Серия «Математика»
2019. Т. 27. С. 55—70

Онлайн-доступ к журналу:
http://mathizv.isu.ru

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УДК 518.517
MSC 35L05, 45D05
DOI https://doi.org/10.26516/1997-7670.2019.27.55

Classic Solutions of Boundary Value Problems for Partial Differential Equations with Operator of Finite Index in the Main Part of Equation

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Abstract. This paper is an attempt to give the review of a part of our results in the area of singular partial differential equations. Using the results of the theory of complete generalized Jordan sets we consider the reduction of the PDE with the irreversible linear operator B of finite index in the main differential expression to the regular problems. Earlier we and other authors applied similar methods to the development of Lyapunov alternative method in singular analysis and numerous applications in mechanics and mathematical physics. In this paper, we show how the problem of the choice of boundary conditions is connected with the B-Jordan structure of coefficients of PDE. The estimation of various approaches shows that the most efficient approach for solving this problem is the functional approach combined with the alternative Lyapunov method, Jordan structure coefficients and skeleton decomposition of irreversible linear operator in the main part of the equation. On this base, the problem of the correct choice of boundary conditions for a wide class of singular PDE can be solved. The aggregated theorems of existence and uniqueness of classical solutions can be proved with continuously depending of experimental definite function. The theory is illustrated by considering the solution of some integro - differential equations with partial derivatives.

Keywords: degenerate PDE, Jordan set, Banach space, Noether operator, boundary value problems.

This paper is dedicated to the 100th anniversary of Irkutsk State University

1. Introduction

Let B and A_i, $i = \overline{1,q}$ be closed linear operators from E_1 to E_2 with the dense domains in E_1. Let E_1, E_2 be Banach spaces, $D(B) \subseteq D(A_i)$,

B the operator of finite index with closed range of values, $\dim N(B) = n$, $\dim N(B^\star) = m$, $\nu = n - m < \infty$. The operator

$$L_i\left(\frac{\partial}{\partial x}\right) = \sum_{|k|\le q_i} a_k^i(x)D^k, \;\; i = 1,\dots,q$$

is a partial differential operator of order q_i,

$$q_0 > q_1 > q_2 > \cdots > q_q.$$

Functions $a_k^i(x) : \Omega \in \mathbb{R}^N \to \mathbb{R}^1$ and $f(x) : \Omega \in \mathbb{R}^N \to E_2$ are sufficiently smooth. The differential equation

$$L_0\big(\frac{\partial}{\partial x}\big)Bu + L_1\big(\frac{\partial}{\partial x}\big)A_1u + \cdots + L_q\big(\frac{\partial}{\partial x}\big)A_qu = f(x) \qquad (1.1)$$

is considered below.

Definition 1. *Equation* (1.1) *is a regular equation, if the operator B has bounded inverse. Otherwise we say that* (1.1) *is a singular equation.*

In sections 2, 3 the investigation of singular equation (1.1) with irreversible operator B in the main part is reduced to the regular problems. The special decomposition of the Banach spaces E_1 and E_2 in accordance with the generalized B-Jordan structure of the operator coefficients $A_1,\dots,A_q$ is used. In section 4 this reduction makes it possible to pose boundary value problems for singular equation (1.1) with natural conditions on special projections of the solution.

The methods of this investigation interconnect with the functional approach from [8–11;27] (see also references in [1;2;6;24;25] and the extensive bibliographical review of Loginov school papers [3]).

2. Preliminaries: $P_k,\ Q_k$ – commutability of linear operators in case of Noetherian operator B

Let $E_1 = M_1 \oplus N_1,\ E_2 = M_2 \oplus N_2$, P be a projector on M_1 along N_1, Q a projector on M_2 along N_2, A be a linear closed operator from E_1 to E_2, $\overline{D}(A) = E_1$, $A \in \{A_1,\dots,A_q\}$.

Definition 2. *If $u \in D(A)$, $APu = QAu$, then A (P,Q) –commutes.*

Let $\{\varphi_1^{(1)},\dots,\varphi_n^{(1)}\}$ be a basis in $N(B)$, $\{\psi_1^{(1)},\dots,\psi_m^{(1)}\}$ a basis in $N(B^\star)$. Suppose the following condition is satisfied:

1. The Noether operator B has a complete A_1 –Jordan set $\phi_i^{(j)}$, $i = \overline{1,n}$, $j = \overline{1,p_i}$, and $B^\star$ has a complete $A_1^\star$ -Jordan set $\psi_i^{(j)}$, $i = \overline{1,m}$, $j = \overline{1,p_i}$, and

the systems $\gamma_i^{(j)} \equiv A_i^\star \psi_i^{(p_i+1-j)}$, $z_i^{(j)} \equiv A_1 \phi_1^{(p_i+1-j)}$, where $i = \overline{1,l}, j = \overline{1,p_i}$, $l = \min(m,n)$ corresponding to them are biortogonal [27].

The projectors

$$P_k = \sum_{i=1}^{l} \sum_{j=1}^{p_i} < \cdot, \gamma_i^{(j)} > \phi_i^{(j)} \stackrel{\text{def}}{\equiv} (< \cdot, \gamma >, \Phi), \tag{2.1}$$

$$Q_k = \sum_{i=1}^{l} \sum_{j=1}^{p_i} < \cdot, \psi_i^{(j)} > z_i^{(j)} \stackrel{\text{def}}{\equiv} (< \cdot, \Psi >, Z), \tag{2.2}$$

where $k = p_1 + \cdots + p_l$ is a root number, generate the direct decomposition

$$E_1 = E_{1k} \oplus E_{1\infty-k}, \;\; E_2 = E_{2k} \oplus E_{2\infty-k}.$$

Corollary 1. *The bounded pseudoinverse operator $B^+(P_k, Q_k)$-commutes, the operator AB^+ (Q_k, Q_k)–commutes, B^+A (P_k, P_k)-commutes, $E_{1\infty-k}$, $E_{1,k}$ are the invariant subspaces of operator B^+A, $E_{2\infty-k}$, and E_{2k} are the invariant subspaces of operator AB^+.*

Suppose the operator A (P_k, Q_k)–commutes, where P_k, Q_k are defined by formulas (2.1), (2.2). Then there is a matrix $\mathcal{A}$, such that $A\Phi = \mathcal{A}Z$, $A^\star \Psi = \mathcal{A}'\gamma$. This matrix is called the matrix of (P_k, Q_k)–commutability.

Corollary 2. *The operators B and A (P_k, Q_k)–commute and the matrices of (P_k, Q_k)–commutability are the symmetrical cell-diagonal matrices:*

$$\mathcal{A}_B = diag(B_1, \cdots, B_l) \;\; \mathcal{A}_A = diag(\mathcal{A}_1, \cdots, \mathcal{A}_l),$$

where

$$B_i = \begin{bmatrix} 0 & . & . & . & 0 \\ 0 & . & . & . & 1 \\ . & . & . & . & . \\ 0 & 1 & . & . & 0 \end{bmatrix} \;\; \mathcal{A}_i = \begin{bmatrix} 0 & \cdots & 1 \\ & \cdots & \\ 1 & \cdots & 0 \end{bmatrix}$$

See the detailed proof in the preprint [10] .

Definition 3. *The operator G (P_k, Q_k)–commutes quasitriangularly, if $\mathcal{A}_G$ is upper quasitriangular matrix, whose diagonal blocks $\mathcal{A}_{ii}$ of dimension $p_i \times p_i$ are lower right triangular matrices.*

3. The reduction of equation (1.1) to the regular PDE

Suppose:

2. The operators $A_2, \cdots, A_q$ (P_k, Q_k)–commute. Then there are matrices $\mathcal{A}_i$ $i = \overline{1,q}$, such that

$$A_i\Phi = \mathcal{A}_i Z, \;\; A_i^{\star}\Psi = \mathcal{A}_i'\gamma$$

and $\mathcal{A}_i = (\mathcal{A}_{11}, \cdots, \mathcal{A}_{l1})$ is a cell-diagonal matrix ,where

$$\mathcal{A}_{i1} = \begin{bmatrix} 0 & \cdots & 1 \\ & \cdots & \\ 1 & \cdots & 0 \end{bmatrix}, \;\; i = \overline{1,l}.$$

Let us consider the case $m \leq n$.

We introduce the projectors P_k, Q_k based on formulas (2.1), (2.2) and the projector

$$P_{n-m} = \sum_{i=m+1}^{n} < \cdot, \gamma_i > \phi_i$$

generating the direct decompositions

$$E_1 = E_{1k} \oplus span\{\phi_{m+1}, \cdots, \phi_n\} \oplus E_{1\infty-(k+n-m)}, E_2 = E_{2k} \oplus E_{2\infty-k}.$$

Note that $B^+ : E_{2\infty-k} \to E_{1\infty-(k+n-m)} \subset E_{1\infty-k}$, $B^+ : E_{2k} \to E_{1k+n-m}$.

We shall seek the solution of equation (1.1) in the following form

$$u(x) = B^+v(x) + (C(x), \Phi) + \sum_{i=m+1}^{n} \lambda_i(x)\phi_i, \tag{3.1}$$

where B^+ is a bounded pseudoinverse operator for B, $v \in E_{2\infty-k}$, $C(x) = (C_1(x), \cdots, C_m(x))'$, $C_i(x) = (C_{i1}(x), \cdots, C_{ip}(x))$, $\Phi = (\Phi_1, \cdots, \Phi_m)'$, $\Phi_i = (\phi_i^1, \cdots, \phi_i^{(p_i)})$, $i = \overline{1,m}$.

Substituting the expression (3.1) in equation (1.1) and noting that $BB^+v = v$, because $v \in E_{2\infty-k} \subset E_{2\infty-k} \subset E_{2\infty-m}$, we obtain

$$\begin{gathered} L_0(\frac{\partial}{\partial x})v + \sum_{i=1}^{q} L_i(\frac{\partial}{\partial x})A_iB^+v + L_0(\frac{\partial}{\partial x})B(C,\Phi) + \sum_{i=1}^{q} L_i(\frac{\partial}{\partial x})A_i(C,\Phi) + \\ + \sum_{j=1}^{q}\sum_{i=m+1}^{n} L_j(\frac{\partial}{\partial x})A_j\phi_i\lambda_i(x) = f(x). \end{gathered} \tag{3.2}$$

The operator B^+ (Q_k, P_k)–commutes, so from condition 2 and corollary 1 it follows that $Q_kA_iB^+(I - Q_k) = 0$, $(I - Q_k)A_iB^+Q_k = 0$. Hence, $Q_kA_iB^+v = 0$, $\forall v \in E_{2\infty-k}$. According to corollary 2 $B\Phi = \mathcal{A}_B Z$, where $\mathcal{A}_B = (B_1, \cdots, B_m)$ is a symmetrical cell-diagonal matrix. Consequently,

$$(I - Q_k)B\Phi = 0, \;\; (I - Q_k)A_i\Phi = 0, \;\; i = \overline{1,q}, \tag{3.3}$$

because $(I - Q_k)Z = 0$. The following equalities are fulfilled:

$$(A_i(C, \Phi), \Psi) \stackrel{\text{def}}{\equiv} A_i' C, \;\; (B(C, \Phi), \Psi) \stackrel{\text{def}}{\equiv} \mathcal{A}_B C. \tag{3.4}$$

Projecting equation (3.2) onto $E_{2\infty - k}$ by virtue of (3.3) we obtain the regular PDE

$$\tilde{\mathcal{L}} v = (I - Q_k) f(x) - \sum_{j=1}^{q} \sum_{i=m+1}^{n} L_j(\frac{\partial}{\partial x}) A_j \phi_i \lambda_i(x), \tag{3.5}$$

where

$$\tilde{\mathcal{L}} = L_0(\frac{\partial}{\partial x}) + \sum_{i=1}^{q} L_i(\frac{\partial}{\partial x}) A_i B^+ \tag{3.6}$$

is a regular differential operator of order q_0. In order to determine the vector-function $C(x) : R^N \to R^k$, we project the equation (3.2) onto E_{2k} and by virtue of (3.4) we obtain the PDE-system

$$L_0(\frac{\partial}{\partial x}) \mathcal{A}_B C + \sum_{i=1}^{q} L_i(\frac{\partial}{\partial x}) \mathcal{A}_i' C = < f(x) - \sum_{j=2}^{q} \sum_{i=m+1}^{n} L_j(\frac{\partial}{\partial x}) A_j \phi_i \lambda_i(x), \Psi > . \tag{3.7}$$

So we have proved

Theorem 1. *Suppose conditions 1 and 2 are satisfied, $m \leq n$, $f : \Omega \subset R^N \to E_2$ is a sufficiently smooth function. Then any solution of equation (1.1) can be represented in the form*

$$u = B^+ v + (C, \Phi) + \sum_{i=m+1}^{n} \lambda_i \phi_i,$$

where v satisfies regular equation (3.5), and the vector $C(x)$ is defined by system (3.7). The functions $\lambda_i(x)$, $i = \overline{m+1, n}$ remain arbitrary functions.

Theorem 1 allows generalizations. Suppose the operators $A_2(x), \cdots, A_q(x)$ with the domains independent from x, are subject to the operators B, and for any $x \in \Omega$ satisfy condition 2. Then theorem 1 remains valid.

Let us consider system (3.7) with a unknown vector-function $C(x)$.

Lemma 1. *Suppose conditions 1,2 are satisfied, operators A_i, $i = \overline{1, q}$, (P_k, Q_k)–commute quasitriangularly. Then system (3.7) is a recurrent sequence of linear differential equations of order q_1 with the regular differential operators of the form*

$$\tilde{\mathcal{L}}_{ks} = L_1(\frac{\partial}{\partial x}) + \sum_{i=2}^{q} a^{ik}_{pk-s+1,s} L_i(\frac{\partial}{\partial x}).$$

In particular, if condition 1 is satisfied and $A_2 = \cdots = A_q = 0$, system (3.7) takes the form

$$L_1(\frac{\partial}{\partial x})C_{ip_i}(x) =< f(x), \psi_1^{(1)} >,$$

$$L_1(\frac{\partial}{\partial x})C_{ip_i-s}(x) =< f(x), \psi_i^{(s+1)} > -L_0(\frac{\partial}{\partial x})C_{ip_i-s+1}(x),$$
$s = \overline{1, p_i - 1}$, $i = \overline{1, m}$.

Corollary 3. *Let equation (1.1) has the form*

$$L_0(\frac{\partial}{\partial x})Bu + A_1 u = f(x)$$

and condition 1 be satisfied. Then the vector $C(x)$ is defined by simple recursion.

Proof. The proof is obvious, because in this case $L_i(\frac{\partial}{\partial x}) = 1$ and $A_2 = \cdots = A_q = 0$ in equation (1.1).

Let us consider the second case $m > n$.

In this case we use the direct decompositions:

$$E = E_{1k} \oplus E_{1\infty-k}, E_2 = E_{2k} \oplus span\{z_{n+1}, \cdots, z_m\} \oplus E_{2\infty-(k+m-n)}$$

and also $B^+ : E_{2\infty-(k+m-n)} \to E_{1\infty-k}, B^+ : E_{2k+m-n} \to E_{1k}$. We shall seek the solution of equation (1.1) in the following form

$$u(x) = B^+ v(x) + (C(x), \Phi), \tag{3.8}$$

where $v \in E_{2\infty k}, C(x) = (C_1(x), \cdots, C_n(x))'$, $C_i(x) = (C_{i1}(x), \cdots, Cip_i(x),)$

$$\Phi = (\Phi_1, \cdots, \Phi_n)', \;\; \Phi_i = (\phi_i^{(1)}, \cdots, \phi_i^{(p_i)}), \;\; i = \overline{1, n}.$$

By substituting (3.8) in equation (1.1) we obtain

$$L_0(\frac{\partial}{\partial x})(I - Q_{m-n})v + \sum_{i=1}^{q} L_i(\frac{\partial}{\partial x})A_i B^+ v + L_0(\frac{\partial}{\partial x})B(C, \Phi) + \tag{3.9}$$

$$+ \sum_{i=1}^{q} L_i(\frac{\partial}{\partial x})A_i(C, \Phi) = f(x)$$

with the condition $< v, \psi_i >= 0$, $i = \overline{1, n}$. Let condition 2 is satisfied. By projecting equation (3.9) onto the subspaces E_{2k}, E_{2m-n}, $E_{2\infty-(k+m-n)}$ we obtain

$$L_0(\frac{\partial}{\partial x})\mathcal{A}_B C + \sum_{i=1}^{q} L_i(\frac{\partial}{\partial x})\mathcal{A}'_i C =< f(x), \psi >, \tag{3.10}$$

$$\sum_{i=1}^{q} L_i(\frac{\partial}{\partial x})Q_{m-n}A_iB^{+}v = Q_{m-n}f(x), \tag{3.11}$$

$$L_0(\frac{\partial}{\partial x})(I-Q_{m-n})v + \sum_{i=1}^{q} L_i(\frac{\partial}{\partial x})Q_{m-n}A_iB^{+}v = (I-Q_k-Q_{m-n})f(x), \tag{3.12}$$

where $v = E_{2\infty-k}, \Psi = (\psi_1, \cdots, \psi_n)', \psi_i \stackrel{\text{def}}{\equiv} (\psi_i^{(1)}, \cdots, \psi_i^{(p_i)})$. The element v can be found from the regular equation

$$\tilde{\mathcal{L}}v = (I-Q_k)f \tag{3.13}$$

in the subspace $E_{2\infty-k} \cap E_{2\infty-(m-n)}$. Indeed, if $Q_{m-n}v = 0$, then by virtue of $Q_{m-n}Q_k = 0$ the solution v of the equation 3.13 satisfies equations (3.11), (3.12). □

Thus we obtain the following result

Theorem 2. *Let $n < m$, conditions 1, 2 be satisfied, and $f : \Omega \subset R^N \to E_2$ be a sufficiently smooth function. Then any solution of equation (1.1) can be represented in the form (3.8), where $v \in E_{2\infty-k} \cap E_{2\infty-(m-n)}$ is the solution of equation (3.13), the vector C is defined by the system (3.10).*

4. Examples

Let operator B be Fredholm ($m = n$). Then basing on theorems 1, 2 the choice problem of correct boundary conditions for equations (3.2), (3.7) and (3.13), (3.10) can be solved for a number of differential operators $L_0(\frac{\partial}{\partial x})$ and $L_1(\frac{\partial}{\partial x})$.

Example 1

Consider the equation

$$\frac{\partial^2}{\partial x \partial y}Bu(x,y) + Au(x,y) = f(x,y) \tag{4.1}$$

This equation with usual Goursat conditions $u|_{x=0} = 0,\ u|_{y=0} = 0$ and an arbitrary right part evidently has no the classical solution.

Let operators B and A satisfy condition 1, $k = p_1 + \cdots + p_n$, p_i are the lengths of A-Jordan chains of operator B.

Then in accordance with our theory we can impose the following conditions on the projections of the solution:

$$(I-P_k)u(x,y)|_{x=0} = 0,\ \ (I-P_k)u(x,y)|_{y=0} = 0 \tag{4.2}$$

As a result we can construct the following unique classical solution

$$u(x,y)=$$
$$=\int_0^x\int_0^y \Gamma\sum_{r=0}^{\infty}(-1)^r(A\Gamma)^r\frac{(x_1-x)^r}{r!}\frac{(y_1-y)^r}{r!}(I-Q_k)f(x_1,y_1)dy_1dx_1+$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{p_i}C_{ij}(x,y)\phi_i^{(j)}, \quad (4.3)$$

where $\Gamma=(B+\sum_{i=1}^{n}<\cdot,\gamma_i>z_i)^{-1}$ is the bounded operator (see Scmidt lemma in [27]) The functions $C_{ij}(x,y)$ are defined recursively

$$C_ip_i(x,y)=\beta_{i1}(x,y),$$

$$C_ip_{i-1}(x,y)=\beta_{i2}(x,y)-\frac{\partial^2}{\partial x\partial y}C_ip_i(x,y)$$

$$C_ip_{i-2}(x,y)=\beta_{i3}(x,y)-\frac{\partial^2}{\partial x\partial y}C_ip_{i-1}(x,y),$$
$$\cdots$$

where $\beta_{is}(x,y)=<f(x,y),\psi_i^{(s)}>,\ \ i=\overline{1,n},\ \ s=\overline{1,p_i}$.

Evidently our solution of this special Goursat problem continuously depends on the right part if $f(x)\in C^{(p)}$, where $p=max(p_1,\cdots,p_n)$.

Example 2

Consider the equation

$$\frac{\partial u(t,x)}{\partial t}-3\int_0^1 xs\frac{\partial u(t,s)}{\partial t}ds=u(t,x)+f(t,x) \quad (4.4)$$

with the condition

$$u(0,x)-3\int_0^1 xsu(0,s)ds=0. \quad (4.5)$$

According to (3.8) we can construct the solution as the sum $u(t,x)=v(t,x)+c(t)x$, where $\int_0^1 xv(t,x)dx=0$, $c(t)=-3\int_0^1 xf(t,x)dx$

$$\frac{\partial v}{\partial t}=v+f(t,x)-3\int_0^1 xsf(t,s)ds,$$

$$v|_{t=0}=0.$$

Therefore, we have the unique solution of example 2

$$u(t,x)=\int_0^t e^{t-z}\Big((f(z,x)-3\int_0^1 xsf(z,s)ds)\Big)dz-3\int_0^1 xsf(t,s)ds.$$

Example 3

Consider the integro-differential equation of order 2

$$\frac{\partial^2 u(t,x)}{\partial t^2} - 3\int_0^1 xs\frac{\partial^2 u(t,s)}{\partial t}ds = \frac{\partial u(t,x)}{\partial t} + f(t,x). \qquad (4.6)$$

The continuous function $f(t,x)$ is defined under $x \in [0,1]$, $t \geq 0$. The Cauchy problem with standard conditions $u|_{t=0} = 0$

$$\frac{\partial u}{\partial t}|_{t=0} = 0$$

is unsolvable under an arbitrary function $f(t,x)$.

Projector $P = 3\int_0^1 xs[\cdot]ds$ corresponds to the Fredholm operator

$$B = I - 3\int_0^1 xs[\cdot]ds.$$

We can use theorem 1. Therefore introducing special conditions $u|_{t=0} = 0$,

$$(I-P)\frac{\partial u}{\partial t}\bigg|_{t=0} = 0 \qquad (4.7)$$

we can construct the solution as the sum

$$u(t,x) = v(t,x) + c(t)x,$$

where $Pv = 0$. Functions $v(t,x)$ and $c(t)$ can be found from two the simplest Cauchy problems

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} = \frac{\partial v}{\partial t} + \frac{dc}{dt}x + f(t,x) \\ v|_{t=0} = 0 \;\; \frac{\partial v}{\partial t}|_{t=0} = 0, \end{cases}$$

$$\begin{cases} \frac{dc}{dt} + Pf = 0 \\ c(0) = 0. \end{cases}$$

Conditions (4.7) were induced by our theorem 1. As a result we can easily construct the desired classical solution of the problem (4.6), (4.7)

$$u(t,x) = \int_0^t (e^{t-s}-1)f(s,x)ds - 3x\int_0^t\int_0^1 e^{t-s}xf(s,x)dxds.$$

Example 4

Consider the equation

$$\frac{\partial^2}{\partial x^2}Bu(x,y) + \frac{\partial}{\partial y}Au(x,y) = f(x,y) \qquad (4.8)$$

Let B be a Fredholm operator, $(\phi_1, \cdots, \phi_n)$ be a basis in $N(B)$, and $(\psi_1, \cdots, \psi_n)$ a basis in $N(B^\star)$.

Let

$$< A\phi_i, \psi_k >= \begin{cases} 1, \ i = k \\ 0, \ i \neq k, \end{cases} \quad P = \sum_1^n < \cdot, A^\star \psi_i > \phi_i, \ Q = \sum_1^n < \cdot, \psi_i > A\phi_i.$$

Then according to theorem 1 we can impose the following conditions on projections:

$$(I - P)u|_{x=0} = 0, \ \ (I - P)\frac{\partial u}{\partial x}|_{x=0} = 0, \ \ Pu|_{y=0} = 0. \tag{4.9}$$

As a result we have the unique classical solution in the form of the sum

$$u(x, y) = \Gamma v(x, y) + \sum_1^n \int_0^y < f(x, y), \psi_i > dy\phi_i, \tag{4.10}$$

where

$$\Gamma = (B + \sum_1^n < \cdot, A^\star \psi_i > A\phi_i)^{-1}$$

is the bounded operator. The function $v(x, y)$ is the unique solution of the regular Cauchy problem

$$\frac{\partial^2 v(x, y)}{\partial x^2} + A\Gamma \frac{\partial v(x, y)}{\partial y} = (I - Q)f(x, y), \ \ v|_{x=0} = 0, \ \ \frac{\partial v}{\partial x}|_{x=0} = 0.$$

If $f(x, y)$ is an analytic function, then

$$v(x, y) = \sum_{i=2}^{\infty} C_i(y)x^i,$$

where

$$C_2(y) = \frac{1}{2!}(I - Q)f(0, y),$$

$$C_3(y) = \frac{1}{3!}(I - Q)\frac{\partial f(x, y)}{\partial x}|_{x=0},$$

$$C_4(y) = \frac{1}{4!}(I - Q)\frac{\partial^2 f(x, y)}{\partial x^2}|_{x=0} - \frac{1}{12}A\Gamma\frac{dC_2(y)}{dy},$$

$$\dots$$

Therefore, we have the following asymptotics of the solution

$$u(x, y) = \frac{1}{2}x^2\Gamma f(0, 0) + (y - \frac{x^2}{2})\sum_1^n < f(0, 0), \psi_i > \phi_i + O(y^2 + |x|^3).$$

Example 5

Consider the equation of 5th order

$$\frac{\partial^3}{\partial t^3}\left(\frac{\partial^2}{\partial x^2}+1\right)u(x,y,t)+\left(\frac{\partial^2}{\partial y^2}+\lambda\right)u(x,y,t)=f(x,y,t) \tag{4.11}$$

with boundary conditions

$$u|_{x=0}=0,\ \ u|_{x=\pi}=0, \tag{4.12}$$

$$u|_{y=0}=0,\ \ u|_{y=\pi}=0. \tag{4.13}$$

Introduce initial conditions

$$(I-P)\frac{\partial^i u(x,y,t)}{\partial t^i}\bigg|_{t=0}=0,\ \ i=0,1,2, \tag{4.14}$$

where

$$P=\frac{2}{\pi}\int_0^{\pi}\sin x\sin s[\cdot]ds$$

is the projector on $Ker\ B$.

The operator $B=\frac{\partial^2}{\partial x^2}+1$ with condition (4.12) maps from $C^{\circ(2)}_{[0,1]}$ in $C_{[0,1]}$.

Let $\lambda\neq n^2$, $f(x,y,t)$ be a continuous function in the domain $0\leq x\leq 1$, $0\leq y\leq 1$, $t\geq 0$.

Then from the proof of theorem 1 it follows that equation (4.11) with the initial and boundary conditions (4.12), (4.13), (4.14) has the unique classical solution.

Example 6

Consider the differential-difference integral equation

$$\frac{\partial u(t,x)}{\partial t}-3\int_0^1 xs\frac{\partial u(t,s)}{\partial t}ds=u(t,x)+\lambda u(t-\Delta,x)+f(t,x),\ t\geq 0, \tag{4.15}$$

$$u(t,x)\bigg|_{-\Delta\leq t\leq 0}=0.$$

Following example 2, without loss of generality, we are looking for a solution in the form of a sum

$$u(t,x)=v(t,x)+c(t)x, \tag{4.16}$$

where

$$\int_0^1 sv(t,s)ds=0.$$

Using the steps method we uniquely define the functions $v(t,x)$ and $c(t)$ from the following problems

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = v(t,x) + \lambda v(t-\Delta, x) + f(t,x) - 3\int_0^1 xsf(t,s)ds, \\ v\Big|_{-\Delta\le t\le 0} = 0, \end{cases}$$

$$\begin{cases} c(t) + \lambda c(t-\Delta) = -3\int_0^1 sf(t,s)ds, \\ c(t)\Big|_{-\Delta\le t\le 0} = 0 \end{cases}$$

If $\lambda = 0$, then (4.16) satisfies the initial condition (4.5), and we come to the result discussed in Example 2.

5. Conclusion

The very first results concerning nonclassical correct boundary conditions for a degenerate differential equation with the Fredholm operator were established in paper [21].

In papers [8–10] the general approach how to construct the set of correct boundary condition for equation (1.1) was considered. For example, some authors effectively exploited Showalter-Sidorov boundary conditions for the mathematical modeling of complex problems (see [25;26]). Such conditions can be obtained as a special case of the approach established above. Of particular interest is finding the solution of a irregular PDE system (1.1), where operator B is assumed to enjoy the skeleton decomposition [15].

In paper [15] the first version of skeleton chains of a linear operator is introduced. In this case the problem of finding the solution of a singular PDE (1.1) also can be reduced to the regular split systems of equations. The corresponding systems also can be solved with the respect of taking into account certain initial and boundary conditions. However the effective use of the concept of skeleton chains for applications will be in the future. The development and applications of our functional approach to other linear and nonlinear integral and integro-differentional systems can be found in [12]– [23] and in mathematical reviews (for example, see MR3721762, MR1959647, MR 0810400, MR3343641, MR2920089, MR279574, MR2675324, MR3201397, etc.)

It should be noted that the interest in this field was stimulated by the important problems first posed in the famous article by L. A. Lusternik "Some issues of nonlinear functional analysis" [4].

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Received 21.01.19

Классические решения граничных задач для дифференциальных уравнений в частных производных с оператором конечного индекса в главной части

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Аннотация. В статье дается обзор результатов в области нерегулярных уравнений в банаховых пространствах с частными производными с необратимым оператором в главной части уравнения. Некоторые результаты статьи ранее анонсировались в препринтах. Показано как, используя информацию об обобщенной в смысле Келдыша жордановой структуре операторных коэффициентов уравнения, можно сводить нерегулярные задачи к регулярным. На этой основе демонстрируется решение проблемы подбора корректных граничных условий для широкого класса нерегулярных уравнений в частных производных высокого порядка. Общие теоремы о редукции, приведенные в статье, дают возможность не только получать достаточные условия существования и единственности классических решений, но и строить решения с входными данными, взятыми из эксперимента. Теория абстрактных уравнений в банаховых пространствах на содержательном уровне иллюстрируется решением граничных задач для ряда конкретных интегро-дифференциальных и разностных уравнений различных порядков.

Ключевые слова: вырожденные дифференциальные уравнения, банаховы пространства, нетеров оператор, граничная задача.

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Поступила в редакцию 21.01.19