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The Generalized Splitting Theorem for Linear Sobolev type Equations in Relatively Radial Case

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Abstract. Sobolev type equations now constitute a vast area of nonclassical equations of mathematical physics. Those called nonclassical equations of mathematical physics, whose representation in the form of equations or systems of equations partial does not fit within one of the classical types (elliptic, parabolic or hyperbolic). In this paper we prove a generalized splitting theorem of spaces and actions of the operators for Sobolev type equations with respect to the relatively radial operator. The main research method is the Sviridyuk theory about relatively spectrum. Abstract results are applied to prove the unique solvability of the multipoint initial–final problem for the evolution equation of Sobolev type, as well as to explore the dichotomies of solutions for the linearized phase field equations.

Apart from the introduction and bibliography article comprises three parts. The first part provides the necessary information regarding the theory of p-radial operators, the second contains the proof of main result about generalized splitting theorem for strongly (L,p)-radial operator M. The third part contains the results of the application of the preceding paragraph for different tasks, namely to prove the unique solvability of the multipoint initial–final problem for Dzektser and to explore the dichotomies of solutions of the linearized phase field equations. References not purport to, and reflects only the authors' tastes and preferences.

Keywords: linear Sobolev type equations, generalized splitting theorem, dichotomies of solutions, multipoint initial-final problem.

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, the operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ (linear and continuous) and $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$ (linear, closed and densely defined). Let the operator M be a relative p-radial with respect to operator L (or shortly, (L,p)-radial operator), $p \in \{0\} \cup \mathbb{N}$ (terminology and the results see in sec. 3 [12]). We note only that the concept of (L,p)-radial operator M was introduced by G.A. Sviridyuk in [9].

Let us consider a linear *Sobolev type equation*

$$L\dot{u} = Mu. \tag{0.1}$$

Sobolev type equations now constitute a vast area of nonclassical equations of mathematical physics. Detailed historical review of these studies and an extensive bibliography can be found in the book [1].

In this paper we prove a generalization of the splitting theorem of the spaces $\mathfrak{U}$ and $\mathfrak{F}$ on invariant subspaces in accordance to the splitting of relative L-spectrum of operator M. As you know, firstly the splitting theorem formulated and proved by G.A. Sviridyuk [8] in the case of (L, p)-bounded and (L, p)-sectorial operators M. A.V. Keller developed these results in [10] and applied them to the study of the dichotomies solutions [3]. Splitting theorem in the case of (L, p)-radial operator M with respect to the dichotomy of the solutions appeared, for example, in [5] (see in more detail in [6]). First proof of the generalized splitting theorem in the case of (L, p)-bounded operator M appeared in [13].

The need for generalized splitting theorem appeared in the study of multipoint initial–final conditions

$$P_j(u(\tau_j) - u_j) = 0, \ j = \overline{0, n}, \tag{0.2}$$

for linear equations of Sobolev type (0.1). In these parts $\tau_j \in \mathbb{R}$ $(\tau_j < \tau_{j+1})$, $u_j \in \mathfrak{U}$ and P_j is a *relative spectral projections* (talking about them will go to claim 1 this article), $j = \overline{0, n}$. Note that if $n = 1$ then the condition (0.2) take a simpler form

$$P_0(u(\tau_0) - u_0) = P_1(u(\tau_1) - u_1) = 0, \tag{0.3}$$

then there will be a *initial-final* condition [14]. Problem (0.1), (0.3) has been very extensively studied at various aspects in [4], [15], [16].

Apart from the introduction and bibliography article comprises three parts. The first part provides essential information regarding the theory of relatively p-radial operators [12], the second one contains the main result, namely the proof of the generalized splitting theorem in the case of (L, p)-radial operator M. The third part contains the application of the preceding paragraph for different tasks, namely to the prove the unique solvability of the multipoint initial-final problem for Dzektser and to explore the dichotomies of solutions of the linearized phase field equations. References not purport and reflects only the authors' tastes and preferences.

Finally, we note that all considerations are conducted in real Banach spaces, but when considering the «spectral issues» introduced their natural complexification. All circuits oriented counterclockwise movement and limit area lying to the left in such a motion. Symbols of $\mathbb{O}$ and $\mathbb{I}$ denote, respectively, the «zero» and «identity» operators whose domain is clear from the context.

In conclusion, the authors consider it their pleasant duty to express their sincere gratitude to G.A. Sviridyuk for his interest and active creative discussions.

1. Relatively p-radial operators

Recall the standard notation of the theory of relatively p-radial operators [9], [12].

Suppose, as above, $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ (linear and continuous) and $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$ (linear, closed and densely defined).

Denote

$$p^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}, \qquad \sigma^L(M) = \mathbb{C} \setminus p^L(M),$$

$$R^L_\mu(M) = (\mu L - M)^{-1} L, \quad L^L_\mu(M) = L(\mu L - M)^{-1}, \quad \mu \in p^L(M),$$

$$R^L_{(\lambda,p)}(M) = \prod_{k=0}^{p} R^L_{\lambda_k}(M),\ L^L_{(\lambda,p)}(M) = \prod_{k=0}^{p} L^L_{\lambda_k}(M),\ \ \lambda_k \in p^L(M)\ (k = \overline{0,p}).$$

Definition 1. Operator M is called *p-radial by relative* to operator L (or shortly (L,p)*-radial*) if

(i) $\exists \beta \in \mathbb{R}\ (\beta, +\infty) \subset p^L(M)$;

(ii) $\exists K > 0\ \forall \mu = (\mu_0, \mu_1, \ldots, \mu_p) \in (\beta, +\infty)^{p+1}\ \forall n \in \mathbb{N}$

$$\max\{\|(R^L_{(\mu,p)}(M))^n\|_{\mathcal{L}(\mathfrak{U})}, \|(L^L_{(\mu,p)}(M))^n\|_{\mathcal{L}(\mathfrak{F})}\} \leq \frac{K}{\prod\limits_{k=0}^{p} (\mu_k - \beta)^n}.$$

Also introduce the notation

$$\mathfrak{U}^0 = \ker R^L_{(\mu,p)}(M),\ \mathfrak{F}^0 = \ker L^L_{(\mu,p)}(M),\ L_0 = L\Big|_{\mathfrak{U}^0},\ M_0 = M\Big|_{\operatorname{dom} M \cap \mathfrak{U}^0}.$$

By $\mathfrak{U}^1$ ($\mathfrak{F}^1$) denote the closure of the lineal $\operatorname{im} R^L_{(\mu,p)}(M)$ ($\operatorname{im} L^L_{(\mu,p)}(M)$). Also by $\tilde{\mathfrak{U}}$ ($\tilde{\mathfrak{F}}$) denote the closure of the lineal $\mathfrak{U}^0 \dot{+} \operatorname{im} R^L_{(\mu,p)}(M)$ ($\mathfrak{F}^0 \dot{+} \operatorname{im} L^L_{(\mu,p)}(M)$) in the norm $\mathfrak{U}$ ($\mathfrak{F}$).

Definition 2. Vector-valued function $u \in C^1(\overline{\mathbb{R}}_+;\mathfrak{U})$ is called the *solution of the equation* (0.1) if it satisfy to this equation on $\overline{\mathbb{R}}_+ = \{0\} \cup \mathbb{R}_+$.

Definition 3. The closed set $\mathfrak{P} \subset \mathfrak{U}$ is called the *phase space* of the equation (0.1), if

(i) any solution $u(t)$ of the equation (0.1) implies in $\mathfrak{P}$ (pointwise);

(ii) there the unique solution of the Cauchy problem

$$u(0) = u_0 \tag{1.1}$$

for the equation (0.1) with any u_0 of a lineal $\overset{\circ}{\mathfrak{P}}$ which is dense in $\mathfrak{P}$.

Together with the equation (0.1) consider the equivalent equation

$$L(\alpha L - M)^{-1}\dot{f} = M(\alpha L - M)^{-1} f, \qquad \alpha \in \rho^L(M). \tag{1.2}$$

Theorem 1. *Let the operator M be a (L,p)-radial. Then the phase space of equation* (0.1) ((1.2)) *is set* $\mathfrak{U}^1$ ($\mathfrak{F}^1$).

Definition 4. The family of operators $U^\bullet : \overline{\mathbb{R}}_+ \to \mathcal{L}(\mathfrak{U})$ is called a *resolving semigroup* of equation (0.1), if

(i) $U^s U^t = U^{s+t}$ $\forall s, t \in \overline{\mathbb{R}}_+$;

(ii) $u(t) = U^t u_0$ is a solution of this equation for any u_0 from some lineal witch dense in $\mathfrak{U}$;

(iii) restriction of the semigroup identity on the phase space $\mathfrak{P}$ of the equation is $U^0\Big|_{\mathfrak{P}} = \mathbb{I}$.

Semigroup $\{U^t \in \mathcal{L}(\mathfrak{U}) : t \in \overline{\mathbb{R}}_+\}$ called *exponentially bounded* with constants C, β if $\exists C > 0$ $\exists \beta \in \mathbb{R}$ $\forall t \in \overline{\mathbb{R}}_+$ $\|U^t\|_{\mathcal{L}(\mathfrak{U})} \leq Ce^{\beta t}$.

Theorem 2. *Let the operator M be a (L,p)-radial. Then there exists strongly continuous resolving semigroup of equation* (0.1) ((1.2)) *considered on subspace* $\tilde{\mathfrak{U}}$ ($\tilde{\mathfrak{F}}$). *And it is exponentially bounded with constants K, β from definition* 1.

Remark 1. Operators of the resolving semigroup for (0.1) ((1.2)) with $t > 0$, as amended, are discussed in [7], can be represented as

$$U^t = s\text{-}\lim_{k\to\infty}\left(\left(L - \frac{t}{k}M\right)^{-1} L\right)^k = s\text{-}\lim_{k\to\infty}\left(\frac{k}{t} R^L_{\frac{k}{t}}(M)\right)^k$$

$$\left(F^t = s\text{-}\lim_{k\to\infty}\left(L\left(L - \frac{t}{k}M\right)^{-1}\right)^k = s\text{-}\lim_{k\to\infty}\left(\frac{k}{t} L^L_{\frac{k}{t}}(M)\right)^k\right).$$

Remark 2. The identity of the semigroup $\{U^t \in \mathcal{L}(\tilde{\mathfrak{U}}) : t \in \overline{\mathbb{R}}_+\}$ ($\{F^t \in \mathcal{L}(\tilde{\mathfrak{F}}) : t \in \overline{\mathbb{R}}_+\}$) is a projector $P = \lim\limits_{t\to 0+} U^t$ ($Q = \lim\limits_{t\to 0+} F^t$) along $\mathfrak{U}^0$ ($\mathfrak{F}^0$) on $\mathfrak{U}^1$ ($\mathfrak{F}^1$).

Definition 5. Operator M is called *strongly* (L,p)*-radial* if for any $\lambda, \mu_0, \mu_1, ..., \mu_p > \beta$ the next conditions are fulfilled

(i) there is dense in $\mathfrak{F}$ lineal $\overset{\circ}{\mathfrak{F}}$ such that for all $f \in \overset{\circ}{\mathfrak{F}}$

$$\|M(\lambda L - M)^{-1} L^L_{(\mu,p)}(M) f\|_{\mathfrak{F}} \leq \frac{\text{const}(f)}{(\lambda - \beta)\prod\limits_{k=0}^{p}(\mu_k - \beta)};$$

(ii) $$\|R^L_{(\mu,p)}(M)(\lambda L - M)^{-1}\|_{\mathcal{L}(\mathfrak{F};\mathfrak{U})} \le \frac{K}{(\lambda-\beta)\prod\limits_{k=0}^{p}(\mu_k-\beta)}.$$

Theorem 3. *Suppose that M is strongly (L,p)-radial. Then*
(i) $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$, $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$;
(ii) $L_k = L\Big|_{\mathfrak{U}^k} \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$, $\quad M_k = M\Big|_{\mathrm{dom}M_k} \in \mathcal{C}l(\mathfrak{U}^k;\mathfrak{F}^k)$,
$\mathrm{dom}M_k = \mathrm{dom}M \cap \mathfrak{U}^k$, $k = 0, 1$;
(iii) *there exist operators* $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$ *u* $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$.

2. Generalized splitting theorem

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ and $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$, where the operator M is strongly (L,p)-radial, $p \in \{0\} \cup \mathbb{N}$. We introduce the condition

$$\left.\begin{array}{c} \sigma^L(M) = \bigcup\limits_{j=0}^{n} \sigma_j^L(M),\ n \in \mathbb{N},\ \text{and } \sigma_j^L(M) \neq \emptyset, \text{ there is} \\ \text{closed loop } \gamma_j \subset \mathbb{C} \text{ and } \gamma_j = \partial D_j, \text{ where } D_j \supset \sigma_j^L(M), \\ \text{such that } \overline{D_j} \cap \sigma_0^L(M) = \emptyset \text{ and } \overline{D_k} \cap \overline{D_l} = \emptyset \\ \text{for all } j,k,l = \overline{1,n}, k \neq l. \end{array}\right\} \tag{2.1}$$

Consider operators $P_j \in \mathcal{L}(\mathfrak{U})$ and $Q_j \in \mathcal{L}(\mathfrak{F})$, $j = \overline{j,n}$, that because of the relative spectral theorem [3] have the form

$$P_j = \frac{1}{2\pi i}\int_{\gamma_j} R^L_\mu(M)d\mu, \quad Q_j = \frac{1}{2\pi i}\int_{\gamma_j} L^L_\mu(M)d\mu, \quad j = \overline{1,n}.$$

By the results of [5] operators P_0 and Q_0 are of the form :

$$P_0 = P - \sum_{j=1}^{n} P_j, \qquad Q_0 = Q - \sum_{j=1}^{n} Q_j.$$

Lemma 1. *Let the operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ and $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$. The operator M is strongly (L,p)-radial, $p \in \{0\} \cup \mathbb{N}$ and the condition (2.1) holds. Then the operators $P_j : \mathfrak{U} \to \mathfrak{U}$, $(j = \overline{1,n})$ are projectors and*
(i) $PP_j = P_jP = P_j$, $j = \overline{1,n}$;
(ii) $P_kP_l = P_lP_k = \mathbb{O}$, k, $l = \overline{1,n}$, $k \neq l$.

Proof. (i)

$$PP_j = \frac{1}{(2\pi i)^2}\int\limits_{\gamma}\int\limits_{\gamma_j} R^L_\mu(M)R^L_\lambda(M)d\mu d\lambda =$$

$$= \frac{1}{(2\pi i)^2}\left(\int\limits_{\gamma}\frac{d\mu}{\mu-\lambda}\int\limits_{\gamma_j}R^L_\lambda(M)d\lambda + \int\limits_{\gamma}R^L_\mu(M)d\mu\int\limits_{\gamma_j}\frac{d\lambda}{\lambda-\mu}\right) = Q_j$$

by the Fubini's theorem, the theorem deductions and the right Hilbert relative resolvent identity

$$R^L_\lambda(M) - R^L_\mu(M) = (\mu-\lambda)R^L_\mu(M)R^L_\lambda(M).$$

The second equality in (i) and proved similarly.

(ii) Now applies the same theorem and the same identity as in (i) when $k \neq l$ and we obtain

$$P_kP_l = \frac{1}{(2\pi i)^2}\int\limits_{\gamma_k}\int\limits_{\gamma_l}R^L_\mu(M)R^L_\lambda(M)d\mu d\lambda =$$

$$= \frac{1}{(2\pi i)^2}\left(\int\limits_{\gamma_k}\frac{d\lambda}{\lambda-\mu}\int\limits_{\gamma_l}R^L_\mu(M)d\mu + \int\limits_{\gamma_k}R^L_\lambda(M)d\lambda\int\limits_{\gamma_l}\frac{d\mu}{\mu-\lambda}\right) = \mathbb{O}.$$

□

Remark 3. Projector P_0 by the results of [6] hold similar equalities

(i) $PP_0 = P_0P = P_0$;

(ii) $P_0P_j = P_jP_0 = \mathbb{O}$, $j = \overline{1,n}$.

Remark 4. Clearly that for projectors Q_j ($j = \overline{0,n}$) the relations of Lemma 1 and Remark 3 are truth by the construction.

By Lemma 1 and Remarks 3, 4 the projectors P_j and Q_j, $j = \overline{0,n}$ call *relative spectral projectors.*

We introduce the subspace $\mathfrak{U}^{1j} = \mathrm{im}P_j$, $\mathfrak{F}^{1j} = \mathrm{im}Q_j$, $j = \overline{0,n}$. By construction,

$$\mathfrak{U}^1 = \bigoplus_{j=0}^{n}\mathfrak{U}^{1j} \qquad \text{and} \qquad \mathfrak{F}^1 = \bigoplus_{j=0}^{n}\mathfrak{F}^{1j}.$$

By the L_{1j} denote the restriction of L on $\mathfrak{U}^{1j}$, $j = \overline{0,n}$ and by M_{1j} denote the restriction of M on $\mathrm{dom}M \cap \mathfrak{U}^{1j}$, $j = \overline{0,n}$. It's easy to show that $P_j\varphi \in \mathrm{dom}\,M$, if $\varphi \in \mathrm{dom}\,M$ then the domain of $\mathrm{dom}\,M_{1j} = \mathrm{dom}\,M \cap \mathfrak{U}^{1j}$ is dense in $\mathfrak{U}^{1j}$, $j = \overline{0,n}$.

Theorem 4 (generalized spectral theorem). *Let the operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ and $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$, the operator M is strongly (L,σ)-radial and the condition* (2.1) *holds. Then*

(i) *operators $L_{1j} \in \mathcal{L}(\mathfrak{U}^{1j};\mathfrak{F}^{1j})$, $M_{1j} \in \mathcal{C}l(\mathfrak{U}^{1j};\mathfrak{F}^{1j})$, $j = \overline{0,n}$;*

(ii) *there exist operators $L^{-1}_{1j} \in \mathcal{L}(\mathfrak{F}^{1j};\mathfrak{U}^{1j})$, $j = \overline{0,n}$.*

Proof. The proof is based on ideological Sviridyuk proof of splitting theorem [10] so is given in abbreviated form.

$$\text{(i)} \qquad L_{1j}P_j = \frac{1}{2\pi i}\int\limits_{\gamma_j} L_{1j}(\mu L_{1j} - M_{1j})^{-1}L_{1j}P_j d\mu = Q_j L_{1j} P_j,$$

where $j = \overline{0,n}$. Next let $\varphi \in \operatorname{dom} M_{1j}$ then

$$M_{1j}P_j\varphi = \frac{1}{2\pi i}\int\limits_{\gamma_j} M_{1j}(\mu L_{1j} - M_{1j})^{-1}L_{1j}P_j\varphi d\mu = Q_j M_{1j} P_j\varphi.$$

Since

$$M_{1j}(\mu L_{1j} - M_{1j})^{-1}L_{1j} = \mu L_{1j}(\mu L_{1j} - M)^{-1}L_{1j} - L_{1j},$$

and the rights of equality is a continuous operator, densely defined on $\mathfrak{U}^{1j}$. These continuous operators $M_{1j}P_j$ can be uniquely complemented to a continuous operator defined on all subspace $\mathfrak{U}^{1j}$. Through $M_{1j}P_j$ denote this extension $j = \overline{0,n}$.

(ii) Since

$$L_{1j}\left(\frac{1}{2\pi i}\int\limits_{\gamma_j}(\mu L - M)^{-1}Q_j d\mu\right) = \frac{1}{2\pi i}\int\limits_{\gamma_j} L_{1j}(\mu L_{1j} - M_{1j})^{-1}Q_j d\mu = Q_j$$

and

$$\left(\frac{1}{2\pi i}\int\limits_{\gamma_j}(\mu L - M)^{-1}Q_j d\mu\right) L_{1j} = \frac{1}{2\pi i}\int\limits_{\gamma_j}(\mu L_{1j} - M_{1j})^{-1}L_{1j}P_j d\mu = P_j,$$

the operatorws $L_{1j}^{-1} \in \mathcal{L}(\mathfrak{F}^{1j};\mathfrak{U}^{1j})$ are the restriction operators

$$\frac{1}{2\pi i}\int\limits_{\gamma_j}(\mu L M)^{-1}Q_j d\mu$$

onto the subspaces $\mathfrak{F}^{1j}$ $j = \overline{0,n}$. □

Under the conditions of Theorem 4 exist operators $S_j = L_{1j}M_{1j} \in \mathcal{C}l(\mathfrak{U}^{1j})$, $j = \overline{0,n}$.

So let the operator M is strongly (L,p)-radial and the condition (2.1) is fulfilled $\tau_j \in \mathbb{R}_+$, $(\tau_j < \tau_{j+1})$, $u_j \in \mathfrak{U}$, $j = \overline{0,n}$. Take $f \in C^\infty(\mathbb{R}_+;\mathfrak{F})$, $j = \overline{0,n}$. Consider the Sobolev type equations

$$L\dot{u} = Mu + f. \tag{2.2}$$

We acton to this equation by the series of projectors $\mathbb{I}-Q$ and Q_j, $j=\overline{0,n}$ and obtain an equivalent system of equations

$$H\dot{u}^0 = u^0 + M_0^{-1}f^0, \tag{2.3}$$

$$\dot{u}_j^1 = S_{1j}u_j^1 + L_{1j}^{-1}f_j^1, \tag{2.4}$$

where $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$ is a nilpotent operator of degree $p \in \{0\} \cup \mathbb{N}$, $S_{\mathrm{I}j} = L_{1j}^{-1}M_{\mathrm{I}j} \in \mathcal{C}l(\mathfrak{U}_j^1)$ and range of $\sigma(S_j) = \sigma_j^L(M)$; $f^0 = (\mathbb{I}-Q)f$, $f_j^1 = Q_jf$, $u^0 = (\mathbb{I}-P)u$, $u_j^1 = P_ju$, $j=\overline{0,n}$.

3. Applications of the generalized splitting theorem

3.1. Multipoint initial-final problem for the Dzektser equation

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ and $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$. Where the operator M is strongly (L,p)-radial, $p \in \{0\} \cup \mathbb{N}$. Consider the multipoint initial-final problem

$$P_j(u(\tau_j) - u_j) = 0, \quad j=\overline{0,n} \tag{3.1}$$

for the equation (2.2).

Definition 6. Vector-valued function $u \in C([0,\tau_n];\mathfrak{U}) \cap C^1((0,\tau_n);\mathfrak{U})$, satisfying (2.2), called a *solution of the multipoint initial-final problem* (2.2), (3.1) if it satisfies the equation (2.2) and the terms of $\lim\limits_{t\to\tau_0+} P_0(u(t)-u_0) = 0$, $P_j(u(\tau_j)-u_j) = 0$, $j=\overline{1,n}$.

Lemma 2. *Let operator M is strongly (L,p)-radial ($p \in \{0\}\cup\mathbb{N}$) and part of the spectrum $\sigma_{1j}^L(M)$ is bounded, $j=\overline{1,n}$. Then for any vector function $f^0 \in C^{p+1}((0,\tau);\mathfrak{F}^0)$ there exists a unique solution $u^0 \in C^1([0,\tau];\mathfrak{U}^0)$ of equation* (2.3) *which also has the form*

$$u^0(t) = -\sum_{q=0}^{p} H^q M_0^{-1} \frac{d^q}{dt^q} f^0(t).$$

Proof. By substituting the vector function $u^0 = u^0(t)$ in (2.3) verify the existence of solutions. Uniqueness obtained by successive differentiation of the homogeneous equation (2.3) $0 = H^p u^{0(p)} = \ldots = H\dot{u}^0 = u^0$. Lemma proved. □

Lemma 3. *Under the conditions of Lemma 2 for any vector $u_j \in \mathfrak{U}$ and for any vector-valued function $f_j^1 \in C([0,\tau_n];\mathfrak{F}_j^1)$ there exists an unique solution*

$u^1_{1j} \in C([0,\tau_n];\mathfrak{U}^1_j) \cap C^1((0,\tau_n);\mathfrak{U}^1_{1j})$ *of the problem* $P_{1j}(u(\tau_j)-u_j)=0$ *for the equation* (2.4)*, which also has the form*

$$u^1_j(t) = U_j^{t-\tau_j}u_{\tau_j} - \int_t^{\tau} U^{ts}_{1j}L^{-1}_{1j}f^1_j(s)ds.$$

Proof. If $j=0$ then declaration of Lemma is a classical result by the radiality of operator S_0.

Let's prove the uniqueness. Using the substitution we see that the vector-valued $u^1_{1j} = u^1_{1j}(t)$ is a solution to this problem. Let $v=v(t)$, $t\in[0,\tau]$ be an another solution of this problem. Construct vector-valued function $w(s,t) = L_{1j}U^{ts}_{1j}v(s)$. By construction

$$\frac{\partial w(s,t)}{\partial s} = L_{1j}\frac{\partial U^{ts}_{1j}}{\partial s}v(s) + L_{1j}U^{ts}_{1j}\frac{\partial v(s)}{\partial s} = 0.$$

Hence $w(\tau,t)=w(t,t)$, ie, $U^{t-\tau}_{1j}v(t)-v(t)=0$. □

Thus we have proved

Theorem 5. *For any vectors* $u_j \in \mathfrak{U}$ *and any vector function* $f:[0,\tau_n]\to\mathfrak{F}$*, which satisfies conditions of Lemmas* 2, 3*, there exists an unique solution* $u\in C([0,\tau_n];\mathfrak{U})\cap C^1((0,\tau_n);\mathfrak{U})$*, which also has the form*

$$u(t) = u^0(t) + \sum_{j=0}^{n} u^1_j(t).$$

We now consider the *Dzektser model* for evolution of the free surface of the filtered fluid.

Let $\Omega\subset\mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class C^∞. Consider the equation

$$(\lambda-\Delta)u_t = \alpha\Delta u - \beta\Delta^2 u + f, \tag{3.2}$$

where $\lambda\in\mathbb{R}$, $\alpha,\beta\in\mathbb{R}_+$, with the boundary conditions

$$u(x,t)=0,\ (x,t)\in\partial\Omega\times(0,\tau_n). \tag{3.3}$$

This system modells the evolution of the free surface filtered fluid (see [11] and references there in). We reduce equation(3.2) to the equation (2.2). For this we take the functional spaces

$$\mathfrak{U} = \{u\in W^k_p(\Omega): u(x)=0,\ x\in\partial\Omega\},\quad \mathfrak{F}=W^k_p(\Omega),$$

where $k\in\{0\}\cup\mathbb{N}$, $p\in(1,+\infty)$ and $W^l_q(\Omega)$ are Sobolev spaces. We define the operators $L\in\mathcal{L}(\mathfrak{U};\mathfrak{F})$, $M\in\mathcal{C}l(\mathfrak{U};\mathfrak{F})$ by formulas $L=\lambda-\Delta$, $M=\alpha\Delta-\beta\Delta^2$, where $\operatorname{dom} M = \{u\in W^{k+2}_p(\Omega): u(x)=\Delta u(x)=0,\ x\in\partial\Omega\}$.

Lemma 4. *For any $\lambda \in \mathbb{R}\backslash\{0, \alpha\cdot\beta^{-1}\}$ operator M is strongly $(L, 0)$-radial.*

We denote by $\{\lambda_k\}$ sequence of eigenvalues of the homogeneous Dirichlet problem for the Laplace operator Δ in Ω. The sequence $\{\lambda_k\}$ is numbered by non-increasing with according to multiplicity. We denote by $\{\varphi_k\}$ orthonormal (in the sense of $L_2(\Omega)$) sequence corresponding eigenfunctions, $\varphi_k \in C^\infty(\Omega)$, $k \in \mathbb{N}$. The L-spectrum of M has the form

$$\sigma^L(M) = \left\{\mu_k = \frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k},\ k \in \mathbb{N} \setminus \{l : \lambda_l = \lambda\}\right\}.$$

Clearly that for such a set we can pick up the contour $\gamma \in \mathbb{C}$, which satisfying the condition (2.1). Construct projectors

$$P_j = \sum_{k:\mu_k \in \sigma_j^L(M)} \langle \cdot, \varphi_k \rangle \varphi_k,\ j = \overline{0, n}.$$

Take $\tau_j \in \mathbb{R}_+$ $(\tau_j < \tau_{j+1})$, $u_j \in \mathfrak{U}$, $j = \overline{0, n}$ and will be in the cylinder $\Omega \times (0, \tau_n)$ seek a solution of (3.2), satisfying the boundary condition (3.3) and conditions

$$P_j = \sum_{k:\mu_k \in \sigma_j^L(M)} \langle (u(\tau_j) - u_j), \varphi_k \rangle \varphi_k = 0,\ j = \overline{0, n} \tag{3.4}$$

of multipoint initial-final problem. For simplicity, only the case where f is independen to t, ie $f =$ const. Theorem 5 and Lemma 4 follows

Theorem 6. *For any $\lambda \in \mathbb{R} \setminus \{0, \alpha \cdot \beta^{-1}\}$, $\beta \in \mathbb{R}_+$, $u_0 \in \mathrm{dom}M$, $u_\tau \in \mathfrak{U}$, $f \in \mathfrak{F}$ exists an unique solution $u \in C([0, \tau]; \mathfrak{U}) \cap C^1((0, \tau); \mathfrak{U})$ of problem* (3.4), (3.3) *for the equation* (3.2).

Remark 5. By Lemmas 2, 3 and Theorem 5, you can get the kind of solutions, however, due to the bulkiness it descends.

3.2. Dichotomy of solutions of the linearized phase field equations

Let the operator M is strongly (L, p)-radial with constant $\beta > 0$. Introduce the condition

$$\left.\begin{array}{c} \text{There is } \omega > 0, \text{ that } \sigma^L(M) \cap \{\mu \in \mathbb{C} : -\omega \leq \mathrm{Re}\mu \leq \omega\} = \emptyset. \\ \text{Denote } \mathbb{C}_+ = \{\mu \in \mathbb{C} : \mathrm{Re}\mu > 0\},\ \mathbb{C}_- = \{\mu \in \mathbb{C} : \mathrm{Re}\mu < 0\}, \\ \sigma_\pm = \sigma^L(M) \cap \mathbb{C}_\pm \text{ and let set } \sigma_+ \textit{ is bounded.} \end{array}\right\} \tag{3.5}$$

Remark 6. Condition (3.5) is a special case of condition (2.1) for $n = 1$. However, when considering this condition dichotomies enhanced condition of separability of the imaginary axis. Further more, the description of this case can be introduce by the simpler notation.

Due to the relative isolation of the spectrum there is a finite loop $\Gamma_+ \subset p^L(M) \cap \mathbb{C}_+$ and bounded region containing σ_+.

According to the relative spectral theorem [3] spaces $\mathfrak{U}^1$ and $\mathfrak{F}^1$ split: $\mathfrak{U}^1 = \mathfrak{U}^+ \oplus \mathfrak{U}^-$, $\mathfrak{F}^1 = \mathfrak{F}^+ \oplus \mathfrak{F}^-$. This splitting of the corresponding projection

$$P_+ = \frac{1}{2\pi i}\int\limits_{\Gamma_+} (\mu L - M)^{-1} L d\mu, \qquad P_- = P - P_+,$$

and

$$Q_+ = \frac{1}{2\pi i}\int\limits_{\Gamma_+} L(\mu L - M)^{-1} d\mu, \qquad Q_- = Q - Q_+.$$

Denote $L_\pm = L\Big|_{\mathfrak{U}^\pm}$, $M_\pm = M\Big|_{\mathrm{dom} M_\pm}$, $\mathrm{dom} M_\pm = \mathrm{dom} M \cap \mathfrak{U}^\pm$. By Lemma 2 [3] $L_\pm \in \mathcal{L}(\mathfrak{U}^\pm; \mathfrak{F}^\pm)$, $M_\pm \in \mathcal{C}l(\mathfrak{U}^\pm; \mathfrak{F}^\pm)$. Furthermore, by the Theorem 1 [3] we have $\sigma^{L_\pm}(M_\pm) = \sigma_\pm$, so the operator M_+ is (L_+, p)-bounded [6], and M_- is (L_-, p)-radial with constant $\beta \leq -\omega < 0$.

Construct a semigroup $\{U^t_\pm \in \mathcal{L}(\mathfrak{U}^\pm) : t \in \overline{\mathbb{R}}_+\}$

$$U^t_\pm = s\text{-}\lim_{k\to\infty} \left(\frac{k}{t} R^{L_\pm}_{\frac{k}{t}}(M_\pm)\right)^k.$$

Due to the fact that the operator M_+ is (L_+, p)-bounded, semigroup $\{U^t_+ \in \mathcal{L}(\mathfrak{U}^+) : t \in \overline{\mathbb{R}}_+\}$ can be extended to the group $\{U^t_+ \in \mathcal{L}(\mathfrak{U}^+) : t \in \mathbb{R}\}$. By Remark 2 operators of the resolving semigroup of equation (0.1) can be represented as

$$U^t = U^t P = U^t(P_+ + P_-) = U^t_+ P_+ + U^t_- P_-.$$

Definition 7. Let $\mathfrak{P}$ is a phase space of equation (0.1). The set $\mathfrak{P}^1 \subset \mathfrak{P}$ is called an *invariant subspace* of this equation, if there exists an unique solution $u = u(t)$ problem (0.1), (1.1) for any of the u_0 from dense in $\mathfrak{P}^1$ lineal, and it has the form $u(t) = U^t u_0 \in \mathfrak{P}^1$ $\forall t \in \mathbb{R}_+$.

Theorem 7. *Suppose that M is strongly (L, p)-radial and holds* (3.5), *then the subspace $\mathfrak{U}^+$ and $\mathfrak{U}^-$ are invariant spaces of equation* (0.1).

Definition 8. Let $\mathfrak{P} \subset \mathfrak{U}$ is a phase space of the equation (0.1) and $\mathfrak{P} = \mathfrak{P}^1 \oplus \mathfrak{P}^2$, where $\mathfrak{P}^k$ is an invariant subspace, $k = 1, 2$. We say that equation (0.1) has an *exponential dichotomy* if its solutions satisfired to the following conditions:

(i) $\exists N_1, \nu_1 > 0$ $\|u^1(t)\| \leq N_1 e^{-\nu_1(s-t)}\|u^1(s)\|$, $s \geq t$;

(ii) $\exists N_2, \nu_2 > 0$, $\|u^2(t)\| \leq N_2 e^{-\nu_2(t-s)}\|u^2(s)\|$, $\forall s \in \mathbb{R}$ $t \in [s, +\infty)$,
where the solution of the equation $u^k(t) \in \mathfrak{P}^k$, $k = 1, 2$.

Remark 7. In other words, the existence of exponential dichotomies of solutions means that if the solution lie in one of invariant subspace then it grow exponentially and if it lie in another one then it is exponentially decrease.

Theorem 8. *Let the operator M is strongly (L,p)-radial and holds* (3.5). *Then the equation* (0.1) *has an exponential dichotomy.*

Let us consider the *linearized phase-field equations.*
Consider the system of equations

$$\theta_t(x,t) + \psi_t(x,t) = \Delta\theta(x,t), \quad (x,t) \in \Omega \times \overline{\mathbb{R}}_+, \tag{3.6}$$

$$\Delta\psi(x,t) + \alpha\psi(x,t) + \beta\theta(x,t) = 0, \quad (x,t) \in \Omega \times \overline{\mathbb{R}}_+, \tag{3.7}$$

equipped with the boundary conditions

$$\frac{\partial\theta}{\partial n}(x,t) + \lambda\theta(x,t) = 0, \quad (x,t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \tag{3.8}$$

$$\frac{\partial\psi}{\partial n}(x,t) + \lambda\psi(x,t) = 0, \quad (x,t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \tag{3.9}$$

which is the linearization at zero of phase-field equations describing within mesoscopic theory of phase transitions of the first order [2]. Here $\Omega \subset \mathbb{R}^s$ is a bounded domain with boundary $\partial\Omega$ of class C^∞, $\lambda \in \mathbb{R}$, $\alpha, \beta \in \mathbb{C}$. Unknown functions are $\theta(x,t)$,$\psi(x,t)$.

We reduce the system (3.6)–(3.9) to equation (2.2). Let's make the replacement $\theta(x,t) + \psi(x,t) = u(x,t)$,$\psi(x,t) = v(x,t)$. Then the system takes the form

$$u_t(x,t) = \Delta u(x,t) - \Delta v(x,t), \quad (x,t) \in \Omega \times \overline{\mathbb{R}}_+, \tag{3.10}$$

$$\Delta v(x,t) + (\alpha - \beta)v(x,t) + \beta u(x,t) = 0, \quad (x,t) \in \Omega \times \overline{\mathbb{R}}_+, \tag{3.11}$$

$$\frac{\partial u}{\partial n}(x,t) + \lambda u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \tag{3.12}$$

$$\frac{\partial v}{\partial n}(x,t) + \lambda v(x,t) = 0, \quad (x,t) \in \partial\Omega \times \overline{\mathbb{R}}_+. \tag{3.13}$$

Let $\mathfrak{U} = \{(u,v) \in (H^2(\Omega))^2 : \left(\frac{\partial}{\partial n} + \lambda\right)u(x) = \left(\frac{\partial}{\partial n} + \lambda\right)v(x) = 0, x \in \partial\Omega\}$, $\mathfrak{F} = (L_2(\Omega))^2$.

$$L = \begin{pmatrix} I & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}, \quad M = \begin{pmatrix} \Delta & -\Delta \\ \beta I & (\alpha-\beta)I + \Delta \end{pmatrix}.$$

Thus constructed statements is $L, M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$. Moreover, if

$$\mathfrak{U}' = \{w \in H^2(\Omega) : \frac{\partial w}{\partial n}(x) + \lambda w(x) = 0,\, x \in \partial\Omega\}$$

then $\ker L = \{0\} \times \mathfrak{U}'$.

Let $Aw = \Delta w$ then $A \in \mathcal{L}(\mathfrak{U}', L_2(\Omega))$. Through $\{\varphi_k : k \in \mathbb{N}\}$ denote orthonormal in the sense of the scalar product $\langle \cdot, \cdot \rangle$ in $L_2(\Omega)$ eigenfunctions of the operator A, numbered by non-increasing eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$ with respect to their multiplicities.

Lemma 5. *Let $\beta - \alpha \notin \sigma(A)$. Then the operator M is strongly $(L, 0)$-radial.*

In this case, L-spectrum of M has the form

$$\sigma^L(M) = \left\{ \mu_k = \frac{(\alpha + \lambda_k)\lambda_k}{\alpha + \lambda_k - \beta}, \ k \in \mathbb{N} \setminus \{l : \lambda_l = \beta - \alpha\} \right\}.$$

Construct projectors

$$P_+ = \begin{pmatrix} \sum\limits_{k:\mathrm{Re}\mu_k > 0} \langle \cdot, \varphi_k \rangle \varphi_k & \mathbb{O} \\ \sum\limits_{k:\mathrm{Re}\mu_k > 0} \frac{\beta \langle \cdot, \varphi_k \rangle \varphi_k}{\beta - \alpha - \lambda_k} & \mathbb{O} \end{pmatrix}, \qquad P_- = \begin{pmatrix} \sum\limits_{k:\mathrm{Re}\mu_k < 0} \langle \cdot, \varphi_k \rangle \varphi_k & \mathbb{O} \\ \sum\limits_{k:\mathrm{Re}\mu_k < 0} \frac{\beta \langle \cdot, \varphi_k \rangle \varphi_k}{\beta - \alpha - \lambda_k} & \mathbb{O} \end{pmatrix}.$$

Theorem 9. *Let $\beta - \alpha, -\alpha, 0 \notin \sigma(A)$. Then the solution of the problem* (3.10) – (3.13) *have an exponential dichotomy.*

More detailed justification assertions in this paragraph can be found in [6].

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Обобщенная теорема о расщеплении для линейных уравнений соболевского типа в относительно радиальном случае

Аннотация. Уравнения соболевского типа в настоящее время составляют обширную область среди неклассических уравнений математической физики. Неклассическими называют те уравнения математической физики, чьи представления в виде уравнений или систем уравнений в частных производных не укладываются в рамках одного из классических типов — эллиптического, параболического или гиперболического. В данной работе доказывается обобщенная теорема о расщеплении пространств и действий операторов для уравнения соболевского типа с относительно радиальным оператором. Отметим, что необходимость в обобщенной теореме о расщеплении появилась при изучении многоточечных начально-конечных для линейных уравнений соболевского типа. В настоящее время эти задачи нашли

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свое применение в теории управляемости и оптимального управления. Основным методом исследования является теория Свиридюка относительного спектра.

Статья кроме введения и списка литературы содержит три части. В первой части приводятся необходимые сведения теории относительно p-радиальных операторов, вторая содержит основной результат статьи — доказательство обобщенной теоремы о расщеплении в случае сильно (L, p)-радиального оператора M. Третья часть содержит применение результатов предыдущего пункта для различных задач, а именно для доказательства однозначной разрешимости многоточечной начально-конечной задачи для уравнения Дзекцера и для исследования дихотомий решений линеаризованной системы уравнений фазового поля. Список литературы не претендует на полноту и отражает лишь вкусы и пристрастия авторов.

Ключевые слова: линейные уравнения соболевского типа, обобщенная теорема о расщеплении, дихотомии решений, многоточечная начально-конечная задача.

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