top(s)\phi(s)ds$, c is an arbitrary vector from $\mathbf{R}^n$, if and only if

$$\phi(t) = M(t)\mathcal{G}^-\theta. \tag{1.10}$$

2. Structure of general solutions and index of linear DAEs

It is known [7, p.335] that for any pencil of constant $m \times n$ matrices $\lambda A + B$ there exist constant matrices $P,\ Q$ of suitable dimension such that: $\det\ P \det\ Q \neq 0$,

$$P(\lambda A + B)Q = \{\lambda E_d + J,\quad \lambda N + E_\mu,\quad \lambda L + M,\quad \lambda L^\star + M^\star,\quad 0\}, \tag{2.1}$$

where J is some $(d \times d)$-block, E_* is an identity matrix of dimension $*$,

$$\lambda N + E_\mu = \{\lambda N_{k_1} + E_{k_1}, \cdots, \lambda N_{k_p} + E_{k_p}\},$$
$$\lambda L + M = \{\lambda L_{\eta_1} + M_{\eta_1}, \cdots, \lambda L_{\eta_q} + M_{\eta_q}\},$$
$$\lambda L^\star + M^\star = \{\lambda L^\star_{\nu_1} + M^\star_{\nu_1}, \cdots, \lambda L^\star_{\nu_v} + M^\star_{\nu_v}\},$$

$$N_{k_j} = \begin{pmatrix} 0 & E_{n_j-1} \\ 0 & 0 \end{pmatrix},\ j = \overline{1,p},\ L^\star_{\nu_1} = \begin{pmatrix} E_{\nu_j-1} \\ 0 \end{pmatrix},\ M^\star_{\nu_j} = \begin{pmatrix} 0 \\ E_{\nu_j-1} \end{pmatrix},\ j = \overline{1,v},$$

$$L_{\eta_1} = \begin{pmatrix} E_{\eta_j-1} & 0 \end{pmatrix},\ M_{\eta_j} = \begin{pmatrix} 0 & E_{\eta_j-1} \end{pmatrix},\ j = \overline{1,q}.$$

In the blocks $L,\ M$ the zero sub-blocks are either columns or rows. The representation (2.1) is called a canonical structure of the pencil and was first introduced in the papers by Weierstrass and Kronecker. Based on (2.1), [7] gives necessary and sufficient solvability conditions for stationary systems. Moreover, the form of (2.1) contains full information about properties of stationary systems. Transformation of the DAE (1.1) to the Weierstrass-Kronecker canonical form faces a serious difficulty: generally, the canonical structure of the pencil $\lambda A(t) + B(t),\ t \in T$ may not coincide with the canonical structure of the pencil for the system $A(t)d[Q(t)y]/dt + B(t)Q(t)y = f(t),\ t \in T$, where $Q(t)$ is a non-singular matrix from $\mathbf{C}^1(T)$ (some examples can be found in [15]).

In 1982, S.L. Campbell S.L. and L.R. Petzold (see, for example, [4] and the references listed therein) introduced a notion of the central canonical form (CCF) for closed DAEs: it is assumed that there exist smooth matrices $P(t),\ Q(t)$, det $P(t)Q(t) \neq 0\ \forall t \in T$ with the property

$$P(t)A(t)d[Q(t)y]/dt + P(t)B(t)Q(t)y = \tag{2.2}$$

$$= \mathrm{diag}\{E_d, N(t)\}\dot{y} + \mathrm{diag}\{J(t), E_{n-d}\}y = P(t)f(t),$$

where $J(t),\ N(t)$ are some square blocks, $N(t)$ has an upper-triangular form with a zero diagonal for any $t \in T$. Using (2.2), we can write down a general solution for the closed DAE (1.1). A specific structural form was introduced for non-closed DAEs in [9]. In particular, it is different to the canonical structure for stationary systems. In the case of variable smooth matrices, it is generally impossible to split system (1.1) with the same detalization as it can be done for SS. Moreover, we cannot always guarantee the reduction to the CCF.

Example 2. Consider the index 2 DAE from [4]:

$$\Lambda_1 x = A(t)\dot{x} + x = \begin{pmatrix} 0 & w(t) \\ v(t) & 0 \end{pmatrix}\dot{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = \phi,\ t \in T,$$

where $v(t),\ w(t) \in \mathbf{C}^\infty(T),\ v(t)w(t) = 0\ \forall t \in T$. This DAE cannot be reduced to the CCF on the whole T, because it is impossible to find even a continuous matrix with the properties $\det P(t) \neq 0,\ P(t)A(t) = \begin{pmatrix} A_1(t) \\ 0 \end{pmatrix}\ \forall t \in T$.

Assume that 1) all entries of (1.1) are continuous on T; 2) there exist non-singular for any $t \in T$ matrices $P(t) \in \mathbf{C}(T)$, $Q(t) \in \mathbf{C}^1(T)$, such that

$$P(t)A(t)Q(t)\dot{z} + P(t)[B(t)Q(t) + A(t)\dot{Q}(t)]z =$$

$$= \begin{pmatrix} L_{11} & L_{12}(t) & L_{13}(t) & L_{14}(t) & L_{15}(t) \\ 0 & L_{22}(t) & L_{23}(t) & L_{24}(t) & L_{23}(t) \\ 0 & 0 & L_{33}(t) & L_{34}(t) & L_{35}(t) \\ 0 & 0 & 0 & L_{44}(t) & L_{45}(t) \\ 0 & 0 & 0 & 0 & 0_{m_3\times n_3} \end{pmatrix}\dot{z}+$$

$$+\begin{pmatrix} M_{11} & M_{12}(t) & M_{13}(t) & M_{14}(t) & M_{15}(t) \\ 0 & M_{22}(t) & M_{23}(t) & M_{24}(t) & M_{25}(t) \\ 0 & 0 & M_{33}(t) & M_{34}(t) & M_{35}(t) \\ 0 & 0 & 0 & M_{44}(t) & M_{45}(t) \\ 0 & 0 & 0 & 0 & 0_{m_3\times n_3} \end{pmatrix} z = Pf,\ t \in T, \tag{2.3}$$

where $x = Q(t)z,\ L_{ij}(t),\ M_{ij}(t)$ are some blocks whose dimension is defined by the blocks located on the diagonal, $i = \overline{1,4},\ j \geq i.$ Here blocks $L_{11}(t),\ M_{11}(t)$ are $(d \times d)$-matrices, det $L_{11}(t) \neq 0\ \forall t \in T$, $L_{22}(t),\ M_{22}(t)$ are $(d_1 \times d_1)$-matrices, and the solution to the system

$$L_{22}(t)\dot{w} + M_{22}(t)(t)w = g(t),\ t \in T,$$

is unique and has the form

$$w = \hat{\Lambda}_{l-1} g = \sum_{j=0}^{l-1} \mathcal{C}_j(t)(d/dt)^j g(t)\ \forall g(t) \in \mathbf{C}^l(T), \tag{2.4}$$

where $\mathcal{C}_j(t)$ are $(d_1 \times d_1)$-matrices from $\mathbf{C}^1(T)$. The subsystem of (2.2), in particular, has such a solution :

$$N(t)\dot{w} + w = g,\ t \in T,$$

$$w(t) = g + \mathcal{T}g + \cdots + \mathcal{T}^{l-1}g,\ \mathcal{T} = -N(t)(d/dt). \tag{2.5}$$

If the solution to a DAE has the form (2.4), then it can be found by the formula

$$w(t) = \begin{pmatrix} E_{d_1} & 0 & \dots & 0 \end{pmatrix} D^{-}_{l-1}[L_{22}(t),\ M_{22}(t)]\mathrm{d}_{l-1}[f(t)] = \sum_{j=0}^{l-1} \mathcal{C}_j(t) g^{(j)}(t),$$

where we used notations from (1.5). For a pair of matrices $L_{33}(t),\ M_{33}(t)$, $L_{44}(t),\ M_{44}(t)$ of dimensions $(m_1 \times n_1)$ and $(m_2 \times n_2)$, correspondingly, the following inequalities are valid: $m_1 < n_1,\ \ m_2 > n_2,$

$$\text{rank } L_{33}(t) = \min\{m_1, n_1\},\ \text{rank } L_{44}(t) = \min\{m_2, n_2\}\ \forall t \in T. \tag{2.6}$$

Here $d + d_1 + m_1 + m_2 + m_3 = m,\ d + d_1 + n_1 + n_2 + n_3 = n.$

Definition 7. *The first part of (2.3) is said to be the generalized split form (GSF) of the DAE (1.1), whereas the number l in (2.4) is called index.*

Introduce the following splittings

$$z = \begin{pmatrix} z_1^\top & z_2^\top & z_3^\top & z_4^\top & z_5^\top \end{pmatrix}^\top,$$

$$\begin{pmatrix} f_1^\top & f_2^\top & f_3^\top & f_4^\top & f_5^\top \end{pmatrix}^\top = \begin{pmatrix} P_1^\top & P_2^\top & P_3^\top & P_4^\top & P_5 \end{pmatrix}^\top f = Pf, \tag{2.7}$$

and study the structure of solutions to subsystems of system (2.3). For the components from (2.7), we have:

$$L_{11}(t)\dot{z}_1 + M_{11}(t)z_1 = \tilde{f}_1,\ L_{22}(t)\dot{z}_2 + M_{22}(t)z_2 = \tilde{f}_2, \tag{2.8}$$

$$L_{33}(t)\dot{z}_3 + M_{33}(t)z_3 = \tilde{f}_3,\ L_{44}(t)\dot{z}_4 + M_{44}(t)z_4 = \tilde{f}_4, \tag{2.9}$$

$$0_{m_3 \times n_3}\dot{z}_5 + 0_{m_3 \times n_3}z_5 = f_5, \tag{2.10}$$

where $\tilde{f}_i = f_i - \sum_{j=1}^{5-i} \Big[L_{i,6-j}(t)\dot{z}_{6-j} + M_{i,6-j}(t)z_{6-j}\Big]$, $i = 4,3,2,1$. Then, from (2.10)- (2.8) we find z_5, z_4, z_3, z_2, z_1. Under the assumption that $f_5(t) \equiv 0,\ t \in T$, in (2.3) we have

$$z_5(t) = w_2(t), \tag{2.11}$$

where $w_2(t)$ is an arbitrary vector function of dimension n_3.

The matrix $L_{44}(t)$, where $m_2 > n_2$, is full rank for any $t \in T$. Using (1.7) and (1.8), write down a system of ODEs

$$\dot{z}_4 = -L_{44}^{-}(t)M_{44}(t)z_4 + L_{44}^{-}(t)\tilde{f}_4,$$

with the projector $\tilde{\Pi}_{44}(t) = \Big[E_{n_2} - L_{44}^{-}(t)L_{44}(t)\Big] \equiv 0$ and the consistency condition

$$\Pi_{44}(t)\Big[M_{44}(t)z_4 + \tilde{f}_4\Big] = 0,\ \Pi_{44}(t) = \Big[E_{m_2} - L_{44}(t)L_{44}^{-}(t)\Big], \tag{2.12}$$

It follows that

$$z_4(t) = Z_2(t)c_2 + \psi_2(t),\ \psi_2(t) = \int_{\alpha}^{t} Z_2(t)Z_2^{-1}(s)L_{44}^{-}(s)\tilde{f}_4(s)ds, \tag{2.13}$$

where $Z_2(t)$ is the Cauchy matrix of the system $\dot{v} = -L_{44}^{-}M_{44}v$, c_2 is an arbitrary constant vector from $\mathbf{R}^{n_2}$. Due to the consistency condition (2.12) and by formula (1.9), we have

$$c_2 = \mathcal{G}^{-}\theta + [E_n - \mathcal{G}^{-}\mathcal{G}]\mathrm{c}. \tag{2.14}$$

In (1.9) we assume

$$M(t) = \Pi_{44}(t)M_{44}(t)Z_2(t), \phi(t) = -\tilde{f}_4(t) - \Pi_{44}(t)M_{44}(t)\psi_2(t), \mathrm{c} \in \mathbf{R}^{n_2}.$$

According to [2], the relation (2.14) is valid if and only if

$$\phi(t) = M(t)\mathcal{G}^{-}\theta. \tag{2.15}$$

Further, the matrix $L_{33}(t)$, where $m_1 < n_1$, is full rank for any $t \in T$. According to (1.7) and (1.8)

$$\dot{z}_3 = -L_{33}^{-}(t)M_{33}(t)z_3 + L_{33}^{-}(t)\tilde{f}_3 + \tilde{\Pi}_{33}(t)w_1(t),$$

$$\tilde{\Pi}_{33}(t) = \Big[E_{n_1} - L_{33}^{-}(t)L_{33}(t)\Big],$$

where the condition (2.12) is absent, since $\Pi_{33}(t) \equiv 0$; $w_1(t)$ is an arbitrary vector function of dimension n_1. Then,

$$z_3(t) = Z_1(t)c_1 + \int_{\alpha}^{t} Z_1(t)Z_1^{-1}(s)\Big[L_{33}^{-}(s)\tilde{f}_3(s) + \tilde{\Pi}_{33}(s)w_1(s)\Big]ds, \tag{2.16}$$

where $Z_1(t)$ is the Cauchy matrix of the system $\dot{v} = -L_{33}^{-}M_{33}v$, c_1 is an arbitrary constant vector from $\mathbf{R}^{n_1}$.

Subsystems with matrices $L_{22}(t), L_{11}(t)$ have general solutions of the form

$$z_2(t) = \sum_{j=0}^{l-1} \mathcal{C}_j(t)(d/dt)^j \tilde{f}_2,\ z_1(t) = Z(t)c + \int_{\alpha}^{t} Z(t)Z^{-1}(s)\tilde{f}_1(s)ds, \quad (2.17)$$

where $Z(t)$ is the Cauchy matrix of the system $\dot{v} = -L_{11}^{-1}M_{11}v$, c is an arbitrary constant vector from $\mathbf{R}^d$.

Lemma 1. *If the DAE (1.1) can be reduced to its GSF, then 1) the DAE has LRO and RRO, and the left and the right indices are equal; 2)* $f \in \ker \Lambda_1$ *if and only if the condition (2.15) holds and in (2.7) we have* $f_5(t) \equiv 0,\ t \in T$.

Proof. Now prove the fist point of the theorem. Here we can define the LRO and the RRO as

$$\Omega_l = \{E_d, (d/dt)E_{d_1} \circ \hat{\Lambda}_{l-1}, E_{m_1}, E_{m_2}, E_{m_3}\}P(t),$$

$$\Omega_r = Q(t)\{E_d, \hat{\Lambda}_{l-1} \circ (d/dt)E_{d_1}, E_{n_1}, E_{n_2}, E_{n_3}\},\ r = l,$$

where $P(t),\ Q(t)$ are matrices reducing the DAE to the GSF, $\hat{\Lambda}_{l-1}$ is an opertator from (2.4).

The second point of the theorem follows from the form of the GSF and the consistency condition (2.12), which is equivalent to (2.15) . □

Now formulate the key statement of this section.

Theorem 1. *Let the DAE (1.1) be reduced to the GSF and*

$$f \in \ker \Lambda_1 \bigcap \mathbf{C}^l(T),$$

where $l \geq 1$. *Then, there exist smooth in their domains matrices* $X_\nu(t)$, $Y(t), K_1(t,s),\ C_j(t),\ j = \overline{0, l-1},\ \tilde{C}(t),\ K_2(t,s)$, *such that any linear combination*

$$x(t,c) = X_\nu(t)c + Y(t)\mathcal{G}^{-}\theta + \varphi(t) + v(t),\ t \in T, \quad (2.18)$$

is a solution to (1.1), and T does not contain any other solutions. Here

$$\varphi(t) = \int_{\alpha}^{t} K_1(t,s)f(s)ds + \sum_{j=0}^{l-1} C_j(t)(d/dt)^j f(t),$$

$$v(t) = \tilde{C}(t)w(t) + \int_{\alpha}^{t} K_2(t,s)w(s)ds,\ t \in T,$$

c is an arbitrary constant vector from $\mathbf{R}^\nu$,

$$\nu = d + n_2 + n_4, n_4 = \operatorname{rank}\,[E_{n_2} - \mathcal{G}^{-}\mathcal{G}],\ w(t) = (w_1^\top(t)\ w_2^\top(t))^\top$$

is an arbitrary vector-function of dimension $n_2 - m_2 + n_3$, rank $X_\nu(t) = \nu \ \forall t \in T$.

Proof. Write down the expressions for general solutions to (2.11), (2.13), (2.16), (2.17) and, using the formula for the GSF (2.3), we obtain the expression for the vector function $z = \begin{pmatrix} z_1^\top & z_2^\top & z_3^\top & z_4^\top & z_5^\top \end{pmatrix}^\top$. Then, we restore vector functions $x = Qz$ and f and arrive at the desired formula (2.18). $\square$

3. Existence of the Generalized Split Form

In this section, we discuss conditions under which the DAE (1.1) can be reduced to its GSF.

Theorem 2. *Suppose that (1.1) satisfies the conditions:* $A(t), B(t) \in \mathbf{C}^{2n}(T), f \in \mathbf{C}^n(T)$.

Then, the following assumptions are equivalent:
1) the DAE has a Cauchy type solution and in formula (1.3)

$$\psi(t) = \int_\alpha^t K_1(t,s) f(s) ds + \sum_{j=0}^{l-1} C_j(t)(d/dt)^j f(t),$$

where $K_1(t,s)$, $C_j(t)$, $j = \overline{0, l-1}$ *are some* $(n \times n)$*-matrices;*
2) the DAE has the left and the right indices on T *equal to* l*;*
3) the DAE has an GSF on T *and (2.3) has only two subsystems*

$$L_{11}(t)\dot{z}_1 + M_{11}(t) z_1 = \tilde{f}_1, \ L_{22}(t)\dot{z}_2 + M_{22}(t) z_2 = \tilde{f}_2;$$

4) starting with some $i \geq l$*, the following conditions hold:*

$$\operatorname{rank} \Gamma_i[A(t), B(t)] = const,$$

$$\Gamma_i^-[A(t), B(t)]\Gamma_i[A(t), B(t)] = \begin{pmatrix} E_n & 0 \\ Z_{21}(t) & Z_{22}(t) \end{pmatrix}, \ t \in T,$$

where $\Gamma_i[A(t), B(t)]$ *is a matrix from (1.6),* $Z_{21}(t)$, $Z_{22}(t)$ *are some blocks of appropriate dimension.*

The proof of points 1), 2), 4) of the theorem can be found in [3]. The first n rows of the matrix $\Gamma_i^-[\mathbf{A}(t)]$, split into $(n \times n)$-blocks, can be taken as the LRO coefficients $L_j(t)$, $j = \overline{0, l}$. Point 3) of the theorem was proven in [6]. If $A(t)$, $B(t) \in \mathbf{C}^A(T)$, then a DAE can be reduced to its CCF and in formula (2.2) $P(t)$, $Q(t) \in \mathbf{C}^A(T)$.

Lemma 2. *Suppose that system (1.1) satisfies the conditions* $A(t)$, $B(t) \in \mathbf{C}^{r+1}(T)$, $r = \max\{\operatorname{rank} A(t), \ t \in T\}$. *Then, any subsegment* $T_0 = [\alpha_0, \ \beta_0] \subset T$ *includes a segment* $T_* = [\alpha_*, \ \beta_*] \subset T_0$*, where the system can be reduced to its CCF.*

Proof. Let there be given a matrix $M(t),\ t \in T$. Introduce a segment $T_0 \subset T$. Since the matrix rank is a bounded whole number, there exists a point $t_0 \in T_0$: $r_0 = \text{rank}\ A(t_0) = \max\{\text{rank}\ A(t),\ t \in T_0\}$. By continuity, there exists a segment $T_1 = [\alpha_1\ \beta_1] \subset T_0$ and a sub-matrix of $M_{11}(t),\ t \in T_1$, where $\det\ M_{11}(t) \neq 0\ \forall t \in T_1$ and the following relationship holds

$$LM = \begin{pmatrix} E_r & 0 \\ -M_{11}^{-1}M_{21} & E_{n-r} \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ 0 & 0 \end{pmatrix},\ t \in T_1$$

Now transform the DAE (1.1). Any subsegment $[\alpha_0,\ \beta_0] \subseteq T$ includes the subsegment $T_1 = [\alpha_1,\ \beta_1]$, in which an $(m \times m)$-matrix $L_1 \in C^{r+1}(T_1)]$ is defined, such that $\det\ L_1 \neq 0$,

$$L_1 \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \dot{x} \\ x \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ 0 & B_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ x \end{pmatrix} = L_1 f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},\ t \in T_1,$$

where ranks of the matrices $A_1,\ B_2$ are full on T_1. If B_2 is a $(n \times n)$-matrix, then $x = B_2^{-1} f_2$, the process terminates and the lemma is valid. If $r_1 = \text{rank}\ B_2(t) < n$, then there exists a subsegment $T_2 \subset T_1$, on which the matrix R_1 is defined, $B_2 R_1 = \begin{pmatrix} 0 & E_{n-r} \end{pmatrix}$. R_1 is designed similarly to the matrix L_1 and has the following properties: $\det R_1 \neq 0$,

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ 0 & E_{n-r_1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},\ t \in T_2,$$

where $x = R_1 y,\ y = \begin{pmatrix} y_1^\top & y_2^\top \end{pmatrix}^\top$. If $\text{rank}\ A_{11},\ t \in T_2$ is full, then the process terminates and the lemma is valid. Otherwise, following the scheme described, we continue the process for the subsystem $A_{11}\dot{y}_1 + B_{11}y_1 + A_{12}\dot{y}_2 + B_{12}y_2 = f_1$.

We arrive at a new system

$$\begin{pmatrix} A_{1,11} & A_{1,12} & A_{1,13} \\ 0 & 0 & A_{2,13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} + \begin{pmatrix} B_{1,11} & B_{1,12} & B_{1,13} \\ 0 & E_{r_1-r_2} & B_{2,13} \\ 0 & 0 & E_{n-r_1} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \tilde{f},\ t \in T_3,$$

where T_3 is some subsegment of the segment T_2, $\tilde{f}$ is a suitably transformed vector function f. If $\text{rank}\ A_{1,11},\ t \in T_3$ is full, then, by introducing the notion $N = \begin{pmatrix} 0 & A_{1,13} \\ 0 & 0 \end{pmatrix}$, we obtain the desired structure. Otherwise, following the scheme described, we continue the process and in a finite number of steps we will build the GSF on a sequence of nested segments. □

4. Conclusion

In this paper, we proposed a structural form for linear systems of ODEs, which can be represented as a quasi-upper-triangular form. We called this form a generalized split form (GSF). Consistently solving the subsystems of the GSF and using generalized inverse matrices, we can write down the general solution of the original system of ODEs. It was proven that any system of ODEs with sufficiently smooth entries can be reduced to its GSF. For the case of closed systems, we established equivalence of requirements for the form of the general solution on T and for existence of the GSF on T. At present, for the construction of a complete theory, it is necessary to find out whether the existence conditions of GSFs are equivalent to those of the general solution (2.18) on T, as it was proven in Theorem 2.

References

1. Belov A.A., Kurdyukov A.P. *Descriptor systems and control problems*. Moscow, Fizmatlit Publ., 2015, 272 p.(in Russian).
2. Boyarintsev Y.E. *Regular and singular systems of linear ordinary differential equations*. Novosibirsk, Nauka Publ., 1980, 222 p.
3. Boyarintsev Yu.E., Chistyakov V.F. *Algebro-differencial'nye sistemy. Metody resheniya i issledovaniya* [Algebraic Differential Systems. Methods for Investigation and Solution]. Novosibirsk, Nauka Publ., 1998, 224 p.(in Russian)
4. Brenan K.E., Campbell S.L., Petzold L.R. *Numerical solution of initial-value problems in differential-algebraic equations (classics in applied mathematics; 14)*. Philadelphia, SIAM, 1996.
5. Bulatov M.V., Chistyakov V.F. Odin metod chislennogo resheniya lineinich singularnich sistem ODU indeksa vishe edinitsi. *Chislenie metodi analiza i ix prilozeniya*. Irkutsk, SEI SO AN SSSR Publ., 1987, pp. 100-105. (in Russian).
6. Bulatov M.V., Chistyakov V.F. A numerical method for solving differential-algebraic equations. *Comput. Math. Math. Phys.*, 2002, vol. 42, no. 4, pp. 439–449.
7. Gantmacher F. R. *The Theory of Matrices*. AMS Chelsea Publishing: Reprinted by American Mathematical Society, 2000, 660 p.
8. Hairer E., Lubich C., Roche M. The numerical solution of differential-algebraic system by Runge - Kutta methods, Report CH-1211, Dept. de Mathematiques, Universite de Geneve, Switzerland, 1988.
9. Kunkel P., Mehrmann V. Canonical forms for linear differential-algebraic equations with variable coefficients. *J. Comput. Appl. Math*, 1995, vol. 56, pp. 225-251. https://doi.org/10.1016/0377-0427(94)90080-9
10. Kurina G.A. On regulating by descriptor systems in an innite interval. *Izvestija RAN, Tekhnicheskaja Kibernetika*, 1993, no. 6, pp. 33–38. (in Russian).
11. Lamour R., Marz R., Tischendorf C. Differential-Algebraic Equations: A Projector Based Analysis. Springer-Verlag, 2013. https://doi.org/10.1007/978-3-642-27555-5
12. Sidorov N. A. The Cauchy problem for a certain class of differential equations. *Differ. Equations*, 1972, vol. 8, no. 8. pp. 1521–1524 (in Russian).
13. Sidorov N. A. The branching of the solutions of differential equations with a degeneracy. *Differ. Equations*, 1973, vol. 9, no. 8. pp. 1464–1481. (in Russian)

14. Sidorov N. A. A study of the continuous solutions of the Cauchy problem in the neighborhood of a branch point. *Soviet Math.*, 1976, vol. 20, no. 9, pp. 77–87. (in Russian).
15. Chistyakov V.F. *Algebraic differential operators with finite-dimensional kernel.* Novosibirsk, Nauka Publ., 1996, 278 p.(in Russian).

Mikhail Bulatov, Doctor of Sciences (Physics and Mathematics), Matrosov Institute for System Dynamics and Control Theory of SB RAS, Post Box 292, 134, Lermontov st., Irkutsk, 664033, Russian Federation, tel.: (3952)427100 (e-mail: mvbul@icc.ru)

Victor Chistyakov, Doctor of Sciences (Physics and Mathematics), Institute of System Dynamics and Control Theory of SB RAS, 134, Lermontov st., Irkutsk, 664033, Russian Federation, tel.: 8(3932)453029 (e-mail: chist@icc.ru)

Received 25.04.19

Об индексе и расщепленных формах линейных дифференциально-алгебраических уравнений

М. В. Булатов

Институт динамики систем и теории управления им. В.М. Матросова СО РАН, Иркутск, Российская Федерация

В. Ф. Чистяков

Институт динамики систем и теории управления им. В.М. Матросова СО РАН, Иркутск, Российская Федерация

Аннотация. Рассматриваются линейные системы обыкновенных дифференциальных уравнений (ОДУ) с прямоугольными матрицами коэффициентов, включая случай, когда матрица перед производной искомой вектор-функции имеет неполный ранг для всех значений аргумента из области определения. Системы такого вида, принято называть дифференциально-алгебраическими уравнениями (ДАУ). Получены критерии существования неособенных преобразований, расщепляющих систему на подсистемы, для которых с помощью аппарата обобщенных обратных матриц можно выписать общие решения в виде конечных формул. Эта форма названа обобщенной расщепленной формой ДАУ. Она является некоторым аналогом канонической формы Вейерштрасса – Кронекера и совпадает с ней для пучков матриц с постоянными элементами. В частности, показано, что произвольные ДАУ с прямоугольными матрицами коэффициентов приводимы локально к обобщенной расщепленной форме. Структура этих форм (если она определена на отрезке интегрирования) полностью определяет структуру общих решений систем. При анализе обозначенного выше класса систем ОДУ выявлено наличие целочисленных характеристик систем, называемые размерность пространства решений и индекс. Размерность пространства решений определяет произвол многообразия общего решения. Индекс определяет порядок производных входных данных, от которых зависит решение задачи. Указаны способы вычисления этих характеристик.

Ключевые слова: дифференциально-алгебраические уравнения, каноническая форма, расщепленная форма, пространство решений, индекс, особые точки.

Список литературы

1. Белов А. А., Курдюков А. П. Дескрипторные системы и задачи управления. М. : Физматлит, 2015. 272 с.
2. Бояринцев Ю. Е. Регулярные и сингулярные системы линейных обыкновенных дифференциальных уравнений. Новосибирск : Наука, 1980. 222 с.
3. Бояринцев Ю. Е., Чистяков В. Ф. Алгебро-дифференциальные системы. Методы решения и исследования. Новосибирск : Наука, 1998. 224 с.
4. Brenan K. E., Campbell S. L., Petzold L. R. Numerical solution of initial-value problems in differential-algebraic equations (classics in applied mathematics; 14). Philadelphia, SIAM, 1996.
5. Булатов М.В., Чистяков В.Ф. Один метод численного решения линейных сингулярных систем ОДУ индекса выше единицы // Численные методы анализа и их приложения. Иркутск : СЭИ СО АН СССР, 1987. С. 100–105.
6. Bulatov M. V., Chistyakov V. F. A numerical method for solving differential-algebraic equations // Comput. Math. Math. Phys. 2002. Vol. 42, N 4. P. 439–449.
7. Гантмахер Ф. Р. Теория матриц. 3-изд. М. : Наука, 1966. 576 с.
8. Hairer E., Lubich C., Roche M. The numerical solution of differential-algebraic system by Runge – Kutta methods, Report CH-1211, Dept. de Mathematiques, Universite de Geneve, Switzerland, 1988.
9. Kunkel P., Mehrmann V. Canonical forms for linear differential-algebraic equations with variable coefficients // J. Comput. Appl. Math. 1995. Vol. 56. P. 225–251. https://doi.org/10.1016/0377-0427(94)90080-9
10. Kurina G.A. On regulating by descriptor systems in an innite interval // Izvestija RAN, Tekhnicheskaja Kibernetika. 1993. N 6, P. 33–38 (in Russian).
11. Lamour R., Marz R., Tischendorf C. Differential-Algebraic Equations: A Projector Based Analysis. Springer-Verlag, 2013. https://doi.org/10.1007/978-3-642-27555-5
12. Сидоров Н. А. Задача Коши для одного класса дифференциальных уравнений // Дифференц. уравнения. 1972. Т. 8, № 8. С. 1521–1524
13. Сидоров Н. А. О ветвлении решений дифференциальных уравнений с вырождением // Дифференц. уравнения. 1973. Т. 9, № 8. С. 1464–1481.
14. Сидоров Н. А. Исследование непрерывных решений задачи Коши в окрестности точки ветвления // Изв. вузов. Математика. 1976, № 9. С. 99–110.
15. Чистяков В.Ф. Алгебро-дифференциальные операторы с конечномерным ядром. Новосибирск : Наука. Сиб. издат. фирма РАН, 1996. 278 с.

Михаил Валерьянович Булатов, доктор физико-математических наук, главный научный сотрудник, Институт динамики систем и теории управления им. В. М. Матросова СО РАН, 664033, Российская Федерация, г. Иркутск, ул. Лермонтова, 134, а/я 292, тел.: (3952)242210 (e-mail: mvbul@icc.ru)

Виктор Филимонович Чистяков, доктор физико-математических наук, главный научный сотрудник, Институт динамики систем и теории управления им. В. М. Матросова СО РАН, 664033, Российская Федерация, г. Иркутск, ул. Лермонтова, 134, а/я 292, тел.: (3952)242210 (e-mail: chist@icc.ru)

Поступила в редакцию 25.04.19