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Left-Right Cleanness and Nil Cleanness in Unital Rings

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Abstract. We introduce the notions of left and right cleanness and nil cleanness in rings showing their close relationships with the classical concepts of cleanness and nil cleanness. Specifically, it is proved that strongly clean rings are both L-clean and R-clean as well as strongly nil clean rings are both L-nil clean and R-nil clean. These two assertions somewhat strengthen well-known results due to Nicholson (Comm. Algebra, 1999) and Diesl (J. Algebra, 2013). Moreover, it is shown that L-nil cleanness (respectively, R-nil cleanness) is preserved modulo nil Jacobson radical as well as that this is still true for L-cleanness (respectively, R-cleanness), provided the Jacobson radical is nil.

Keywords: clean rings, nil clean rings, L-clean rings, R-clean rings, L-nil clean rings, R-nil clean rings.

1. Introduction and Background

Everywhere in the text of the current short paper, all rings P under consideration are assumed to be associative, containing the identity element 1, which differs from the zero element 0 of P, and all subrings are unital (i.e., containing the same identity as that of the former ring). Our standard notations and the terminology are mainly in agreement with [6]. Concretely, $U(P)$ denotes the set of all units in P, $Id(P)$ the set of all idempotents in P, $Nil(P)$ the set of all nilpotents in P and $J(P)$ the Jacobson radical of P. For the defined in the sequel *left* (resp., *right*) cleanness and nil cleanness notions, we shall use for our convenience the letters "L" and "R", respectively.

Referring to [7] a ring P is said to be *clean* if, for each $r \in P$, there exist $e \in Id(P)$ and $u \in U(P)$ such that $r = e + u$. If, in addition, $eu = ue$, P is called *strongly clean*. The class of clean rings contains all unit-regular rings (see [2]), whereas the class of strongly clean rings encompasses all strongly π-regular rings (see [7]); note that strongly π-regular rings are themselves unit-regular whenever they are (von Neumann) regular. Recently, it was constructed a concrete example in [9] of a unit-regular ring which is *not* strongly clean.

On the other hand, mimicking [5], a ring P is said to be *nil clean* if, for every $r \in P$, there exist $e \in Id(P)$ and $q \in Nil(P)$ such that $r = e + q$. If, in addition, $eq = qe$, P is called *strongly nil clean*. The class of (strongly) nil clean rings is properly contained in the class of (strongly) clean rings as well as strongly nil clean rings are themselves strongly π-regular (cf. [5]).

Our brief article is motivated by one more fundamental property of cleanness and nil cleanness related to one-sided ideals in rings. The newly obtained properties somewhat increase the description of the so-difficult general structure of clean and nil clean rings.

2. L-clean and R-clean Rings

We start here with some new variations of cleanness.

Definition 1. *Let P be a ring. We will say that P is L-clean if, for any $x \in P$, there are $e \in (1-x)P \cap Id(P)$ and $u \in U(P)$ such that $x = e + u$. Analogically, if $e \in P(1-x) \cap Id(P)$, P is said to be R-clean.*

By definition, both L-clean and R-clean rings are necessarily clean. Moreover, the critical elements 0 and 1 have the following trivial representations as both L-clean and R-clean elements: $0 = 0 + 1$ with $0 = 1.0 = 0.1 \in 1P \cap P1$; $1 = 0 + 1$ with $0 = 0.1 = 1.0 \in 0P \cap P0$.

What we next may comment, is that Definition 1 is tantamount to the following equivalent reformulations:

For any $y \in P$, setting $x = 1 - y$, it must be that $e \in yP \cap Id(P)$ and that $y = (1-e) + v$, where $v = -u \in U(P)$. By replacing $1 - e$ with f, we detect that it amounts to $1 - f \in yP \cap Id(P)$ with $y = f + v$.

By analogy, $e \in Py \cap Id(P)$ and $y = (1-e) + v$, where $v = -u \in U(P)$. By substituting $1-e$ with f, we receive that it amounts to $1-f \in Py \cap Id(P)$ with $y = f + v$. With the aid of these equivalencies, idempotents and units can be presented in the sense of L-clean and R-clean elements as follows: $e = (1-e) + (2e-1)$ with $e = e.1 = 1.e \in eP \cap Pe$; $u = 0 + u = (1-1) + u$ with $1 = uu^{-1} = u^{-1}u \in uP \cap Pu$, as required.

What can be observed at once as a valuable example of such rings is that strongly regular rings are both L-clean and R-clean (for a common generalization, see Corollary 1 stated below). Indeed, if P is a strongly regular ring, then one can write $x = eu = ue$ for some $e \in Id(P)$ and $u \in U(P)$ such that $x = (1 - e) + (ue - 1 + e) \in Id(P) + U(P)$ with $e = xu^{-1} = u^{-1}x \in xP \cap Px$, as wanted.

Nevertheless, the following implication, which extends the last observation, could be useful.

Proposition 1. *Strongly clean rings are both L-clean and R-clean.*

Proof. For any element x of such a ring P, one writes that $1-x = e+u$ with $eu = ue$ for some idempotent e and unit u from P. Thus $x = (1 - e) - u$ and $xe = ue$, so that one can verify that $e = u^{-1}xe = xu^{-1}e \in xP$, because x and u do commute, as asked for.

The case of R-cleanness can be handled in a similar way. □

As an immediate consequence, we derive the following statement. However, we will give a new more conceptual confirmation of its validity.

Corollary 1. *Strongly π-regular rings are both L-clean and R-clean.*

Proof. According to [7, Proposition 1], one deduces for such a ring P that $x^n = eu = ue$ for some $n \in \mathbb{N}$, $e \in Id(P)$ and $u \in U(P)$, whenever x is an arbitrary element from P. Thus $e = x^n u^{-1} \in xP$ and one can concludes as demonstrated in the already cited Proposition 1 from [7] that there is $w = x - 1 + e \in U(P)$ such that $x = (1 - e) + w$, as desired.

Paralleling, R-cleanness follows as well. □

In accordance with [10, Corollary 4], for all $n \in \mathbb{N}$, the upper triangular $n \times n$ matrix ring $\mathbb{T}_n(K)$ over the ring $K = \{\frac{m}{n} \in \mathbb{Q} \ : \ n \text{ is odd}\}$, where $\mathbb{Q}$ is the field of all rationals, is strongly clean but neither strongly π-regular nor local.

The following folklore fact, pertaining to a little more specific lifting of idempotents modulo a nil ideal, is a key instrument for our next theorem: *If K is a ring with a nil ideal I, $d \in K$ and $d+I \in Id(K/I)$, then $d+I = e+I$ for some $e \in Id(K) \cap dK$ with $de = ed$.*

We are now in a position to prove the following statement.

Theorem 1. *Suppose that P is a ring with nil $J(P)$. Then P is L-clean (resp., R-clean) if, and only if, $P/J(P)$ is L-clean (resp., R-clean).*

Proof. "**Necessity.**" Setting that $\overline{P} = P/J(P)$ and, for all $x \in P$, that $\overline{x} = x + J(P)$, we obtain $\overline{x} \in \overline{P}$. If we, furthermore, write $x = 1 - e + u$ for $e \in Id(P) \cap xP$ and $u \in U(P)$, then one derives that $\overline{x} = \overline{1 - e} + \overline{u}$. Likewise, $\overline{e} = e + J(P) = xy + J(P) = (x + J(P))(y + J(P)) = \overline{x}\,\overline{y} \in \overline{x}\overline{P}$, for some $y \in P$, as asserted.

"**Sufficiency.**" Given $x \in P$, we have $\overline{x} \in \overline{P}$ and write that $\overline{x} = [\overline{1} - \overline{a}] + \overline{b}$ with $\overline{a} \in \overline{x}\overline{P}$, for some idempotent $\overline{a}$ and some unit $\overline{b}$ in $\overline{P}$. Therefore, one writes that $x + J(P) = [1 - a + J(P)] + [b + J(P)] = (1 - a + b) + J(P)$, where it is clear that $b \in U(P)$ since $1 + J(P) \leq U(P)$. Also, it is not too hard to check that $U(P) + J(P) = U(P)$ holds. Consequently, one deduces that $x = 1 - a + u$ for some $u \in Nil(P)$. Besides, $a + J(P) = (x + J(P))(c + J(P)) = xc + J(P)$ is an idempotent and hence, as $J(P)$ is by assumption nil, we utilize the posted above folklore fact, to find that $a + J(P) = e + J(P)$ for some idempotent $e \in P$ having the property that $e \in (xc)P \subseteq xP$. Thus $a \in e + J(P)$ and, finally, $x \in 1 - e + U(P)$, as claimed.

In parallel to above arguments, R-cleanness also follows. $\square$

Remark 1. In connection with Corollary 1, it seems that unit-regular rings are eventually hardly L-clean (resp., R-clean). In fact, it was showed in [2] that a ring K is unit-regular if, and only if, $\forall\ x \in K\ :\ x = e + u$ for some $e \in Id(K)$ and $u \in U(K)$ such that $xR \cap eR = \{0\}$. If, additionally, they were L-clean (or R-clean, respectively), it must be that $e \in (1 - x)K$. With the exchange property at hand, $1 - e \in xK$. All of these inclusions maybe should interpret some contradiction. However, a concrete example of a unit-regular ring which is not L-clean (resp., R-clean) is not presently constructed yet. In case this can be made, in virtue of Proposition 1 that construction will substantially refine the aforementioned corresponding example from [9] of a unit-regular non strongly clean ring (compare with Problem 2 quoted below).

3. L-nil clean and R-nil clean Rings

We begin here with some new variations of nil cleanness.

Definition 2. *Let P be a ring. We will say that P is L-nil clean if, for any $x \in P$, there are $e \in xP \cap Id(P)$ and $q \in Nil(P)$ such that $x = e + q$. Analogously, if $e \in Px \cap Id(P)$, P is said to be R-nil clean.*

By definition, both L-nil clean and R-nil clean rings are of necessity nil clean. The truthfulness of the converse implication is in question yet (compare with Problem 1 posed below).

Obvious examples of both L-nil clean and R-nil clean rings are the boolean rings B, the indecomposable ring $\mathbb{Z}_4$, the upper triangular 2×2 matrix ring $\mathbb{T}_2(\mathbb{Z}_2)$. A reason for this is that their elements are idempotents, nilpotents and unipotents only, and these special elements have the following L-nil clean and R-nil clean presentations in an arbitrary ring P:

- For any $e \in Id(P)$, we write:

$e = e + 0$ and $e = e.1 \in eP$, as required.

$-e = e + (-2e)$, where the second term is a nilpotent as so is 2, and $e = (-e).(-1) \in (-e)P$, as required.

- For any $q \in Nil(P)$, we write:

$q = 0 + q$ and $0 = q.0 \in qP$, as required.

$1 + q = 1 + q$ and $1 = (1 + q).(1 + q)^{-1} \in (1 + q)P$, as required.

Even nil clean units can be presented thus: $u = e + q$ with $e = u - q = u(1 - u^{-1}q) \in uP$ and $e = u - q = (1 - qu^{-1})u \in Pu$, as required.

What we can further see is that L-nil clean rings (resp., R-nil clean rings) are always L-clean (resp., R-clean). In fact, writing as above that $x = e + q$ with $e \in Id(P) \cap xP$ and $q \in Nil(P)$, one observes that $x = (1-e)+(2e-1+q) \in Id(P)+U(P)$ as $2 \in Nil(P)$ and so $2e+q \in Nil(P)$. Similarly can be processed in the case of R-nil cleanness.

The following folklore fact, already used above in Theorem 1, concerning a little more special lifting of idempotents modulo a nil ideal, is crucial for our next theorem: *If K is a ring with a nil ideal I, $d \in K$ and $d + I \in Id(K/I)$, then $d + I = e + I$ for some $e \in Id(K) \cap dK$ with $de = ed$.*

We are now ready to prove the following assertion.

Theorem 2. *Suppose P is a ring. Then P is L-nil clean (resp., R-nil clean) if, and only if, $P/J(P)$ is L-nil clean (resp., R-nil clean) and $J(P)$ is nil.*

Proof. "**Necessity.**" That the ideal $J(P)$ is nil follows at once applying [5], because as just already noticed L-nil clean and R-nil clean rings are both nil clean. Putting now that $\overline{P} = P/J(P)$ and, for all $x \in P$, that $\overline{x} = x + J(P)$, we have $\overline{x} \in \overline{P}$. If we, furthermore, write $x = e + q$ for $e \in Id(P) \cap xP$ and $q \in Nil(P)$, then one detects that $\overline{x} = \overline{e} + \overline{q}$. Likewise, $\overline{e} = e + J(P) = xy + J(P) = (x + J(P))(y + J(P)) = \overline{x}\ \overline{y} \in \overline{x}\overline{P}$, for some $y \in P$, as expected.

We can process by the same token and for R-nil cleanness.

"**Sufficiency.**" Given $x \in P$, we have $\overline{x} \in \overline{P}$ and write that $\overline{x} = \overline{a} + \overline{b}$ with $\overline{a} \in \overline{x}\overline{P}$, for some idempotent $\overline{a}$ and some nilpotent $\overline{b}$ in $\overline{P}$. Therefore, one writes that $x+J(P) = [a+J(P)]+[b+J(P)] = (a+b)+J(P)$, where it is clear that $b \in Nil(P)$ since $J(P)$ is nil. Also, it is not too hard to check that $b+J(P) = \{b+j \mid j \in J(P)\} \subseteq Nil(P)$ holds. Consequently, one infers that $x = a+q$ for some $q \in Nil(P)$. Besides, $a+J(P) = (x+J(P))(c+J(P)) = xc + J(P)$ is an idempotent and hence, utilizing the listed above folklore fact, we get that $a+J(P) = e+J(P)$ for some idempotent $e \in P$ having the property that $e \in (xc)P \subseteq xP$. Thus $a \in e+J(P)$ and, finally, $x \in e+J(P)$, as promised.

We can process in the same manner and for R-nil cleanness. □

As an immediate consequence, we yield:

Corollary 2. *Strongly nil clean rings are both L-nil clean and R-nil clean.*

Proof. It was established in [4] (see [8] too) that a ring P is strongly nil clean if, and only if, $P/J(P)$ is a boolean ring and $J(P)$ is a nil ideal. As aforementioned, boolean rings are always L-nil clean and R-nil clean, so that we can apply Theorem 2 to arrive at our claim. □

Remark 2. It is worthwhile noticing that the upper triangular matrix 2×2 ring $\mathbb{T}_2(\mathbb{Z}_2)$ is always strongly nil clean. In view of the stated above element-wise presentations accomplished with results from [1], a non strongly nil clean example of both L-nil clean and R-nil clean rings is the full matrix 2×2 ring $\mathbb{M}_2(\mathbb{Z}_2)$ whose elements consist of only nil clean units, idempotents and nilpotents. However, $\mathbb{M}_2(\mathbb{Z}_2)$ is strongly clean.

Moreover, it follows directly from Theorem 2 that $\mathbb{M}_2(\mathbb{Z}_4)$ is both L-nil clean and R-nil clean. Indeed, one sees that $\mathbb{Z}_4/J(\mathbb{Z}_4) \cong \mathbb{Z}_2$ and so $\mathbb{M}_2(\mathbb{Z}_2) \cong \mathbb{M}_2(\mathbb{Z}_4/J(\mathbb{Z}_4)) \cong \mathbb{M}_2(\mathbb{Z}_4)/\mathbb{M}_2(J(\mathbb{Z}_4)) = \mathbb{M}_2(\mathbb{Z}_4)/J(\mathbb{M}_2(\mathbb{Z}_4))$, where $J(\mathbb{Z}_4) = \{0, 2\}$ is nil whence so does $J(\mathbb{M}_2(\mathbb{Z}_4)) = \mathbb{M}_2(J(\mathbb{Z}_4))$. As already noted above, $\mathbb{M}_2(\mathbb{Z}_2)$ is both L-nil clean and R-nil clean, so that we are set.

On the other vein, Corollary 2 could be proved more transparently as follows: Writing in the presence of above notations that $x = e + q$ with $eq = qe$, and thus $xe = ex$, we quickly obtain by expanding $x - e$ with k, where $q^k = 0$, $k \in \mathbb{N}$, that is $(x - e)^k = 0$, that $e \in xP$ as well as that $e \in Px$, as needed.

We end our work with the following two questions of some interest and importance:

Problem 1. *Is it true that nil clean rings are L-nil clean or, respectively, R-nil clean?*

Certainly, for any nil clean ring P with $x \in P$ one writes that $x = e + q$, where $e \in P$ is an idempotent and $q \in P$ is a nilpotent, but does there exist a record $x = f + t$ for an idempotent f and a nilpotent t such that $f \in xP$ (resp., $f \in Px$) is not too elementary.

Problem 2. *Does it follow that unit-regular rings are L-clean (resp., R-clean)? Generally, does there exist a clean ring which is neither L-clean nor R-clean?*

It is worth noticing that, owing to Proposition 1, such a ring has to be not strongly clean. In this observation in mind, let we consider the subring $K = \{\frac{m}{n} \in \mathbb{Q} \ : \ n \text{ is odd}\}$ of the field of rational numbers $\mathbb{Q}$ and also the 2×2 full matrix ring $\mathbb{M}_2(K)$ over K. It was shown in [10, Example 1] that $\mathbb{M}_2(K)$ is a semiperfect (and hence clean by [3, Theorem 9]) ring which is not strongly clean. Let now we test $\mathbb{M}_2(K)$ for L-cleanness and R-cleanness, expecting that this will definitely be false. To that goal, let us consider the non-invertible matrix $\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix}$, which must be presented as the sum of a non-trivial idempotent and a unit as the matrix $\begin{pmatrix} 7 & 6 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is nonunit as well. Simple calculations show that the non-trivial idempotents are of the form $\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$, where $a, b, c \in K$ with $bc = a - a^2$. Thus

$$\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} + \begin{pmatrix} 8-a & 6-b \\ 3-c & 6+a \end{pmatrix}.$$

On the other vein, a few more not too straightforward computations should lead to a contradiction that

$$\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} \in \begin{pmatrix} -7 & -6 \\ -3 & -6 \end{pmatrix} \mathbb{M}_2(K),$$

but at this stage this is impossible to be verified.

4. Concluding Discussion

We shall state here a brief summary of all established above results, namely:

(i) **All strongly clean (and, in particular, strongly π-regular) rings are both L-clean and R-clean.**

(ii) **All strongly nil clean rings are both L-nil clean and R-nil clean.**

(iii) **Let K be a ring with nil $J(K)$. Then K is L-clean (respectively, R-clean) $\iff$ $K/J(K)$ is L-clean (respectively, R-clean).**

(iv) **Let K be a ring. Then K is L-nil clean (respectively, R-nil clean) $\iff$ $K/J(K)$ is L-nil clean (respectively, R-nil clean) and $J(K)$ is nil.**

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L-чистота, R-чистота и нильпотентная чистота унитарных колец

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Аннотация. В статье определены и исследованы два новых понятия, а именно классы L-чистых и R-чистых колец, а также классы L-нильпотентно и R-нильпотентно чистых колец. Полученные результаты уточняют некоторые классические результаты Николсона (Comm. Algebra, 1999) и Дизеля (J. Algebra, 2013).

Ключевые слова: чистые кольца, нильпотентно чистые кольца, L-чистота кольца, R-чистота кольца.

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