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Ways of obtaining topological measures on locally compact spaces

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Abstract. Topological measures and quasi-linear functionals generalize measures and linear functionals. Deficient topological measures, in turn, generalize topological measures. In this paper we continue the study of topological measures on locally compact spaces. For a compact space the existing ways of obtaining topological measures are (a) a method using super-measures, (b) composition of a q-function with a topological measure, and (c) a method using deficient topological measures and single points. These techniques are applicable when a compact space is connected, locally connected, and has a certain topological characteristic, called "genus", equal to 0 (intuitively, such spaces have no holes). We generalize known techniques to the situation where the space is locally compact, connected, and locally connected, and whose Alexandroff one-point compactification has genus 0. We define super-measures and q-functions on locally compact spaces. We then obtain methods for generating new topological measures by using super-measures and also by composing q-functions with deficient topological measures. We also generalize an existing method and provide a new method that utilizes a point and a deficient topological measure on a locally compact space. The methods presented allow one to obtain a large variety of finite and infinite topological measures on spaces such as $\mathbb{R}^n$, half-spaces in $\mathbb{R}^n$, open balls in $\mathbb{R}^n$, and punctured closed balls in $\mathbb{R}^n$ with the relative topology (where $n \geq 2$).

Keywords: topological measure, deficient topological measure, solid-set function, super-measure, q-function.

1. Introduction

This paper belongs to a series of papers devoted to study of topological measures, deficient topological measures, and their corresponding non-linear functionals on locally compact spaces. The main focus of this paper is techniques for generating new topological measures on a locally compact, locally connected and connected space whose one-point compactification has genus 0. Such spaces include $\mathbb{R}^n$, half-planes in $\mathbb{R}^n$, open balls in $\mathbb{R}^n$, and punctured closed balls in $\mathbb{R}^n$ with the relative topology ($n \geq 2$).

The study of topological measures (initially called quasi-measures) and corresponding quasi-linear functionals began with papers by J. F. Aarnes [1–3]. Deficient topological measures were first defined and used by A. Rustad and Ø. Johansen in [10], and later independently rediscovered by M. Svistula, see [14] and [15]. Application of topological measures and quasi-linear functionals to symplectic topology has been studied in numerous papers (beginning with [12]) and a monograph [13]. All this work is done for compact spaces.

Topological measures, deficient topological measures and some ways to obtain them when X is locally compact are studied by the author in [7] and [8]. In this paper we develop analogs on locally compact spaces of techniques that exist for compact spaces with genus 0. These are methods for generating new topological measures from super-measures, via q-functions, and by utilizing a deficient topological measure and a point. When X is compact, the method of super-measures was first developed in [4]; the method of q-functions first appeared in [5], and was discussed in [10] and [6]. One method that utilizes a deficient topological measure and a point first appeared in [10].

In this paper X is a locally compact, connected, and locally connected space. By a component of a set we always mean a connected component. We denote by $\overline{E}$ the closure of a set E. A set $A \subseteq X$ is called *bounded* if $\overline{A}$ is compact. A set $A \subseteq X$ is called *solid* if A is connected and $X \setminus A$ has only unbounded components. We denote by $\bigsqcup$ a union of disjoint sets.

Several collections of sets are used often. These include: $\mathscr{O}(X)$, the collection of open subsets of X; $\mathscr{C}(X)$, the collection of closed subsets of X; $\mathscr{K}(X)$, the collection of compact subsets of X; and $\mathscr{A}(X) = \mathscr{C}(X) \cup \mathscr{O}(X)$. By $\mathscr{K}_0(X)$ we denote the collection of finite unions of disjoint compact connected sets. $\mathscr{P}(X)$ is the power set of X. We use subscripts s or c to indicate (open, compact) sets that are, respectively, solid or connected. For example, $\mathscr{K}_c(X)$ is the collection of compact connected subsets of X. Given any collection $\mathscr{E} \subseteq \mathscr{P}(X)$, we denote by $\mathscr{E}^*$ the subcollection of all bounded sets belonging to $\mathscr{E}$. For example, $\mathscr{A}_s^*(X) = \mathscr{O}_s^*(X) \cup \mathscr{K}_s(X)$ is the collection of bounded open solid and compact solid sets.

Definition 1. *Let X be a topological space and μ be a set function on $\mathcal{E}$, a family of subsets of X. We say that μ is finite if* $\sup\{|\mu(A)| : A \in \mathcal{E}\} \leq M < \infty$*; μ is compact-finite if $|\mu(K)| < \infty$ for any $K \in \mathscr{K}(X)$; μ is simple if it assumes only values* 0 *and* 1*.*

We consider set functions that are not identically ∞.

2. Preliminaries

We will need the following two results (see, for example, section 2 in [7]).

Lemma 1. *Let $K \subseteq U$, $K \in \mathscr{K}(X)$, $U \in \mathscr{O}(X)$ in a locally compact, locally connected space X. If either K or U is connected there exist a bounded open connected set V and a compact connected set C such that $K \subseteq V \subseteq C \subseteq U$. One may take $C = \overline{V}$.*

Lemma 2. *Let X be a locally compact and locally connected space. Suppose $K \subseteq U$, $K \in \mathscr{K}(X)$, $U \in \mathscr{O}(X)$. Then there exists $C \in \mathscr{K}_0(X)$ such that $K \subseteq C \subseteq U$.*

The next two lemmas can be found in section 3 of [7].

Lemma 3. *If $K \subseteq U$, $K \in \mathscr{K}(X)$, $U \in \mathscr{O}^*_s(X)$ then there exists $C \in \mathscr{K}_s(X)$ such that $K \subseteq C \subseteq U$.*

Lemma 4. *Let $K \subseteq V$, $K \in \mathscr{K}_s(X)$, $V \in \mathscr{O}(X)$. Then there exists $W \in \mathscr{O}^*_s(X)$ such that $K \subseteq W \subseteq \overline{W} \subseteq V$.*

Definition 2. *A topological measure on X is a set function $\mu : \mathscr{C}(X) \cup \mathscr{O}(X) \to [0, \infty]$ satisfying the following conditions:*
(TM1) *if $A, B, A \sqcup B \in \mathscr{K}(X) \cup \mathscr{O}(X)$ then $\mu(A \sqcup B) = \mu(A) + \mu(B)$;*
(TM2) *$\mu(U) = \sup\{\mu(K) : K \in \mathscr{K}(X),\ K \subseteq U\}$ for $U \in \mathscr{O}(X)$;*
(TM3) *$\mu(F) = \inf\{\mu(U) : U \in \mathscr{O}(X),\ F \subseteq U\}$ for $F \in \mathscr{C}(X)$.*

Definition 3. *A deficient topological measure on a locally compact space X is a set function ν on $\mathscr{C}(X) \cup \mathscr{O}(X) \longrightarrow [0, \infty]$ which is finitely additive on compact sets, inner compact regular, and outer regular, i.e. :*
(DTM1) *if $C \cap K = \emptyset$, $C, K \in \mathscr{K}(X)$ then $\nu(C \sqcup K) = \nu(C) + \nu(K)$;*
(DTM2) *$\nu(U) = \sup\{\nu(C) :\ C \subseteq U,\ C \in \mathscr{K}(X)\}$ for $U \in \mathscr{O}(X)$;*
(DTM3) *$\nu(F) = \inf\{\nu(U) :\ F \subseteq U,\ U \in \mathscr{O}(X)\}$ for $F \in \mathscr{C}(X)$.*

For a closed set F, $\nu(F) = \infty$ iff $\nu(U) = \infty$ for every open set U containing F.

Remark 1. For more information about topological measures and deficient topological measures on locally compact spaces, their properties, and examples see [7] and [8]. We point out that a deficient topological measure ν is monotone, countably additive on open sets, $\nu(\emptyset) = 0$, and ν is superadditive, i.e. if $\bigsqcup_{t \in T} A_t \subseteq A$, where $A_t, A \in \mathscr{O}(X) \cup \mathscr{C}(X)$, and at most one of the closed sets is not compact, then $\nu(A) \geq \sum_{t \in T} \nu(A_t)$.

Remark 2. Let ν be a deficient topological measure on X. If X is locally compact and locally connected then by Lemma 2 for each open set U

$$\nu(U) = \sup\{\nu(K) :\ K \subseteq U,\ K \in \mathscr{K}_0(X)\}.$$

If X is locally compact, connected, and locally connected, then from Lemma 1

$$\nu(X) = \sup\{\nu(K) : \ K \in \mathscr{K}_c(X)\},$$

and considering for a compact connected set $C \subseteq X$ its solid hull $\tilde{C} \in \mathscr{K}_s(X), C \subseteq \tilde{C}$ (see section 3 in [7] for detail), we also obtain

$$\nu(X) = \sup\{\nu(K) : \ K \in \mathscr{K}_s(X)\}.$$

We denote by $TM(X)$ and $DTM(X)$, respectively, the collections of all topological measures on X, and all deficient topological measures on X. By $M(X)$ we denote the collection of all Borel measures on X that are inner regular on open sets and outer regular (restricted to $\mathscr{O}(X) \cup \mathscr{C}(X)$).

Remark 3. Let X be locally compact. We have:

$$M(X) \subsetneqq TM(X) \subsetneqq DTM(X). \tag{2.1}$$

For proper inclusions in (2.1) and criteria for a deficient topological measure to be a topological measure or a measure in $M(X)$ see sections 4 and 6 in [8], and section 9 in [7].

Definition 4. *A function $\lambda : \mathscr{A}_s^*(X) \to [0, \infty)$ is a solid set function on X if*

(s1) *whenever* $\bigsqcup\limits_{i=1}^{n} C_i \subseteq C, \quad C, C_i \in \mathscr{K}_s(X)$, *we have* $\sum\limits_{i=1}^{n} \lambda(C_i) \le \lambda(C)$;

(s2) $\lambda(U) = \sup\{\lambda(K) : \ K \subseteq U, \ K \in \mathscr{K}_s(X)\}$ *for* $U \in \mathscr{O}_s^*(X)$;

(s3) $\lambda(K) = \inf\{\lambda(U) : \ K \subseteq U, \ U \in \mathscr{O}_s^*(X)\}$ *for* $K \in \mathscr{K}_s(X)$;

(s4) *if* $A = \bigsqcup_{i=1}^n A_i, \quad A, A_i \in \mathscr{A}_s^*(X)$ *then* $\lambda(A) = \sum\limits_{i=1}^{n} \lambda(A_i)$.

Theorem 1, Theorem 2, and Lemma 5 below are proved in [7], section 8.

Theorem 1. *Let X be locally compact, connected, locally connected. A solid set function on X extends uniquely to a compact-finite topological measure on X. If a solid set function λ is extended to a topological measure μ then the following holds: if λ is simple, then so is μ; if* $\sup\{\lambda(K) : \ K \in \mathscr{K}_s(X)\} = M < \infty$ *then μ is finite and $\mu(X) = M$.*

Theorem 2. *The restriction λ of a compact-finite topological measure μ to $\mathscr{A}_s^*(X)$ is a solid set function, and μ is uniquely determined by λ.*

Remark 4. We will summarize the extension procedure for obtaining a topological measure μ from the a solid set function λ. First, for a compact connected set C we have: $\mu(C) = \lambda(\tilde{C}) - \sum_{i \in I} \lambda(B_i)$, where $\tilde{C} = C \sqcup \bigsqcup_{i \in I} B_i$

is a solid hull of C, and $\{B_i : i \in I\}$ is the family of bounded components of $X \setminus C$. The set $\tilde{C}$ is compact solid, and all B_i are bounded open solid sets.

For $C \in \mathscr{K}_0(X)$, that is, for a compact set C which is the union of finitely many disjoint compact connected sets $C_1, \dots, C_n$, we have: $\mu(C) = \sum_{i=1}^n \mu(C_i)$.

For an open set U we have: $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \in \mathscr{K}_0(X)\}$, and for a closed set F let $\mu(F) = \inf\{\mu(U) : F \subseteq U, U \in \mathscr{O}(X)\}$.

Remark 5. When X is compact, a set is called solid if it and its complement are both connected. For a compact space X we define a certain topological characteristic, genus. See [3] for more information about genus g of the space. We are particularly interested in spaces with genus 0. A compact space has genus 0 iff any finite union of disjoint closed solid sets has a connected complement. Another way to describe the "$g = 0$" condition is the following: if the union of two open solid sets in X is the whole space, their intersection must be connected. (See [9].) Intuitively, X does not have holes or loops. In the case where X is locally path connected, $g = 0$ if the fundamental group $\pi_1(X)$ is finite (in particular, if X is simply connected). Knudsen [11] was able to show that if $H^1(X) = 0$ then $g(X) = 0$, and in the case of CW-complexes the converse also holds.

Lemma 5. *Let X be a locally compact space whose one-point compactification $\hat{X}$ has genus 0. If $A \in \mathscr{A}_s^*(X)$ then any solid partition of A is the set A itself.*

Remark 6. From Lemma 5 it follows that for any locally compact space whose one-point compactification has genus 0 the last condition of Definition 4 holds trivially. This is true, for example, for $\mathbb{R}^n, (\mathbb{R}^n)^+$, an open ball in $\mathbb{R}^n$, or for a punctured closed ball in $\mathbb{R}^n$ with the relative topology ($n \geq 2$).

Example 1. Let $X = \mathbb{R}^2$, l be a straight line and p a point of X not on the line l. For $A \in \mathscr{A}_s^*(X)$ define $\mu(A) = 1$ if $A \cap l \neq \emptyset$ and $p \in A$; otherwise, let $\mu(A) = 0$. It is easy to verify the first three conditions of Definition 4. From Remark 6 it follows that μ is a solid set function on X. By Theorem 1 μ extends uniquely to a topological measure on X, which we also call μ. Note that μ is simple. We claim that μ is not a measure. Let F be the closed half-plane determined by l which does not contain p. Then using Remark 4 we have $\mu(F) = \mu(X \setminus F) = 0$, and $\mu(X) = 1$. Failure of subadditivity shows that μ is not a measure.

Example 2. Let X be a locally compact space whose one-point compactification has genus 0. Let n be a natural number. Let P be the set of distinct $2n + 1$ points. For each $A \in \mathscr{A}_s^*(X)$ let $\nu(A) = i/n$ if $\sharp A = 2i$ or $2i+1$, where $\sharp A$ is the number of points in $A \cap P$. We claim that ν is a solid set function. By Remark 6 we only need to check the first three conditions of Definition 4. The first one is easy to see. Using Lemma 3 and Lemma 4 it is easy to verify the next two conditions. The solid set function ν extends to a unique topological measure on X. This topological measure assumes values $0, 1/n, \dots, 1$.

3. Super-measures on a locally compact space

If X is compact, one way to obtain a large collection of topological measures on X is to use super-measures (see [4], for example). In this section we shall generalize this technique to locally compact spaces.

First, we adapt the definition of a super-measure.

Definition 5. *A super-measure on a countable set E is a function $\nu : \mathscr{P}(E) \to [0, \infty]$ such that $\nu(A \sqcup B) \geq \nu(A) + \nu(B)$ and $\nu(A) < \infty$ for any finite subsets A and B of E.*

Note that a super-measure is a monotone set function.

Theorem 3. *Let X be a locally compact, connected, locally connected space whose one-point compactification has genus 0. Let E be a countable subset of X such that each bounded subset of X contains finitely many points from E, and let ν be a super-measure on E. Define function μ on bounded solid subsets of X by*

$$\mu(A) = \nu(A \cap E).$$

Then μ is a solid set function on X which extends uniquely to a compact-finite topological measure on X.

Proof. By Remark 6 we only need to check the first three conditions of Definition 4. Condition (s1) in Definition 4 is satisfied because ν is a super-measure. Lemma 3 and Lemma 4 help to verify conditions (s2) and (s3). It is easy to see that μ is compact-finite. ☐

Example 3. Let X be $\mathbb{R}^n$ or $\mathbb{R}^n_+$, and E be the set of points with integer coordinates. Let $\nu(A) = \left[\frac{1}{2}|A \cap E|\right]$. Here $|A \cap E|$ is the cardinality of the set $A \cap E$, and $[x]$ denotes the whole part of a real number. Then ν is a super-measure on E, and by Theorem 3 we obtain the topological measure μ on X. Note that μ is not a measure as it is not subadditive: it is easy to see that a compact solid set with positive ν-value can be covered by finitely many solid sets each of which has zero ν-value.

4. Q-functions on a locally compact space

In this section we shall generalize the techniques of q-functions for obtaining topological measures on X to the situation where X is locally compact.

We begin by adapting the definition of a q-function.

Definition 6. *A function $f : [0, \infty) \to [0, \infty)$ is called a q-function if*

(i) *f is right-continuous*

(ii) $f(x) + f(y) \leq f(x + y)$

Remark 7. From Definition 6 it follows that
a) $f(0) = 0$
b) f is non-decreasing
c) $\sum_{i=1}^{n} f(x_i) \leq f(\sum_{i=1}^{n} x_i)$.

Definition 7. *The split spectrum of a deficient topological measure ν is the set $sp(\nu) = \{\alpha \in (0, \infty) :$ there exist $C \in \mathscr{K}_s(X),\ U \in \mathscr{O}_s^*(X),\ C \subseteq U$ with $\nu(C) = \nu(U) = \alpha\}$.*

Remark 8. The definition of a split spectrum of a topological measure on a compact space was first given in [6]. It is easy to see that when X is compact and ν is a topological measure Definition 7 is equivalent to Definition 3.4 in [6].

Theorem 4. *Let X be a locally compact, connected, locally connected space whose one-point compactification has genus 0, ν be a compact-finite deficient topological measure on X and f be a q-function. Define function μ on bounded solid subsets of X by letting $\mu(C) = f(\nu(C))$ for $C \in \mathscr{K}_s(X)$ and $\mu(U) = \sup\{f(\nu(C)) : C \subseteq U,\ C \in \mathscr{K}_s(X)\} = f(\nu(U)^-)$ for $U \in \mathscr{O}_s^*(X)$. Then*

(I) *$\mu = f \circ \nu$ defined as above is a solid set function on X and, hence, extends uniquely to a topological measure on X, which we also call μ.*

(II) *f is continuous on the split spectrum of ν.*

Proof. First note that the second equality in the definition of $\mu(U)$ holds because of regularity of ν.

(I) By Remark 6 we only need to check the first three conditions of Definition 4. Suppose that $C_1, \dots C_n, C \in \mathscr{K}_s(X)$ and $C_1 \sqcup \dots \sqcup C_n \subseteq C$. ν is a deficient topological measure, so by superadditivity of ν (see Remark 1) and the monotonicity of a q-function

$$\sum_{i=1}^{n} \mu(C_i) = \sum_{i=1}^{n} f(\nu(C_i)) \leq f(\sum_{i=1}^{n} \nu(C_i)) \leq f(\nu(C)) = \mu(C).$$

By the the definition of $\mu(U)$, we are only left to show that for every $C \in \mathscr{K}_s(X)$

$$\mu(C) = \inf\{\mu(U) : \ C \subseteq U, \ U \in \mathscr{O}_s^*(X).$$

Let $C \in \mathscr{K}_s(X)$, so $\nu(C) < \infty$. Since f is right-continuous and ν is outer regular, given $\epsilon > 0$ there exist $\delta > 0$ and (by Lemma 4) $U \in \mathscr{O}_s^*(X)$, $C \subseteq U$ such that $\nu(U) - \nu(C) < \delta$ and $f(\nu(U)) - f(\nu(C)) < \epsilon$. Since $f(\nu(C)) \le f(\nu(U)^-) \le f(\nu(U))$, we have:

$$\mu(U) - \mu(C) = f(\nu(U)^-) - f(\nu(C)) \le f(\nu(U)) - f(\nu(C)) < \epsilon,$$

which shows property (s3) of Definition 4 for μ. Thus, μ is a solid set function.

(II) Let $\alpha \in sp(\nu)$ and $C \in \mathscr{K}_s(X)$, $U \in \mathscr{O}_s^*(X)$ be such that $C \subseteq U$, $\nu(C) = \nu(U) = \alpha$. Since μ is a topological measure, we have:

$$\begin{aligned} f(\alpha^-) = f(\nu(U)^-) = \mu(U) &= \sup\{\mu(K) : \ K \in \mathscr{K}_s(X), \ K \subseteq U\} \\ &\ge \mu(C) = f(\nu(C)) = f(\alpha). \end{aligned}$$

Thus, $f(\alpha^-) = f(\alpha)$ and f is continuous at α.

□

Remark 9. (a) By Theorem 1 we may take ν to be a solid set function. (b) In the proof of the first part above we only need f to be right-continuous at $\nu(U)$, $U \in \mathscr{O}_s^*(X)$.

Example 4. Here are some examples of q-functions and compositions of q-functions with topological measures. X is a locally compact, connected, locally connected space whose one-point compactification has genus 0.

(i) The easiest one is the function $f(x) = x$. Then $f \circ \nu = \nu$ for any topological measure ν, where topological measure $f \circ \nu$ is as in Theorem 4.

(ii) Let $\epsilon > 0$. Define $f : [0, \infty) \to [0, \infty)$ by $f(x) = 0$ for $x \in [0, \epsilon)$ and $f(x) = x$ for $x \ge \epsilon$. Then f is a q-function. Let m be the Lebesque measure on $\mathbb{R}^n, n \ge 2$ and $\mu = f \circ m$. Then $\mu(A) = 0$ for any compact solid set A with $m(A) < \epsilon$, and for any open solid bounded set with $m(A) \le \epsilon$. Otherwise $\mu(A) = m(A)$. A closed ball is a compact solid set. A closed ball B of radius greater than ϵ has $\mu(B) = m(B) > 0$ and can be covered by finitely many closed balls B_i of radius less than ϵ with $\mu(B_i) = 0$. Thus, μ is not subadditive and, hence, can not be a measure.

(iii) Consider $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = i$ for $x \in [i, i+1)$, where $i \in \mathbb{N}$. Then f is a q-function. Let m be the Lebesque measure on X and $\mu = f \circ m$. Then μ is a topological measure that assumes nonnegative integer values. Note that μ is not finite. As in part (ii) it is easy to show that μ is not subadditive and, hence, is not a measure.

Remark 10. The topological measure ν in Example 2 can be also obtained by a q-function. Let $g = \frac{1}{n} f$, where f is the q-function from part (iii)in Example 4. Let $m = \frac{1}{2n+1}(\delta_1 + \dots \delta_{2n+1})$, where δ_i are point masses at $2n+1$ points which comprise the set P in Example 2. Then $\nu = g \circ m$.

5. DTM and point methods

In this section we will study topological measures obtained by utilizing a deficient topological measure and a point. We call such methods DTM and point methods, and they are presented in Theorem 5 and Theorem 6.

Theorem 5. *Let X be a locally compact, connected, locally connected space whose one-point compactification has genus 0. Let ν be a deficient topological measure on X such that $\nu(X) < \infty$ and let $p \in X$ be an arbitrary point. Define a set function $\nu_p : \mathscr{A}_s^*(X) \to [0, \infty)$ by*

$$\nu_p(A) = \begin{cases} \nu(A), & \text{if } p \notin A \\ \nu(X) - \nu(X \setminus A), & \text{if } p \in A \end{cases}$$

Then ν_p is a solid set function and, hence, extends to a topological measure on X.

Proof. To show that ν_p is a solid set function by Remark 6 we only need to check the first three conditions of Definition 4. Suppose $C_1 \sqcup C_2 \sqcup \ldots \sqcup C_n \subseteq C$. If $p \notin C$, the first condition is just the superadditivity of ν (see Remark 1). Now assume that p is in one of the sets $C_1, \ldots, C_n$, say, $p \in C_1$. Since $(X \setminus C) \bigsqcup C_2 \bigsqcup \ldots \bigsqcup C_n \subseteq X \setminus C_1$, by superadditivity of ν we have: $\nu(X \setminus C) + \nu(C_2) + \ldots + \nu(C_n) \le \nu(X \setminus C_1)$. Then

$$\begin{aligned} \nu_p(C_1) + \nu_p(C_2) + \ldots + \nu_p(C_n) &= \nu(X) - \nu(X \setminus C_1) + \nu(C_2) + \ldots + \nu(C_n) \\ &\le \nu(X) - \nu(X \setminus C) = \nu_p(C) \end{aligned}$$

The case when $p \in C$ but $p \notin C_i$ for $i = 1, \ldots, n$ can be proved similarly by noticing that $(X \setminus C) \bigsqcup C_1 \bigsqcup \ldots \bigsqcup C_n \subseteq X$ and applying the superadditivuty of ν.

Now we shall show inner and outer regularity conditions (s2) and (s3) of Definition 4 for ν_p. Inner and outer regularity is easy to see when a solid set does not contain p. So assume that $p \in C$, where $C \in \mathscr{K}_s(X)$. For an open set $W = X \setminus C$ and $\epsilon > 0$ find compact $K \subseteq W$ for which $\nu(W) - \nu(K) < \epsilon$. With $U = X \setminus K$, we see that $C \subseteq U$ and by Lemma 4 there exists $V \in \mathscr{O}_s^*(X)$ such that $C \subseteq V \subseteq U$. Then

$$\begin{aligned}\nu_p(V) - \nu_p(C) = \nu(X \setminus C) - \nu(X \setminus V) = \nu(W) - \nu(X \setminus V)\\ \le \nu(W) - \nu(X \setminus U) = \nu(W) - \nu(K) < \epsilon,\end{aligned}$$

which shows the outer regularity condition (s3) of Definition 4 for ν_p.

Now we will assume $p \in U$, where $U \in \mathscr{O}_s^*(X)$, and we shall show the inner regularity. For a closed set $F = X \setminus U$ and $\epsilon > 0$ find an open set W such that $F \subseteq W$ and $\nu(W) - \nu(F) < \epsilon$. Since compact $X \setminus W \subseteq U$, by Lemma 3 there exists $K \in \mathscr{K}_s(X)$ such that $X \setminus W \subseteq K \subseteq U$ and $p \in K$. Then

$$\begin{aligned}\nu_p(U) - \nu_p(K) = \nu(X \setminus K) - \nu(X \setminus U) = \nu(X \setminus K) - \nu(F)\\ \le \nu(W) - \nu(F) < \epsilon,\end{aligned}$$

which shows inner regularity (s2) of Definition 4 for ν_p. ☐

Example 5. Let ν be the topological measure on $X = \mathbb{R}^2$ from Example 1, and let ν_p be given by Theorem 5 using p from Example 1. Then for $A \in \mathscr{A}_s^*(X)$

$$\nu_p(A) \;=\; \begin{cases} 0, & \text{if } p \notin A \\ 1, & \text{if } p \in A \end{cases} \tag{5.1}$$

Let $C \in \mathscr{K}_c(X)$. From Remark 4,

$$\nu_p(C) = \nu_p(\tilde{C}) - \sum_{i \in I} \nu_p(B_i). \tag{5.2}$$

where B_i are bounded open solid sets and $\tilde{C}$ is a compact solid set.

If $p \in C$ then $p \in \tilde{C}$ but $p \notin B_i$ for all $i \in I$. Then by (5.1) $\nu_p(C) = 1$. If $p \notin C$ then p may or may not belong to $\tilde{C}$. If $p \notin \tilde{C}$, then $p \notin B_i$ for each i, and $\nu_p(C) = 0$. If $p \notin C$, but $p \in \tilde{C}$, then p is in some component B_j, and $\nu_p(C) = 0$. We see that $\nu_p(A) = 0$ if $p \notin A$ and $\nu_p(A) = 1$ if $p \in A$ for A being compact connected, then a finite disjoint union of compact connected sets, then open, and then closed, by Remark 4. Thus, ν_p on $\mathscr{A}(X)$ is the point mass δ_p.

Example 6. Let $X = \mathbb{R}^2$, and let ν be a topological measure on X as in Example 2 for $P = \{p_1, p_2, p_3, p_4, p_5\}$, where $p_i = (4i-1, 0)$. For $i = 1, \dots, 5$ let U_i be an open disk of radius 3, and let $W = U_1 \cup U_2 \cup \ldots \cup U_5$, $U = U_1 \cup U_2 \cup U_3$. Then $U_1, \dots, U_5, U, W$ are all open bounded solid sets, and $W = U \cup U_4 \cup U_5$. Taking p to be any point not in W, consider the topological measure ν_p given by Theorem 5. Both ν and ν_p assume values $0, 1/2, 1$. We see that $\nu_p(U_4) = \nu_p(U_5) = 0, \nu_p(U) = 1/2$, and $\nu_p(W) = 1$. Thus, ν_p is not subadditive, hence, it is a topological measure which is not a measure. This is in contrast to Example 5.

Theorem 6. *Let X be a locally compact, locally connected, connected space whose one-point compactification has genus 0. Let λ be a compact-finite deficient topological measure on X, and let $p \in X$ be an arbitrary point. Define a set function $\lambda_p : \mathscr{A}_s^*(X) \to [0, \infty)$ by*

$$\lambda_p(A) \; = \; \begin{cases} 0, & \text{if } p \notin A \\ \lambda(A), & \text{if } p \in A \end{cases}$$

Then λ_p is a solid set function and, hence, extends to a topological measure on X. If λ is compact-finite but not finite, then so is λ_p.

Proof. By Remark 6 we only need to check the first three conditions of Definition 4. The first one is easy to see. We shall show the inner and outer regularity conditions of Definition 4 for λ_p. Let $U \in \mathscr{O}_s^*(X)$. The inner regularity is trivial when $p \notin U$. Now let $p \in U$. Since $\lambda(U) < \infty$, for $\epsilon > 0$ choose C such that $p \in C \subseteq U$, $\lambda(U) - \lambda(C) < \epsilon$. By Lemma 3 we may assume that $C \in \mathscr{K}_s(X)$. Then

$$\lambda_p(U) - \lambda_p(C) = \lambda(U) - \lambda(C) < \epsilon.$$

The proof of outer regularity uses Lemma 4 and is similar. □

Example 7. Let $X = \mathbb{R}^n$, $n \geq 2$. The Lebesque measure λ is a compact-finite deficient topological measure on X, so let λ_p be a topological measure on X according to Theorem 6. We claim that λ_p is not a measure. Since $\lambda_p(X) = \sup\{\lambda_p(K) : \; K \in \mathscr{K}_s(X)\}$, taking balls of arbitrarily large radius we see that $\lambda_p(X) = \infty$. Now let X be covered by countably many open balls of the same positive radius: $X = \bigcup_{i=1}^{\infty} B_i$. Only finitely many of B_i contain p, and thus have a positive λ_p measure. Thus, $\sum_{i=1}^{\infty} \lambda_p(B_i) < \infty$, so λ_p is not subadditive and, hence, can not be a measure.

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Способы получения топологических мер на локально компактных пространствах

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Аннотация. Топологические меры и квазилинейные функционалы являются обобщением мер и линейных функционалов. Дефектные топологические меры, в свою очередь, являются обобщением топологических мер. В этой статье мы продолжаем исследование топологических мер на локально компактных пространствах. На компактном пространстве существующие способы получения топологических мер — это (а) метод, использующий супер-меры, (б) композиция q-функции с топологической мерой и (в) метод с использованием дефектных топологических мер и единичных точек. Эти способы применимы, когда компактное пространство является связным, локально связным, а также имеет определённую топологическую характеристику, которая называется «род», равную 0 (интуитивно, у таких пространств нет дыр). Мы обобщаем известные способы на случай, когда пространство локально компактное, связное, локально связное, и его компактификация Александрова имеет род 0. Мы даём определение супер-мер и q-функций на локально компактном пространстве. Затем мы получаем методы построения новых топологических мер, используя супер-меры, а также композиции q-функций с дефектными топологическими мерами. Мы также обобщаем существующий метод и приводим новый метод с использованием точки и дефектной топологической меры на локально компактном пространстве. Представленные способы позволяют получить большое количество разнообразных конечных и бесконечных топологических мер на таких пространствах, как $\mathtt{R}^n$, полупространства в $\mathtt{R}^n$, открытые шары в $\mathtt{R}^n$, и проколотые замкнутые шары в $\mathtt{R}^n$ с индуцированной топологией (где $n \geq 2$).

Ключевые слова: топологические меры, солид-функции, супермеры, q-функции.

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